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LINEAR COMPLEMENTARITY SYSTEMS: ZENO STATES*

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Abstract. A linear complementarity system (LCS) is a hybrid dynamical system defined by a linear time-invariant ordinary differential equation coupled with a finite-dimensional linear complementarity problem (LCP). The present paper is the first of several papers whose goal is to study some fundamental issues associated with an LCS. Specifically, this paper addresses the issue of Zeno states and the related issue of finite number of mode switches in such a system. The cornerstone of our study is an expansion of a solution trajectory to the LCS near a given state in terms of an observability degree of the state. On the basis of this expansion and an inductive argument, we establish that an LCS satisfying the P-property has no strongly Zeno states. We next extend the analysis for such an LCS to a broader class of problems and provide sufficient conditions for a given state to be weakly non-Zeno. While related mode-switch results have been proved by Brunovsky and Sussmann for more general hybrid systems, our analysis exploits the special structure of the LCS and yields new results for the latter that are of independent interest and complement those by these two and other authors.

Key words. linear complementarity systems, Zeno states, P-matrix, complementarity problem

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1. Introduction. A linear complementarity system (LCS) is a special dynamical system defined by an linear ordinary differential equation (ODE) involving an algebraic variable that is required to be a solution of a standard linear complementarity problem (LCP) [11]. While being a special instance of a differential variational inequality, which has recently been studied in great depth in [23], the LCS has itself received an extensive treatment in two excellent Ph.D. theses [5, 13] and in related articles [3, 8, 9, 15, 16, 17]. In addition, the LCS belongs to the broad framework of a hybrid system [18, 20, 30, 34, 35, 36, 26, 28], which is defined by a finite number of smooth ODEs, called *modes*, with transitions between the modes occurring along a state trajectory. Examples of dynamical systems in which the complementarity paradigm has played a prominent role include nonsmooth mechanical systems [2, 24] in general and multibody dynamics simulation under frictional contacts in particular [1, 21, 29, 31, 32], as well as switched electrical networks and switched control systems, e.g., relay systems and variable structure systems [6, 7, 14, 19, 38]. In addition, linear-quadratic dynamic Nash games with linear dynamics and control constraints naturally lead to LCSs with special boundary conditions. For an excellent state-of-the-art review of complementarity systems and their applications in engineering and economics, we refer to the excellent recent article by Schumacher [27].

The LCS occupies a fundamental role in the study of nonsmooth dynamical systems because it is arguably the simplest of such systems. Though seemingly simple, the analysis of the LCS in general is complicated by impulsive and multimodal behavior of its solutions. In the references cited above, such as in the two theses [5, 13],

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the study of the LCS has employed many concepts and results from (constrained) linear systems theory; in particular, the concept of system passivity [37] has played a major role. In contrast to this system-theoretic approach, we feel that a better understanding of the LCS as a basic mathematical model can be achieved by considering the simplest, albeit nontrivial, instance of such a system. Motivated by this contrasting "mathematical programming" approach, we are led to consider an LCS satisfying the "P-property," i.e., where the underlying finite-dimensional LCP has a unique solution for all constant vectors. An immediate consequence of this property is that the LCS is globally equivalent to an ODE with a piecewise linear, thus Lipschitz continuous, right-hand side (which albeit is only implicitly defined). While this is a great simplification, the piecewise linear nature of the right-hand side renders the LCS a nonsmooth system and leads to many important system-theoretic and control issues that require careful study. Several of these topics are the main concern of this and accompanying papers. Extending the class of LCSs with the P-property, we will also consider a broader class of systems and study their (non-)Zeno states.

The organization of the rest of the paper is as follows. In section 2, we formally define the LCS, review some basic results of the LCP, and introduce two new LCP concepts that are useful for our study. Section 3 addresses the question of whether there can be infinitely many mode transitions in any finite time, i.e., the *Zeno behavior* of the LCS. Formal algebraic definitions of (non-)Zeno states and of mode switches that are tailored to the LCS are presented. An expansion based technique is developed to prove non-Zenoness, which is applicable to an LCS with the P-property. Extended Zeno results are presented in section 4. A special bimodal system is considered in section 5. The paper ends with some concluding remarks in the sixth and last section.

2. Preliminary discussion.

Defined by a tuple of four matrices, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{m\times n}$, and $D \in \Re^{m\times m}$, and a vector $x^0 \in \Re^n$, the goal of the LCS is to find trajectories $x(t) \in \Re^n$ and $u(t) \in \Re^m$ satisfying

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ 0 \,\le\, u \,\perp\, & Cx + Du \,\ge\, 0, \\ x(0) &= x^0, \end{aligned} \tag{1}$$

where $\dot{x} \equiv \frac{dx}{dt}$ denotes the time derivative of the trajectory $x(t)$ and $a \perp b$ means that the two vectors a and b are orthogonal, i.e., $a^T b = 0$. While it is in general possible for the above differential and complementarity conditions to hold only at almost all times t, in the present paper, the conditions that we will impose on the tuple (A, B, C, D) will ensure that the x-trajectory is continuously differentiable and the u-trajectory is well defined (albeit not necessarily continuous) on the time interval of interest.

It is clear that LCP theory has a major role to play in the study of the LCS. For this reason, we summarize in the next subsection some of the essential concepts from this theory that are relevant to the developments in this paper. Details of this review can be found in the monograph [11], and the results therein will be used freely. Two new LCP concepts are introduced in subsection 2.2.

2.1. LCP background.

Formally, given a vector $q \in \Re^m$ and a matrix $M \in \Re^{m\times m}$, the aim of the LCP (q, M) is to find a vector $u \in \Re^m$ such that

$$0 \,\le\, u \,\perp\, w \,\equiv\, q + Mu \,\ge\, 0.$$

The solution set of this problem is denoted by $\mathrm{SOL}(q, M)$. Among all matrix classes in LCP theory, the most fundamental one is that of the P-matrices. Specifically, M

is a P-matrix if all its principal minors are positive. This is the class of matrices that will be the starting point in our study of the LCS (1). It is well known that M is a P-matrix if and only if $\mathrm{SOL}(q, M)$ is a singleton for all $q \in \Re^m$; moreover, the unique element of $\mathrm{SOL}(q, M)$, which we denote $u(q)$, is a piecewise linear function of $q \in \Re^m$. This implies in particular that $u(q)$ is (globally) Lipschitz continuous and directionally differentiable. The directional derivative, denoted $u'(q; dq)$, of the solution function $u(q)$ along the direction dq can be expressed as the unique solution of a certain mixed LCP. Specifically, define three fundamental index sets associated with $u(q)$:

$$\begin{aligned}
\alpha &\equiv \{\, i \,:\, u(q)_i \,>\, 0 \,=\, (\, q + Mu(q)\,)_i \,\},\\
\beta &\equiv \{\, i \,:\, u(q)_i \,=\, 0 \,=\, (\, q + Mu(q)\,)_i \,\},\\
\gamma &\equiv \{\, i \,:\, u(q)_i \,=\, 0 \,<\, (\, q + Mu(q)\,)_i \,\}.
\end{aligned}$$

It follows that $u'(q; dq)$ is the unique solution $\widehat{u}$ of the mixed LCP:

$$\begin{aligned}
0 &= (\, dq + M\widehat{u}\,)_\alpha,\\
0 \,\le\, \widehat{u}_\beta &\perp (\, dq + M\widehat{u}\,)_\beta \,\ge\, 0,\\
0 &= \widehat{u}_\gamma.
\end{aligned}$$

The solution set $\mathrm{SOL}(q, M)$ of an LCP is in general the union of finitely many polyhedra, each called a *piece* of this set. Indeed, we have

$$\mathrm{SOL}(q, M) \,=\, \bigcup_{\alpha} \left\{ u \,\in\, \Re^m \,:\, \begin{array}{l} (\, q + Mu\,)_\alpha \,=\, 0, \quad u_\alpha \,\ge\, 0 \\ (\, q + Mu\,)_{\bar\alpha} \,\ge\, 0, \quad u_{\bar\alpha} \,=\, 0 \end{array} \right\},$$

where the union ranges over all subsets α of $\{1, \dots, m\}$. The case where $\mathrm{SOL}(q, M)$ is convex for all $q \in \Re^m$ is particularly important. This case is characterized by the column sufficiency property of the matrix M. Specifically, a matrix M is *column sufficient* if $u \circ Mu \,\le\, 0 \,\Rightarrow\, u \circ Mu \,=\, 0$, where $\circ$ denotes the Hadamard product of two vectors. It is easy to see that the property of column sufficiency is inherited by the principal submatrices of M and also by the principal pivot transforms of M. That is, if M is column sufficient, then so is the principal submatrix $M_{\alpha\alpha}$ for all $\alpha \subseteq \{1, \dots, m\}$; moreover, if $M_{\alpha\alpha}$ is nonsingular, then the matrix below, called the α-principal pivot transform of M,

$$\text{(2)} \qquad \left[\begin{array}{cc} (\, M_{\alpha\alpha}\,)^{-1} & -(\, M_{\alpha\alpha}\,)^{-1} M_{\alpha\bar\alpha} \\ M_{\bar\alpha\alpha}(\, M_{\alpha\alpha}\,)^{-1} & M_{\bar\alpha\bar\alpha} - M_{\bar\alpha\alpha}(\, M_{\alpha\alpha}\,)^{-1} M_{\alpha\bar\alpha} \end{array} \right],$$

is also column sufficient, where $\bar\alpha$ is the complement of α in $\{1, \dots, m\}$. If M is column sufficient, then

$$\mathrm{SOL}(q, M) \,\equiv\, \left\{ u \,\in\, \Re^m \,:\, \begin{array}{l} (\, q + Mu\,)_\alpha \,=\, 0, \quad u_\alpha \,\ge\, 0 \\ (\, q + Mu\,)_{\bar\alpha} \,\ge\, 0, \quad u_{\bar\alpha} \,=\, 0 \end{array} \right\},$$

where α is the set consisting of all indices i for which there exists a solution $u \in \mathrm{SOL}(q, M)$ with $u_i > 0$.

Another known property of the LCP that we need is the "semistability" of its solutions sets. Specifically, by [12, Proposition 5.5.5, Corollary 5.5.9], it follows that for any matrix $M \in \Re^{m\times m}$ and every vector $q \in \Re^m$, there exist positive scalars c and ε such that

$$\| q' - q \| \le \varepsilon \;\Rightarrow\; \mathrm{SOL}(q', M) \subseteq \mathrm{SOL}(q, M) + c \| q - q' \| \mathcal{B},$$

where $\mathcal{B}$ is the (closed) unit ball in $\Re^m$.

2.2. New LCP concepts. While we are unable to directly deal with the entire class of LCSs with a column sufficient matrix D, such matrices provide the motivation to introduce two new LCP concepts to be used later. The first new LCP concept is a broadening of the class of column sufficient matrices that addresses the convexity of the solution sets of the homogeneous problems only.

DEFINITION 1. *The matrix $M \in \Re^{m\times m}$ is said to be* weakly column sufficient *if for every triple of index sets (α, β, γ) that partition $\{1, \dots, m\}$, the set of vectors $u \in \Re^m$ satisfying*

$$\begin{aligned} (Mu)_\alpha &= 0, \\ 0 \le u_\beta \perp (Mu)_\beta &\ge 0, \\ u_\gamma &= 0 \end{aligned} \tag{3}$$

is convex, or equivalently, is polyhedral. □

We leave it to the reader to verify that the convexity of the solution set of (3) is equivalent to its polyhedrality. Besides the class of column sufficient matrices which must be weakly column sufficient, a "nondegenerate matrix," i.e., one whose principal minors are all nonzero, is also weakly column sufficient; indeed if M is nondegenerate, then the only solution to the system (3) is the zero vector. The following result summarizes several properties of weak column sufficiency.

PROPOSITION 2. *Let $M \in \Re^{m\times m}$ be weakly column sufficient. The following statements are valid:*

(a) *For every subset $\tilde\alpha$ of $\{1, \dots, m\}$, the principal submatrix $M_{\tilde\alpha\tilde\alpha}$ is weakly column sufficient.*

(b) *For every subset $\tilde\alpha$ of $\{1, \dots, m\}$ such that $M_{\tilde\alpha\tilde\alpha}$ is nonsingular, the $\tilde\alpha$-principal pivot transform (2) of M is weakly column sufficient.*

(c) *The solution set of the homogeneous LCP $(0, M)$ is polyhedral; in fact,*

$$SOL(0, M) \equiv \left\{ u \in \Re^m : \begin{array}{ll} (Mu)_\alpha = 0, & u_\alpha \ge 0 \\ (Mu)_{\bar\alpha} \ge 0, & u_{\bar\alpha} = 0 \end{array} \right\},$$

where α is the set consisting of all indices i for which there exists $u \in SOL(0, M)$ such that $u_i > 0$ and $\bar\alpha$ is the complement of α.

Proof. Let α', β', and γ' be three index sets partitioning the subset $\tilde\alpha$. A vector $u_{\tilde\alpha}$ satisfies the system

$$\begin{aligned} (M_{\tilde\alpha\tilde\alpha} u_{\tilde\alpha})_{\alpha'} &= 0, \\ 0 \le u_{\beta'} \perp (M_{\tilde\alpha\tilde\alpha} u_{\tilde\alpha})_{\beta'} &\ge 0, \\ u_{\gamma'} &= 0 \end{aligned}$$

if and only if the vector $u \equiv (u_{\tilde\alpha}, 0)$ satisfies the system (3) with $(\alpha, \beta, \gamma) = (\alpha', \beta', \gamma' \cup \hat\alpha)$, where $\hat\alpha$ is the complement of $\tilde\alpha$ in $\{1, \dots, m\}$. Consequently, part (a) holds. To prove (b), let $\widetilde{M}$ be the $\tilde{\alpha}$-principal pivot transform (2) of M. Let α', β', and γ' be three index sets partitioning the set $\{1,\ldots,m\}$. Consider the system

$$(4)\qquad \begin{aligned} (\widetilde{M}\widetilde{u})_{\alpha'} &= 0, \\ 0 \le \widetilde{u}_{\beta'} \perp (\widetilde{M}\widetilde{u})_{\beta'} &\ge 0, \\ \widetilde{u}_{\gamma'} &= 0. \end{aligned}$$

Let $\widetilde{w} \equiv \widetilde{M}\widetilde{u}$. By pivoting on $(M_{\tilde{\alpha}\tilde{\alpha}})^{-1}$ in $\widetilde{M}$, we can recover the original system $w = Mu$, where the variables u and w are related to $\widetilde{u}$ and $\widetilde{w}$ via the identities:

$$w \equiv \begin{pmatrix} w_{\tilde{\alpha}} \\ w_{\hat{\alpha}} \end{pmatrix}, \quad u \equiv \begin{pmatrix} u_{\tilde{\alpha}} \\ u_{\hat{\alpha}} \end{pmatrix},$$

$$\widetilde{w} = \begin{pmatrix} u_{\tilde{\alpha}} \\ w_{\hat{\alpha}} \end{pmatrix}, \quad \widetilde{u} = \begin{pmatrix} w_{\tilde{\alpha}} \\ u_{\hat{\alpha}} \end{pmatrix}.$$

Therefore, system (4) is equivalent to system (3) for some suitable triple (α, β, γ) that partitions $\{1,\ldots,m\}$. From this equivalence, the convexity of the solution set of the former system can be easily proved. This establishes (b). To prove (c), we note that the LCP $(0, M)$ is just the system (3) with $\beta = \{1,\ldots,m\}$. Hence the polyhedrality of $\mathrm{SOL}(0, M)$ follows from the weak column sufficiency of M. The representation of $\mathrm{SOL}(0, M)$ can be proved in the same way as in the case of a column sufficient matrix; see the proof in [11, Theorem 3.5.8] for details. □

To introduce the second new LCP concept, we note that if $q \ne 0$, then for every solution u of the LCP (q, M), there must exist at least one index i such that $u_i > 0$ or $w_i \equiv (q + Mu)_i > 0$. Define two sets of "identifiable indices":

$$\begin{aligned} \mathcal{I}_u &\equiv \{\, i : u_i > 0 \quad \forall\, u \in \mathrm{SOL}(q, M)\,\}, \\ \mathcal{I}_w &\equiv \{\, i : (q + Mu)_i > 0 \quad \forall\, u \in \mathrm{SOL}(q, M)\,\}, \end{aligned}$$

one, or both, of which may be empty in general.

Definition 3. *The LCP (q, M), where $q \ne 0$, is* identifiable *if the following two conditions hold:*

(a) $\mathcal{I}_u \cup \mathcal{I}_w \ne \emptyset$, *and*

(b) *the principal submatrix $M_{\mathcal{I}_u\mathcal{I}_u}$ is nonsingular if $\mathcal{I}_u \ne \emptyset$. (By convention, this condition is vacuously true if $\mathcal{I}_u$ is empty.)* □

If the LCP (q, M), where $q \ne 0$, has a unique solution u, then the LCP is identifiable if $M_{\alpha\alpha}$ is nonsingular, where α is the (possibly empty) support of u. The following lemma asserts a positivity property of the "identifiable variables" of an LCP.

Proposition 4. *For any pair (q, M) with $q \ne 0$, there exists a scalar $\sigma > 0$ such that $u_i \ge \sigma$ and $(q + Mu)_j \ge \sigma$ for all $u \in \mathrm{SOL}(q, M)$ and all $i \in \mathcal{I}_u$ and $j \in \mathcal{I}_w$.*

Proof. We prove the claim only for the u-variable. For each $i \in \mathcal{I}_u$, consider the optimization problem

$$\begin{aligned} &\text{minimize} && u_i \\ &\text{subject to} && u \in \mathrm{SOL}(q, M). \end{aligned}$$

Since the feasible set of this problem is the union of finitely many polyhedra and its objective function is linear and positive (hence bounded below) on this set, it follows from linear programming theory that the above problem attains a finite minimum objective value which must be positive. The desired claim follows readily. □

2.3. Back to the LCS. Returning to the LCS (1), assume that D is a P-matrix. In this case, (1) is equivalent to the ODE

$$\dot{x} = Ax + Bu(Cx), \quad x(0) = x^0, \tag{5}$$

where the right-hand side $Ax + Bu(Cx)$ is a piecewise linear function of x. (Note that $u(0) = 0$.) As such, (1) has a unique solution trajectory $(x(t), u(t))$ defined on $[0, \infty)$ with $x(t)$ being continuously differentiable and $u(t) \equiv u(Cx(t))$ being continuous. In contrast to the above representation (5) which involves the implicit function $u(Cx)$, the right-hand side of the LCS (1) can be represented explicitly using the complementarity cones associated with the matrix D. Specifically, for each index subset δ of $\{1, \dots, m\}$ with complement $\bar{\delta}$, define the polyhedral cone

$$\mathcal{C}_\delta \equiv \left\{ q \in \Re^m : E_\delta \begin{pmatrix} q_\delta \\ q_{\bar{\delta}} \end{pmatrix} \geq 0 \right\}, \tag{6}$$

where

$$E_\delta \equiv \begin{bmatrix} -(D_{\delta\delta})^{-1} & 0 \\ -D_{\bar{\delta}\delta}(D_{\delta\delta})^{-1} & I \end{bmatrix} \in \Re^{m \times m}.$$

Since D is a P-matrix, it is clear that E_δ is well defined and nonsingular. Defining the matrix

$$K_\delta \equiv \begin{bmatrix} -(D_{\delta\delta})^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

we have $u(Cx) = K_\delta Cx$, provided that $Cx \in \mathcal{C}_\delta$. Consequently, the ODE (5) can be written equivalently as

$$\dot{x} = (A + BK_\delta C)x \quad \text{if } E_\delta Cx \geq 0,$$

whose right-hand side is now in an *explicit*, piecewise linear form.

Consider next the case where D is not a P-matrix but the submatrix $D_{\alpha\alpha}$ is nonsingular for some subset α of $\{1, \dots, m\}$. We can define a system that is equivalent to (1) by "pivoting" on $D_{\alpha\alpha}$ as done for a standard LCP [11], i.e., by solving for the variable u_α in the equation

$$w_\alpha = C_{\alpha\cdot}x + D_{\alpha\alpha}u_\alpha + D_{\alpha\bar{\alpha}}u_{\bar{\alpha}}$$

in terms of the other variables w_α, x, and $u_{\bar{\alpha}}$, where $\bar{\alpha}$ is the complement of α in $\{1, \dots, m\}$, and then substituting the resulting expression for u_α into the other conditions in (1). The equivalent LCS, which we call the *α-principal transform* of (1), is

$$\begin{aligned} \dot{x} &= \widetilde{A}x + \widetilde{B}\widetilde{u}, \\ 0 \leq \widetilde{u} &\perp \widetilde{C}x + \widetilde{D}\widetilde{u} \geq 0, \\ x(0) &= x^0, \end{aligned} \tag{7}$$

where $\tilde{u} = (w_\alpha, u_{\bar{\alpha}})$ and

$$
\begin{aligned}
\widetilde{A} &\equiv A - B_{\cdot\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot},\\
\widetilde{B} &\equiv \left[\, B_{\cdot\alpha}(D_{\alpha\alpha})^{-1} \quad B_{\cdot\bar{\alpha}} - B_{\cdot\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar{\alpha}} \,\right],\\
\widetilde{C} &\equiv \begin{bmatrix} -(D_{\alpha\alpha})^{-1}C_{\alpha\cdot} \\ C_{\bar{\alpha}\cdot} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot} \end{bmatrix},\\
\widetilde{D} &\equiv \begin{bmatrix} (D_{\alpha\alpha})^{-1} & -(D_{\alpha\alpha})^{-1}D_{\alpha\bar{\alpha}} \\ D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1} & D_{\bar{\alpha}\bar{\alpha}} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar{\alpha}} \end{bmatrix}.
\end{aligned} \tag{8}
$$

We call the tuple $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ the *α-principal transform* of (A, B, C, D). Of particular importance in the subsequent analysis is the following principal subsystem of this transform:

$$
\begin{aligned}
\dot{x} &= \widetilde{A}x + [\, B_{\cdot\bar{\alpha}} - B_{\cdot\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar{\alpha}}\,]u_{\bar{\alpha}},\\
0 \leq u_{\bar{\alpha}} &\perp [\, C_{\bar{\alpha}\cdot} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot}\,]x + [\, D_{\bar{\alpha}\bar{\alpha}} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar{\alpha}}\,]u_{\bar{\alpha}} \geq 0.
\end{aligned} \tag{9}
$$

To illustrate the role of the latter subsystem, suppose that at some state $x(t_*) = x^*$, the corresponding algebraic vector $u(t_*) = u^*$ is a strongly regular solution [25] of the LCP (Cx^*, D). This means that the submatrix $D_{\alpha_*\alpha_*}$ is nonsingular and the Schur complement $D_{\beta_*\beta_*} - D_{\beta_*\alpha_*}(D_{\alpha_*\alpha_*})^{-1}D_{\alpha_*\beta_*}$ is a P-matrix, where

$$
\begin{aligned}
\alpha_* &\equiv \{\, i : u_i^* > 0 = (Cx^* + Du^*)_i \,\},\\
\beta_* &\equiv \{\, i : u_i^* = 0 = (Cx^* + Du^*)_i \,\},\\
\gamma_* &\equiv \{\, i : u_i^* = 0 < (Cx^* + Du^*)_i \,\}.
\end{aligned}
$$

If the solution trajectory $(x(t), u(t))$ is continuous near t_*, it then follows that for all t sufficiently near t_*, $(Cx(t) + Du(t))_i > 0$ for all $i \in \gamma_*$ and $u_i(t) > 0$ for all $i \in \alpha_*$. This implies that $u_i(t) = 0$ for all $i \in \gamma_*$ and $(Cx(t) + Du(t))_i = 0$ for all $i \in \alpha_*$. Hence, locally for t near t_*, the trajectory $(x(t), u(t))$ must satisfy the following mixed LCS obtained by fixing some variables at zero:

$$
\begin{aligned}
\dot{x} &= Ax + Bu,\\
0 &= (Cx + Du)_i \quad \forall\, i \in \alpha_*,\\
0 \leq u_i &\perp (Cx + Du)_i \geq 0 \quad \forall\, i \in \beta_*,\\
0 &= u_i \quad \forall\, i \in \gamma_*.
\end{aligned} \tag{10}
$$

Since $D_{\alpha_*\alpha_*}$ is nonsingular, we can carry out the pivot operation as described above and deduce that (10) is equivalent to, for all t sufficiently near t_*,

$$
\begin{aligned}
\dot{x} &= \overline{A}x + \overline{B}u_\beta,\\
0 \leq u_\beta &\perp \overline{C}x + \overline{D}u_\beta \geq 0,
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\overline{A} &\equiv A - B_{\cdot\alpha_*}(D_{\alpha_*\alpha_*})^{-1}C_{\alpha_*}, & \overline{B} &\equiv B_{\cdot\beta_*} - B_{\cdot\alpha_*}(D_{\alpha_*\alpha_*})^{-1}D_{\alpha_*\beta_*},\\
\overline{C} &\equiv C_{\beta_*\cdot} - D_{\beta_*\alpha_*}(D_{\alpha_*\alpha_*})^{-1}C_{\alpha_*}, & \overline{D} &\equiv D_{\beta_*\beta_*} - D_{\beta\alpha_*}(D_{\alpha_*\alpha_*})^{-1}D_{\alpha_*\beta_*}.
\end{aligned}
$$

The resulting LCS (11) has the P-property. We summarize this reduction in the following result, which we will later use to deduce an important consequence of the state x^*; see Corollary 15. For further discussion of the above reduction process, see [23, section 5.2].

PROPOSITION 5. *Let* $(x(t), u(t))$ *be a solution trajectory of the LCS* (1) *that is continuous near a time* t_*. *If* $u(t_*)$ *is a strongly regular solution of the LCP* $(Cx(t_*), D)$, *then* $(x(t), u(t))$ *must satisfy the reduced system* (11) *locally near* t_*. ∎

Finally, for any nonsingular constant matrix P, we can consider the change of variables $\bar{x} \equiv Px$ and obtain an LCS in the $\bar{x}$-variable that is equivalent to the original (1). This equivalent LCS is

$$\begin{aligned} \dot{\bar{x}} &= PAP^{-1}\bar{x} + PBu, \\ 0 \le u \perp CP^{-1}\bar{x} + Du &\ge 0, \\ \bar{x}(0) &= Px^0. \end{aligned}$$

Particularly useful for us later (see the proof of Lemma 12) is the transformation so that the pair (PAP^{-1}, CP^{-1}) is of a particular form satisfying a favorable observability condition.

3. Zeno states of an LCS. In what follows, we define two types of Zeno states of a general LCS; see Definition 6. Generally speaking, the presence of a Zeno state in a hybrid system could have an adverse effect on the numerical simulation of a solution trajectory to the system. This issue, which is closely tied to mode switches, has been dealt with extensively in the literature; see, e.g., [4, 9, 19, 33, 38]. In particular, for hybrid systems described by ODEs with piecewise real analytic right-hand sides, the results by Brunovsky [4] and Sussmann [33] show that there is a finite number of mode switches (defined in the sense in the cited references). While the latter results are in principle applicable to the LCS with the P-property, their treatment does not reveal the important complementarity nature of the LCS. (See [5, 9] for some special Zeno results for the LCS where the D matrix is positive definite or satisfies a passifiability assumption.) Because of the fundamental role of the LCS in hybrid system theory, it is useful to have a simplified approach that exploits the characteristics of the LCS. Most importantly, the Zeno concepts defined below are of a refined, algebraic nature that takes into account possible degeneracy of the solutions to the complementarity conditions. Analytically, our proofs of the main Zeno results, Theorems 9 and 21, are based on a local expansion of a solution trajectory to the LCS (Lemma 14) which is a new result by itself and enables us to study systems failing the P-property. Furthermore, this expansion reveals a local property of a solution trajectory of (1) in terms of an "observability degree" of a given state relative to the pair (C, A).

As is well known, an LCS is a special linear hybrid system with finitely many "modes," where a mode is a linear differential algebraic equation (LDAE) defined by a pair of disjoint index sets $(\alpha, \bar{\alpha})$ whose union is the index set $\{1, \dots, m\}$; specifically, such an LDAE is as follows:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ 0 &= (\, Cx + Du \,)_\alpha, \\ 0 &= u_{\bar{\alpha}}. \end{aligned} \tag{12}$$

Every solution trajectory of the LCS (1) must satisfy, at every time instant, the above LDAE for a certain pair $(\alpha, \bar{\alpha})$ that is dependent on the time. Conversely, if $(x(t), u(t))$

is a solution trajectory satisfying the latter LDAE, then $(x(t), u(t))$ satisfies the LCS (1) if $(Cx(t) + Du(t))_{\bar{\alpha}} \geq 0$ and $u(t)_\alpha \geq 0$. In general, it is possible for a solution trajectory $(x(t), u(t))$ of the LCS (1) to satisfy at any given time t the LDAE (12) for multiple pairs of index sets, due to degeneracy of the complementarity conditions. For every such trajectory, at each time t_* that is neither the initial nor the terminal time, there exist (i) an infinite sequence of times $\{t_k^-\}$ converging to t_* from the left and a pair $(\alpha_-, \bar{\alpha}_-)$ of index sets partitioning $\{1, \ldots, m\}$ such that $(x(t_k^-), u(t_k^-))$ satisfies the LDAE (12) corresponding to $(\alpha_-, \bar{\alpha}_-)$ for all k, and (ii) an infinite sequence of times $\{t_k^+\}$ converging to t_* from the right and a pair $(\alpha_+, \bar{\alpha}_+)$ of partitioning index sets such that $(x(t_k^+), u(t_k^+))$ satisfies the LDAE (12) corresponding to $(\alpha_+, \bar{\alpha}_+)$ for all k. An intuitive definition of a *mode switch* at time t_* is that the two pairs of index sets $(\alpha_-, \bar{\alpha}_-)$ and $(\alpha_+, \bar{\alpha}_+)$ are not equal. Roughly speaking, a *Zeno state* of an LCS is a state near which there are infinitely many modes switches.

We now formalize the above informal discussion by defining "weak" and "strong" Zeno states of a solution trajectory $(x(t), u(t))$ of the LCS (1). Both are local properties of a state. The strong Zeno concept is more refined than the weak Zeno concept and is defined in terms of the index sets

$$\alpha(t) \equiv \{\, i \,:\, u_i(t) \,>\, 0 \,=\, (\, Cx(t) + Du(t)\,)_i \,\},$$
$$\beta(t) \equiv \{\, i \,:\, u_i(t) \,=\, 0 \,=\, (\, Cx(t) + Du(t)\,)_i \,\},$$
$$\gamma(t) \equiv \{\, i \,:\, u_i(t) \,=\, 0 \,<\, (\, Cx(t) + Du(t)\,)_i \,\};$$

in contrast, the weak Zeno concept relaxes the strong concept by restricting to the two combined sets $\alpha(t) \cup \beta(t)$ and $\gamma(t) \cup \beta(t)$. The two concepts coincide if $u(t)$ is a nondegenerate solution of the LCP $(Cx(t), D)$ for all t near t_*, in which case, the degenerate set $\beta(t)$ is empty for all such t.

DEFINITION 6. *Let* $(x(t), u(t))$ *be a solution trajectory of* (1) *and let* $x(t_*) = x^*$. *We say that* x^* *is*

(a) strongly left non-Zeno relative to $(x(t), u(t))$ *if a scalar* $\varepsilon_- > 0$ *and a triple of index sets* $(\alpha_-, \beta_-, \gamma_-)$ *exist such that* $(\alpha(t), \beta(t), \gamma(t)) = (\alpha_-, \beta_-, \gamma_-)$ *for every* $t \in [t_* - \varepsilon_-, t_*)$*;*

(b) strongly right non-Zeno relative to $(x(t), u(t))$ *if a scalar* $\varepsilon_+ > 0$ *and a triple* $(\alpha_+, \beta_+, \gamma_+)$ *of index sets exist such that* $(\alpha(t), \beta(t), \gamma(t)) = (\alpha_+, \beta_+, \gamma_+)$ *for every* $t \in (t_*, t_* + \varepsilon_+]$*;*

(c) weakly left non-Zeno relative to $(x(t), u(t))$ *if a scalar* $\varepsilon_- > 0$ *and a pair of index sets* α_- *and* $\bar{\alpha}_-$ *partitioning* $\{1, \ldots, m\}$ *exist such that* $(x(t), u(t))$ *satisfies the LDAE* (12) *corresponding to* $(\alpha_-, \bar{\alpha}_-)$ *for all* $t \in [\, t_* - \varepsilon_-, t_*\,)$*;*

(d) weakly right non-Zeno relative to $(x(t), u(t))$ *if a scalar* $\varepsilon_+ > 0$ *and a pair of index sets* α_+ *and* $\bar{\alpha}_+$ *partitioning* $\{1, \ldots, m\}$ *exist such that* $(x(t), u(t))$ *satisfies the LDAE* (12) *corresponding to* $(\alpha_+, \bar{\alpha}_+)$ *for all* $t \in (t_*, t_* + \varepsilon_+]$.

The state $x^* \equiv x(t_*)$ *is said to be* left (right) Zeno of the first (second) kind relative to the trajectory $(x(t), u(t))$ *if it is* not *strongly (weakly) left (right) non-Zeno relative to the same trajectory. When* x^* *is strongly (weakly) left* and *right non-Zero, then we say that* x^* *is strongly (weakly) non-Zeno; when* x^* *is either left* or *right Zeno of the first (second) kind, then we say that* x^* *is Zeno of the first (second) kind. When the trajectory is clear from the context, we will omit the phrase "relative to the trajectory."* □

Definition 6 is applicable to both the initial and the terminal states of an LCS. Specifically, if x^* is the initial state x^0 of the LCS (1), then we are interested only in the right (non-)Zeno property of x^*; similarly, if x^* is the terminal state $x(T)$ of the LCS

(1) at a prescribed terminal time $T > 0$, then we are interested only in the left (non-) Zeno property of x^*. It is clear that the strongly (left or right) non-Zeno properties must imply the respective weakly (left or right) non-Zeno properties. Nevertheless, the converse is clearly not always true. In essence, the left Zeno properties of a state refer to its reachability from the left, and the right Zeno properties refer to the continuation from the state. Obviously, these properties have important numerical implications when the LCS is solved by a time-stepping method. For instance, the numerical methods discussed in [30] for solving ODEs with discontinuous right-hand sides are based on the presumed absence of Zeno states. Further discussion of such numerical matters is beyond the scope of this paper.

An important remark should be made for Definition 6: namely, this definition pertains to the two trajectories $x(t)$ and $u(t)$ jointly. This is distinct from the treatment in [4, 33] which is applicable to the implicit formulation (5) of the LCS in which the algebraic variable u is eliminated and treated only implicitly. It is not a straightforward task to directly apply the results in these cited references to analyze the Zeno properties of the LCS as described in Definition 6, where the u-trajectory plays a prominent role. In many realistic applications of the LCS (such as in contact mechanics), the role of the algebraic variable u is as important as the differential variable x; thus, an explicit treatment of the former, as emphasized herein, is warranted.

Zeno states are closely tied to mode switches, which we formally define next. For simplicity, we present the definition below only for a time that is neither the initial nor the terminal time of a trajectory. The triple of index sets $(\alpha(t), \beta(t), \gamma(t))$ in this definition are the fundamental index sets associated with the pair $(x(t), u(t))$.

DEFINITION 7. *Let $(x(t), u(t))$ be a solution trajectory of* (1) *and let t_* be an intermediate time of this trajectory. We say that t_* is a*

(a) switch time of the first kind *relative to $(x(t), u(t))$ if there exist two triples of index sets, $(\alpha_-, \beta_-, \gamma_-)$ and $(\alpha_+, \beta_+, \gamma_+)$, and two infinite sequences of times, $\{t_k^-\}$ and $\{t_k^+\}$, the former converging to t_* from the left and the latter converging to t_* from the right, such that, for all k,*

$$(\alpha(t_k^-), \beta(t_k^-), \gamma(t_k^-)) = (\alpha_-, \beta_-, \gamma_-) \neq (\alpha_+, \beta_+, \gamma_+) = (\alpha(t_k^+), \beta(t_k^+), \gamma(t_k^+));$$

(b) switch time of the second kind *relative to $(x(t), u(t))$ if there exist two infinite sequences of times, $\{t_k^-\}$ and $\{t_k^+\}$, the former converging to t_* from the left and the latter converging to t_* from the right, such that for* no *pair of index sets $(\alpha, \bar{\alpha}_-)$ partitioning $\{1, \ldots, m\}$, $(x(t_k^-), u(t_k^-))$ and $(x(t_k^+), u(t_k^+))$ both satisfy the LDAE* (12) *for all k.* □

The following result shows that the absence of Zeno states in a finite time of interval provides a sufficient condition for the finite number of switch times in the interval.

PROPOSITION 8. *Let $(x(t), u(t))$ be a solution trajectory of the LCS* (1) *defined on an open interval containing $[0, T]$. If the trajectory has no Zeno states of the first (second) kind, then there is a finite number of switch times of the first (second) kind relative to $(x(t), u(t))$ in $[0, T]$.*

Proof. We prove the result for the "second" kind only. If $(x(t), u(t))$ contains no Zeno states of the second kind, then for every $t \in [0, T]$, there exist a right neighborhood $\mathcal{N}_t^+ \equiv (t, t + \varepsilon_t)$ and a left neighborhood $\mathcal{N}_t^- \equiv (t - \varepsilon_t, t)$ of t, for some scalar $\varepsilon_t > 0$, and two pairs of index sets, $(\alpha_t^+, \bar{\alpha}_t^+)$ and $(\alpha_t^-, \bar{\alpha}_t^-)$, both partitioning $\{1, \ldots, m\}$, such that for all $t' \in \mathcal{N}_t^+$, the pair $(x(t'), u(t'))$ satisfies the LDAE (12) corresponding to $(\alpha_t^+, \bar{\alpha}_t^+)$, and that for all $t' \in \mathcal{N}_t^-$, the pair $(x(t'), u(t'))$ satisfies the LDAE (12) corresponding to $(\alpha_t^-, \bar{\alpha}_t^-)$. The family

$$\{ (t - \varepsilon_t, t + \varepsilon_t) \,:\, t \,\in\, [0, T] \}$$

constitutes an open covering of the compact interval $[0, T]$. Hence there exists a finite sequence $\{t_0, t_1, \ldots, t_\ell\} \subset [0, T]$ such that

$$[\,0, T\,] \,\subset\, \bigcup_{i=0}^{\ell} [\, t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}\,].$$

By refining the partition on the right-hand side, we may assume without loss of generality that there exist a finite sequence of times $0 = t'_0 < t'_1 < \cdots < t'_k < t'_{k+1} = T$ and a corresponding sequence of index sets α_i of $\{1, \ldots, m\}$ with respective complements $\bar{\alpha}_i$ such that $(x(t), u(t))$ satisfies (12) corresponding to $(\alpha_i, \bar{\alpha}_i)$ for all $t \in (t'_i, t'_{i+1})$, $i = 0, 1, \ldots, k$. Consequently, the only possible switch times of the second type in the interval $[0, T]$ are the times t'_i for $i = 0, 1, \ldots, k+1$. ∎

3.1. The P-matrix case. The following is the main Zeno result for an LCS with the P-property.

THEOREM 9. *If D is a P-matrix, then all states of the LCS* (1) *must be strongly non-Zeno.*

The proof of the above theorem is accomplished via several lemmas. The first such lemma, which is a global and time-invariant version of Proposition 5.3 in [26], gives a decay rate for a Lipschitz continuous system.

LEMMA 10. *Let $\dot{x} = f(x)$, $x(0) = x^0$ be a dynamical system on $\Re^n$, where $f(x)$ is globally Lipschitz continuous in x with Lipschitz constant $L \geq 0$. If $x = 0$ is an equilibrium of the system, i.e., $f(0) = 0$, then*

$$\| \, x^0 \, \|_2 \, e^{-Lt} \,\leq\, \| \, x(t) \, \|_2 \,\leq\, \|x^0\|_2 \; e^{Lt} \quad \forall \, t \,\geq\, 0.$$

The main proof of Theorem 9 is divided into two parts, depending on whether a state x^* in question is observable or unobservable with respect to the pair (C, A). The concept of observability is well known for a linear time-invariant system $\dot{x} = Ax + Bu$ and $y = Cx + Du$ and is briefly reviewed here. A state $x \in \Re^n$ is *unobservable* with respect to (C, A) if $Ce^{At}x \equiv 0$ for all t; otherwise it is called *observable* with respect to (C, A). Without confusion, we usually simply call a state observable/unobservable. The set of all unobservable states is a subspace of $\Re^n$, called the *unobservable subspace* of the pair (C, A). An equivalent condition for a state x being observable is that $CA^k x \neq 0$ for some $0 \leq k \leq n - 1$. The linear system is *observable* if $x = 0$ is the only unobservable state. In such the case, we call (C, A) an observable pair.

The next lemma asserts that the LCS (1) with the P-property is trivial if the initial state x^0 is unobservable.

LEMMA 11. *Let D be a P-matrix. If x^0 is unobservable, then the unique solution trajectory of* (1) *is $(x(t), u(t)) = (e^{At}x^0, 0)$ for all $t \geq 0$. In this case, we have $\beta(t) = \{1, \ldots, m\}$ for all $t \geq 0$; hence, all states $x(t)$ are strongly non-Zeno.*

Proof. This follows easily from the uniqueness of the solution trajectory and the fact that $Ce^{At}x^0 = 0$ for all $t \geq 0$. ∎

Lemma 11 suggests that we may assume without loss of generality that x^0 is observable. The next lemma asserts that in this case, all states on the trajectory $x(t)$ are observable.

LEMMA 12. *Let D be a P-matrix. If $x^0 = x(0)$ is observable, then so is $x(t)$ for all $t \geq 0$.*

Proof. Lemma 10 is applicable to the implicit form (5) of the LCS. It follows that $\|x(t)\| \geq \|x^0\|e^{-Lt}$ for some constant $L \geq 0$. Since x^0 is observable, it is not zero. Hence $x(t) \neq 0$ for all $t \geq 0$. Consequently, we may assume without loss of generality that (C, A) is an unobservable pair. Let $\overline{O}(C, A)$ denote the unobservable subspace whose dimension is n_2 with $1 \leq n_2 \leq n$. According to linear system theory [10, p. 203], there exists a nonsingular matrix $P \in \Re^{n\times n}$ such that the change of variables $\bar{x} = Px$ transforms the original linear system (1) into the observable canonical form

$$\dot{\bar{x}} \;=\; \begin{pmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{uo} \end{pmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{pmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u,$$

$$Cx \;=\; \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{pmatrix} = \bar{C}_o\,\bar{x}_o,$$

and $(\bar{C}_o, \bar{A}_o)$ is an observable pair, $\bar{x} \equiv \left(\begin{smallmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{smallmatrix}\right)$ is the transformed state, $\bar{x}_o \in \Re^{n-n_2}$ and $\bar{x}_{uo} \in \Re^{n_2}$ correspond to the observable part and unobservable part of $\bar{x}$, respectively. Hence, the LCS (1) can be decomposed into the observable dynamics

$$\begin{aligned} \dot{\bar{x}}_o \;&=\; \bar{A}_o\bar{x}_o + \bar{B}_o u, \\ 0 \;\leq\; u \;&\perp\; \bar{C}_o\bar{x}_o + Du \geq 0, \end{aligned} \tag{13}$$

and the unobservable dynamics

$$\dot{\bar{x}}_{uo} = \bar{A}_{21}\bar{x}_o + \bar{A}_{uo}\bar{x}_{uo} + \bar{B}_{uo}u. \tag{14}$$

Moreover, any unobservable state $\widehat{x} \in \overline{O}(C, A)$ is transformed to the following form under the above transformation:

$$P\widehat{x} \;=\; \begin{pmatrix} 0 \\ \bar{x}_{uo} \end{pmatrix}.$$

This means that the observable part of an original unobservable state must be zero and the observable part of an original observable state must not be zero. Since (13) remains an LCS with the P-property, and since $\bar{x}_o(0) \neq 0$ because $x(0)$ is observable, it follows that $\bar{x}_o(t) \neq 0$ for all $t \geq 0$, which means that $x(t)$ must be observable. ∎

A noteworthy remark is that Lemma 12 can be proved using the reverse-time argument.[1] Letting $(x^r(t), u^r(t)) \equiv (x(-t), u(-t))$ for all $t \geq 0$, one easily sees that the pair $(x^r(t), u^r(t))$ satisfies a reverse-time LCS for all $t \geq 0$:

$$\begin{aligned} \dot{x}^r(t) &= -Ax^r(t) - Bu^r(t), \\ 0 \leq u^r(t) &\perp Cx^r(t) + Du^r(t) \geq 0. \end{aligned}$$

Hence, the reverse-time LCS $(-A, -B, C, D)$ preserves the P-property and its solution pair is unique as well. Suppose $x(0)$ is observable but $x(t)$ is not at some $t \geq 0$. Then using the reverse-time LCS and Lemma 11, $x(0) = e^{-At}x(t)$, which is unobservable as well. However, the decomposition of the LCS into the observable dynamics (13)

[1] We thank an anonymous reviewer for bringing this remark to our attention.

and the unobservable dynamics (14) in Lemma 12 has its own interest. Therefore, we present it for this purpose.

Combining the above two lemmas, we have therefore proved Theorem 9 for the unobservable states. We formally state this conclusion in the following corollary, which requires no proof.

COROLLARY 13. *Any unobservable state of an LCS with the P-property is strongly non-Zeno.* □

We next turn our attention to the observable states. The cornerstone of the treatment of these states is an expansion of the solution trajectory near any given time t_*. We will make use of the (unique) solution $u(\pm CA^k x)$ of the LCPs $(\pm CA^k x, D)$. In general, except for the fact that both are nonnegative, the two vectors $u^{k+}(x) \equiv u(CA^k x^*)$ and $u^{k-}(x) \equiv u(-CA^k x^*)$ have very little to do with each other. Since D is a P-matrix, we can speak of the directional derivative of the solution function $u'(q; dq)$ of the LCP (q, D) at the vectors $q \equiv \pm CA^k x$ along the directions $dq \equiv \pm(CA^{k+1}x + CBu^{k\pm}(x))$. Specifically, we will use the following directional derivatives:

$$
\begin{aligned}
&u'(CA^k x; CA^{k+1}x + CBu^{k+}(x)) \\
&\quad = \lim_{\tau \downarrow 0} \frac{u(CA^k x + \tau(CA^{k+1}x + CBu^{k+}(x))) - u(CA^k x)}{\tau}, \\
&u'(CA^k x; -(CA^{k+1}x + CBu^{k+}(x))) \\
&\quad = \lim_{\tau \downarrow 0} \frac{u(CA^k x - \tau(CA^{k+1}x + CBu^{k+}(x))) - u(CA^k x)}{\tau}, \\
&u'(-CA^k x; CA^{k+1}x - CBu^{k-}(x)) \\
&\quad = \lim_{\tau \downarrow 0} \frac{u(-CA^k x + \tau(CA^{k+1}x - CBu^{k+}(x))) - u(-CA^k x)}{\tau}.
\end{aligned}
$$

The reason for distinguishing these derivatives will be evident from the next result. In this result, we use the standard notation $o(f(t))$ to mean a function such that $\lim_{0 \neq t \to 0} \frac{o(f(t))}{t} = 0$; the notation $O(f(t))$ also has the standard meaning.

LEMMA 14. *Let D be a P-matrix. Let $x^* = x(t_*)$ be an arbitrary state of the solution trajectory $(x(t), u(t))$ such that $CA^j x^* = 0$ for all $j = 0, 1, \ldots, k-1$ for some integer $k \geq 0$. The following two statements hold:*

(a) *For all $t > t_*$,*

$$
\begin{aligned}
x(t) = &\sum_{j=0}^{k+2} \frac{(t - t_*)^j}{j!} A^j x^* + \frac{(t - t_*)^{k+1}}{(k+1)!} Bu(CA^k x^*) \\
&+ \frac{(t - t_*)^{k+2}}{(k+2)!} Bu'(CA^k x^*; CA^{k+1}x^* + CBu(CA^k x^*)) + o(|t - t_*|^{k+2}),
\end{aligned}
$$

$$
\begin{aligned}
u(t) = &\frac{(t - t_*)^k}{k!} u(CA^k x^*) + \frac{(t - t_*)^{k+1}}{(k+1)!} u'(CA^k x^*; CA^{k+1}x^* + CBu(CA^k x^*)) \\
&+ o(|t - t_*|^{k+1}).
\end{aligned}
$$

(b) *For all* $t < t_*$,

$$x(t) = \sum_{j=0}^{k+2} \frac{(t - t_*)^j}{j!} A^j x^* + \frac{(t - t_*)^{k+1}}{(k+1)!} Bu(CA^k x^*)$$
$$+ \frac{(t - t_*)^{k+2}}{(k+2)!} Bu'(CA^k x^*; -CA^{k+1} x^* - CBu(CA^k x^*)) + o(|t - t_*|^{k+2}),$$

$$u(t) = \frac{(t - t_*)^k}{k!} u(CA^k x^*) + \frac{|t - t_*|^{k+1}}{(k+1)!} u'(CA^k x^*; -CA^{k+1} x^* - CBu(CA^k x^*))$$
$$+ o(|t - t_*|^{k+1})$$

if k *is even; and if* k *is odd,*

$$x(t) = \sum_{j=0}^{k+2} \frac{(t - t_*)^j}{j!} A^j x^* - \frac{(t - t_*)^{k+1}}{(k+1)!} Bu(-CA^k x^*)$$
$$+ \frac{(t - t_*)^{k+2}}{(k+2)!} Bu'(-CA^k x^*; CA^{k+1} x^* - CBu(-CA^k x^*)) + o(|t - t_*|^{k+2}),$$

$$u(t) = \frac{|t - t_*|^k}{k!} u(-CA^k x^*) + \frac{(t - t_*)^{k+1}}{(k+1)!} u'(-CA^k x^*; CA^{k+1} x^* - CBu(-CA^k x^*))$$
$$+ o(|t - t_*|^{k+1}).$$

Proof. Define

$$z(t) = x(t) - \sum_{j=0}^{k} (t - t_*)^j \frac{A^j x^*}{j!}. \tag{15}$$

Hence, $z(t_*) = 0$ and $z(t)$ satisfies

$$\dot z(t) = Az(t) + \frac{(t - t_*)^k}{k!} A^{k+1} x^* + Bu(t), \tag{16}$$

where $u(t)$ satisfies

$$0 \le u(t) \perp Cz(t) + \frac{(t - t_*)^k}{k!} CA^k x^* + Du(t) \ge 0. \tag{17}$$

Since D is a P-matrix, it follows that there exists a constant $\eta > 0$ such that for all t,

$$\| u(t) \| \le \eta [\, \| z(t) \| + | t - t_* |^k \,].$$

By (16), we deduce the existence of positive constants λ and μ such that for all $t < t_*$,

$$\| z(t) \| \le \lambda (t_* - t)^{k+1} + \mu \int_t^{t_*} \| z(s) \| \, ds.$$

Thus by Gronwall–Bellman inequality, we obtain, for some constant $\mu' > 0$,

$$\| z(t) \| \le \lambda (t_* - t)^{k+1} + \lambda \mu \int_t^{t_*} (t_* - \tau)^{k+1} e^{\mu(t_* - \tau)} d\tau$$
$$\le \lambda (t_* - t)^{k+1} + \mu' (t_* - t)^{k+2}$$

for all $t < t_*$ sufficiently near t_*. A similar bound can be derived for $\|z(t)\|$ for all $t > t_*$ sufficiently near t_*. Consequently, we deduce the existence of a scalar $\lambda' > 0$ such that for all t near t_*,

$$\| z(t) \| \leq \lambda' \, | t - t_* |^{k+1}. \tag{18}$$

Suppose that k is odd. For all $t < t_*$, (17) implies

$$0 \leq v(t) \perp k!\, C \frac{z(t)}{| t - t_* |^k} - CA^k x^* + Dv(t) \geq 0, \tag{19}$$

where $v(t) \equiv k! u(t)/|t - t_*|^k$. Since D is a P-matrix, it follows from the Lipschitz continuity of the solutions to the LCP (q, D) that, for some constant $L > 0$, we have

$$\| v(t) - u(-CA^k x^*) \| \leq L \frac{\| z(t) \|}{| t - t_* |^k}$$

or equivalently

$$\left\| u(t) - \frac{| t - t_* |^k}{k!} u(-CA^k x^*) \right\| \leq L\, k! \, \| z(t) \| \quad \forall\, t < t_* \text{ sufficiently near } t_*. \tag{20}$$

For k odd and for $t < t_*$, we can write (16) as

$$\begin{aligned} \dot{z}(t) &= Az(t) + \frac{(t - t_*)^k}{k!} [A^{k+1} x^* - Bu(-CA^k x^*)] + B\left[u(t) - \frac{| t - t_* |^k}{k!} u(-CA^k x^*) \right] \\ &= \frac{(t - t_*)^k}{k!} [A^{k+1} x^* - Bu(-CA^k x^*)] + B\left[u(t) - \frac{| t - t_* |^k}{k!} u(-CA^k x^*) \right] \\ &\quad + O(|t - t_*|^{k+1}), \end{aligned}$$

where the last inequality is due to (18). Integrating the above and using (20) and (18), we deduce

$$z(t) = z(t_*) + \int_{t_*}^{t} \dot{z}(s)\, ds = \frac{(t - t_*)^{k+1}}{(k + 1)!} [A^{k+1} x^* - Bu(-CA^k x^*)] + O(|t - t_*|^{k+2})$$

for all $t < t_*$ sufficiently near t_*. Substituting into (19), we obtain

$$0 \leq v(t) \perp \frac{t_* - t}{k + 1} C[A^{k+1} x^* - Bu(-CA^k x^*)] + O(|t - t_*|^2) - CA^k x^* + Dv(t) \geq 0.$$

Hence, we deduce

$$v(t) = u(-CA^k x^*) + \frac{t_* - t}{k + 1} u'(-CA^k x^*; CA^{k+1} x^* - CBu(-CA^k x^*)) + o(|t - t_*|),$$

which yields

$$\begin{aligned} u(t) = \frac{| t - t_* |^k}{k!} u(-CA^k x^*) + \frac{(t_* - t)^{k+1}}{(k + 1)!} u'(-CA^k x^*; CA^{k+1} x^* - CBu(-CA^k x^*)) \\ + o(|t - t_*|^{k+1}). \end{aligned}$$

Substituting this into (16), we deduce

$$\dot{z}(t) = Az(t) + \frac{(\,t-t_*\,)^k}{k\,!}\,[\,A^{k+1}x^* - Bu(-CA^kx^*)\,]$$
$$+\ \frac{(\,t-t_*\,)^{k+1}}{(\,k+1\,)\,!}\,Bu'(-CA^kx^*; CA^{k+1}x^* - CBu(-CA^kx^*)) + o(|t-t_*|^{k+1}).$$

Integrating the above equation and recalling $x(t) = z(t) + \sum_{j=0}^{k}(t-t_*)^jA^jx^*/j!$, we obtain the desired expansion for $x(t)$ when k is odd and $t < t_*$.

Next, assume that k is even and consider $t < t_*$ sufficiently near t_*. In this case, instead of (19), we have

$$0 \,\leq\, v(t) \,\perp\, k\,!\,C\,\frac{z(t)}{(\,t-t_*\,)^k} + CA^kx^* + Dv(t) \,\geq\, 0;$$

moreover, (20) is replaced by

$$\left\| u(t) - \frac{(\,t-t_*\,)^k}{k\,!}\,u(CA^kx^*) \right\| \,\leq\, L\,k\,!\,\|\,z(t)\,\| \quad \forall\, t \,<\, t_* \text{ sufficiently near } t_*.$$

Proceeding as above, we obtain from (16)

$$\dot{z}(t) \,=\, \frac{(\,t-t_*\,)^k}{k\,!}\,[\,A^{k+1}x^* + Bu(CA^kx^*)\,] + B\left[u(t) - \frac{(\,t-t_*\,)^k}{k\,!}\,u(CA^kx^*) \right]$$
$$+\,O(|t-t_*|^{k+1}).$$

At this point, we can repeat the above proof and obtain the desired expansion for $(x(t), u(t))$ in this case where k is even. Finally, the proof for statement (a) is similar and therefore omitted. □

On the basis of Lemma 14, we can complete the proof of Theorem 9 by defining the *observability degree* of an observable state x with respect to the pair (C, A), which is defined as the first nonnegative integer k such that $CA^kx \neq 0$.

Proof of Theorem 9. We use induction on m, the dimension of the input variable u. The case $m = 0$ is trivial. Inductively, assume that the theorem is valid for an integer $m \geq 0$. Consider the LCS (1) where the algebraic variable u is of dimension $m+1 \geq 1$. Let $x^* = x(t_*)$ be an arbitrary state. We first prove that x^* is strongly right non-Zeno. By the above arguments, we may assume without loss of generality that x^* is observable. Let $k \geq 0$ be the observability degree of x^*; thus $Cx^* = \cdots = CA^{k-1}x^* = 0$ and $CA^kx^* \neq 0$. The expansion in part (a) of Lemma 14 holds for the trajectory $(x(t), u(t))$ in a small interval $[t_*, t_* + \varepsilon_+]$ for some $\varepsilon_+ > 0$. Since $CA^kx^* \neq 0$, there must exist an index i such that either $u_i(CA^kx^*) > 0$ or $[CA^kx^* + Du(CA^kx^*)]_i > 0$. By part (a) of Lemma 14, this implies that if $u_i(CA^kx^*) > 0$ for some index i, then $u_i(t) > 0$ for all $t > t_*$ sufficiently near t_*, which implies, by complementarity, that $[Cx(t) + Du(t)]_i = 0$ for all such t. Hence, we can solve for $u_i(t)$ from this equation, obtaining

$$u_i(t) = d_{ii}^{-1}\left[C_{i\cdot}x(t) + \sum_{j\neq i} d_{ij}\,u_j(t) \right],$$

which we can then substitute into the remaining conditions in (1). This substitution results in an LCS:

$$\begin{aligned} \dot{x}(t) &= \widehat{A}x(t) + \widehat{B}\widehat{u}(t), \\ 0 \le \widehat{u}(t) &\perp \widehat{C}x(t) + \widehat{D}\widehat{u}(t), \\ x(t_*) &= x^*, \end{aligned} \tag{21}$$

where the matrix $\widehat{D}$ is the Schur complement of the diagonal entry d_{ii} in D and the algebraic variable $\widehat{u}$ is of dimension m, which is one less than that of the original variable u. The original trajectory $(x(t), u(t))$ with the variable u_i removed must satisfy (21) in a small interval $(t_*, t_* + \varepsilon'_+]$ for some $\varepsilon'_+ > 0$. Since $\widehat{D}$ remains a P-matrix, by the induction hypothesis, there exist an index set $(\alpha'_+, \beta'_+, \gamma'_+)$ such that $(\widehat{\alpha}(t), \widehat{\beta}(t), \widehat{\gamma}(t)) = (\alpha'_+, \beta'_+, \gamma'_+)$ for all $t > t_*$ sufficiently near t_*, where $(\widehat{\alpha}(t), \widehat{\beta}(t), \widehat{\gamma}(t))$ are the three fundamental index sets associated with the solution trajectory $(x(t), \widehat{u}(t))$. Clearly, we have $(\alpha(t), \beta(t), \gamma(t)) = (\alpha'_+ \cup \{i\}, \beta'_+, \gamma'_+)$ for all $t > t_*$ sufficiently near t_*.

Next, consider the case where $[CA^k x^* + Du(CA^k x^*)]_i > 0$ for some index i. We then have

$$[\,Cx(t) + Du(t)\,]_i = \frac{(\,t - t_*\,)^k}{k\,!}\,[\,CA^k x^* + Du(CA^k x^*)\,]_i + o(|t - t_*|^{k+1}),$$

which implies that $[\,Cx(t) + Du(t)\,]_i > 0$, and thus $u_i(t) = 0$ by complementarity for all $t > t_*$ sufficiently near t_*. Setting this variable equal to zero and dropping the ith column of B and D and the ith row of C and D, we obtain a principal linear complementarity subsystem of (1) that is satisfied by the trajectory $(x(t), u(t))$ for all $t > t_*$ sufficiently near t_*. The induction hypothesis can be applied to the resulting subsystem whose algebraic variable is of one less dimension than that of the original u. Finally, we can apply part (b) of Lemma 14 to deal with $t < t_*$ and employ similar reductions to complete the inductive proof. □

4. Extended Zeno results. Theorem 9 can be easily extended to the mixed LCS

$$\begin{aligned} \dot{x} &= Ax + B^1 u^1 + B^2 u^2, \\ 0 &= C^1 x + D^{11} u^1 + D^{12} u^2, \\ 0 \le u^2 &\perp C^2 x + D^{21} u^1 + D^{22} u^2 \ge 0, \\ x(0) &= x^0, \end{aligned}$$

provided that the matrix D^{11} is nonsingular and the Schur complement

$$D^{22} - D^{21}(\,D^{11}\,)^{-1} D^{12}$$

is a P-matrix. Instead of presenting the details of this easy extension, we consider a local version of the extension that pertains to an LCS with a "strongly regular" state but which is not of the P-type. Specifically, we call x^* a *strongly regular state* of the LCS (1) if the LCP (Cx^*, D) has a strongly regular solution. The following corollary of Theorem 9 shows that any such state must be strongly non-Zeno. For simplicity, we treat the case where x^* is neither an initial nor a terminal state of a solution trajectory.

Corollary 15. *Any strongly regular state of the LCS* (1) *is strongly non-Zeno relative to a continuous solution trajectory of the system. In fact, if* $(x(t), u(t))$ *is such a trajectory defined on an open interval containing* t_* *and if the LCP* $(Cx(t_*), D)$ *has a strongly regular solution, then* $(x(t), u(t))$ *is the unique continuous solution trajectory passing through* $x^* \equiv x(t_*)$ *for all* t *sufficiently near* t_**, and* x^* *is strongly non-Zeno relative to this trajectory.*

Proof. We first establish the uniqueness of $(x(t), u(t))$. Suppose that $(\widetilde{x}(t), \widetilde{u}(t))$ is another continuous solution trajectory of (1) passing through x^* and defined on the same interval as $(x(t), u(t))$. By the strong regularity of u^*, a neighborhood $\mathcal{V}$ of Cx^*, a neighborhood $\mathcal{U}$ of u^*, and a Lipschitz continuous function $\widehat{u} : \mathcal{V} \to \mathcal{U}$ exist such that for every $q \in \mathcal{V}$, $\widehat{u}(q)$ is the unique solution of the LCP (q, D) in $\mathcal{U}$. Since $(x(t), u(t))$ and $(\widetilde{x}(t), \widetilde{u}(t))$ are both continuous near t_*, it follows that for all t sufficiently near t_*, $(Cx(t), u(t))$ and $(C\widetilde{x}(t), \widetilde{u}(t))$ both belongs to $\mathcal{V} \times \mathcal{U}$. Hence we have $u(t) = \widehat{u}(Cx(t))$ and $\widetilde{u}(t) = \widehat{u}(C\widetilde{x}(t))$. Moreover, a constant $L > 0$ exists such that for all t sufficiently near t_*, we have

$$\|u(t) - \widehat{u}(t)\| \,\leq\, L\,\|\,x(t) - \widehat{x}(t)\|. \tag{22}$$

Since

$$\frac{d\,(x(t) - \widehat{x}(t))}{dt} = A(x(t) - \widehat{x}(t)) + B(u(t) - \widehat{u}(t)),$$

(22) implies that the right-hand side is a Lipschitz function of $x(t) - \widehat{x}(t)$. Since $x(t_*) = \widehat{x}(t_*) = x^*$, it follows that the two trajectories $x(t)$ and $\widehat{x}(t)$, and thus the two trajectories, $u(t)$ and $\widehat{u}(t)$, must coincide in a sufficiently small open interval containing t_*. The uniqueness of $(x(t), u(t))$ therefore follows.

By Proposition 5, it follows that for all t sufficiently near t_*, the trajectory $(x(t), u(t))$ must satisfy the reduced system (11). Since $\overline{D}$, being the Schur complement of $D_{\alpha_*\alpha_*}$ in a principal submatrix of D, remains a P-matrix, it follows that $x(t_*)$ is a strongly non-Zeno state of (11) relative to the trajectory $(x(t), u_{\beta_*}(t))$. Therefore, a scalar $\varepsilon > 0$ and two triples of index sets $(\alpha_{0+}, \beta_{0+}, \gamma_{0+})$ and $(\alpha_{0-}, \beta_{0-}, \gamma_{0-})$ exist such that

$$\left.\begin{array}{l}
\alpha_0(t) \equiv \{\, i \,\in\, \beta_* : u_i(t) \,>\, 0 \,=\, (\,\overline{C}x(t) + \overline{D}u_{\beta_*}(t)\,)_i \} = \alpha_{0+} \\
\beta_0(t) \equiv \{\, i \,\in\, \beta_* : u_i(t) \,=\, 0 \,=\, (\,\overline{C}x(t) + \overline{D}u_{\beta_*}(t)\,)_i \} = \beta_{0+} \\
\gamma_0(t) \equiv \{\, i \,\in\, \beta_* : u_i(t) \,=\, 0 \,<\, (\,\overline{C}x(t) + \overline{D}u_{\beta_*}(t)\,)_i \} = \gamma_{0+}
\end{array}\right\} \quad \forall\, t \,\in\, (t_*, t_* + \varepsilon]$$

and

$$\left.\begin{array}{l}
\alpha_0(t) \,=\, \alpha_{0-} \\
\beta_0(t) \,=\, \beta_{0-} \\
\gamma_0(t) \,=\, \gamma_{0-}
\end{array}\right\} \quad \forall\, t \,\in\, [\, t_* - \varepsilon, t_*\,).$$

Since $0 < u_{\alpha_*}(t) = -(D_{\alpha_*\alpha_*})^{-1}(C_{\alpha_*\cdot}x(t) + D_{\alpha_*\beta_*}u_{\beta_*}(t))$ and $0 = u_{\gamma_*}(t)$, it follows that

$$\overline{C}x(t) + \overline{D}u_{\beta_*}(t) \,=\, (\,Cx(t) + Du(t)\,)_{\beta_*}$$

for all t sufficiently near t_*. Consequently,

$$\left.\begin{array}{l}
\alpha(t) \,=\, \alpha_* \,\cup\, \alpha_{0+} \\
\beta(t) \,=\, \beta_{0+} \\
\gamma(t) \,=\, \gamma_* \,\cup\, \gamma_{0+}
\end{array}\right\} \quad \forall\, t \,\in\, (\,t_*, t_* + \varepsilon\,]$$

and

$$\left.\begin{aligned} \alpha(t) &= \alpha_* \cup \alpha_{0-} \\ \beta(t) &= \beta_{0-} \\ \gamma(t) &= \gamma_* \cup \gamma_{0-} \end{aligned}\right\} \quad \forall\, t \in [\, t_* - \varepsilon, t_* \,).$$

This shows that x^* is a strongly non-Zeno state of (1). □

One important consequence of the P-property is that the u-trajectory must necessarily be unique. In what follows, we present an extended treatment of the Zeno issue for an LCS with nonunique u-trajectories. Specifically, we make several alternative assumptions on the tuple (A, B, C, D), the first of which ensures the existence and uniqueness of a continuously differentiable solution trajectory $x(t)$ corresponding to various subsystems of (1).

(A) For every $x \in \Re^n$ and every triple of index sets (α, β, γ) partitioning $\{1, \ldots, m\}$ with $\beta \neq \emptyset$, the mixed LCP

$$\begin{aligned} 0 &= [\, Cx + Du \,]_\alpha, \\ 0 \leq u_\beta &\perp [\, Cx + Du \,]_\beta \geq 0, \\ 0 &= u_\gamma \end{aligned} \tag{23}$$

has a solution $u \in \Re^m$; moreover, $Bu^1 = Bu^2$ for any two such solutions u^1 and u^2.

The fundamental role of the above assumption is described in the following result.

PROPOSITION 16. *Under assumption* (A), *for every triple of index sets* (α, β, γ) *partitioning* $\{1, \ldots, m\}$ *with* $\beta \neq \emptyset$, *the system*

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ 0 &= [\, Cx + Du \,]_\alpha, \\ 0 \leq u_\beta &\perp [\, Cx + Du \,]_\beta \geq 0, \\ 0 &= u_\gamma, \\ x(0) &= x^0 \end{aligned} \tag{24}$$

has a unique solution trajectory $x(t)$ *for all* $t \in [0, T]$ *for any* $T > 0$*; moreover,* $x(t)$ *is continuously differentiable on its domain.*

Proof. Let $\mathcal{S}(x) \subset \Re^m$ denote the solution set of (23). As a multifunction, the map $\mathcal{S} : x \mapsto \mathcal{S}(x)$ is a polyhedral multifunction; i.e., its graph is the union of finitely many polyhedra. Under assumption (A), the mapping

$$\widehat{B} \,:\, x \,\in\, \Re^n \,\mapsto\, B\mathcal{S}(x)$$

is a single-valued function whose graph is the union of finitely many polyhedra. As such, by a result due originally to Gowda (see [12, Exercise 5.6.14]), it follows that $\widehat{B}$ is a (globally) Lipschitz continuous function on $\Re^n$. In terms of this mapping, the system (24) can be equivalently stated as

$$\dot{x} \,=\, Ax + \widehat{B}(x), \quad x(0) \,=\, x^0.$$

Since the right-hand side of the ODE is Lipschitz continuous, the existence and uniqueness and the continuous differentiability of a solution trajectory $x(t)$ follows from classical ODE theory. □

Before proceeding further, we make several remarks about Proposition 16 and assumption (A). First, if one is interested in the existence of a unique x-trajectory to the single LCS (1), then it suffices to assume that $B\mathrm{SOL}(Cx, D)$ is a singleton for all $x \in \Re^n$. Second, while the x-trajectory is necessarily unique in the proposition, no such uniqueness is asserted for the u-trajectory; no continuity of the u-trajectory is asserted either. This is a significant departure from the P-property under which the u-trajectory exists and is both unique and continuous. Nevertheless, it can be shown that in the case where C has full row rank, assumption (A) implies that D must be a P-matrix. Hence, condition (A) is most interesting when C is deficient in row rank. A class of triples (B, C, D) satisfying assumption (A) with D being non-P is presented in section 5. It should be noted that condition (A) is different from the passifiability property of the triple (B, C, D) used in [5]; the latter property, along with a minimality assumption on (A, B, C, D), yields the existence and uniqueness of a continuous x-trajectory and an $\mathcal{L}_2$ u-trajectory of the LCS (1). Examples of the class of triples (B, C, D) from section 5 can easily be constructed which fail the passifiability condition; conversely, any triple $(B, C, 0)$ with CB symmetric positive definite is passifiable but fails condition (A).

Another difference between assumption (A) and the P-matrix assumption is that (A) does not imply any apparent determinant properties of the principal matrix

$$\begin{bmatrix} D_{\alpha\alpha} & D_{\alpha\beta} \\ D_{\beta\alpha} & D_{\beta\beta} \end{bmatrix};$$

in particular, it could be singular. Assumption (A) does imply

$$\left.\begin{array}{r} 0 \,=\, (\,Du\,)_\alpha \\ 0 \,\leq\, u_\beta \,\perp\, (\,Du\,)_\beta \,\geq\, 0 \\ 0 \,=\, u_\gamma \end{array}\right\} \;\Rightarrow\; Bu \,=\, 0,$$

and is implied by the following more restrictive condition:

$$u \circ Du \,\leq\, 0 \;\Rightarrow\; Bu \,=\, 0.$$

For our purpose, the following invariance properties of assumption (A) are important for the extension of the previous inductive argument to an LCS not satisfying the P-property.

PROPOSITION 17. *Suppose that* (B, C, D) *satisfies condition* (A). *The same condition holds for the following triples of index sets:*

(a) $(B_{\cdot\tilde\alpha}, C_{\tilde\alpha\cdot}, D_{\tilde\alpha\tilde\alpha})$ *for every subset* $\tilde\alpha \subseteq \{1, \ldots, m\}$*;*

(b) *the triple* $(\widetilde{B}, \widetilde{C}, \widetilde{D})$ *associated with the* $\tilde\alpha$*-principal transform* (7) *of* (B, C, D) *for every subset* $\tilde\alpha \subseteq \{1, \ldots, m\}$ *such that* $D_{\tilde\alpha\tilde\alpha}$ *is nonsingular;*

(c) (PB, CP^{-1}, D) *for every nonsingular matrix* P*.*

Proof. Let α', β', and γ' be any three index sets partitioning $\tilde\alpha$, with $\beta' \neq \emptyset$. Let $x \in \Re^n$ be arbitrary. We need to show that the system

$$\begin{aligned} 0 \,&=\, [\,Cx + Du\,]_{\alpha'}, \\ 0 \,\leq\, u_{\beta'} \,\perp\, [\,Cx + Du\,]_{\beta'} \,&\geq\, 0, \\ 0 \,&=\, u_{\gamma'} \end{aligned}$$

has a solution $u_{\tilde{\alpha}}$. Moreover, if $u^1_{\tilde{\alpha}}$ and $u^2_{\tilde{\alpha}}$ are any two such solutions, we must have $B_{\cdot\tilde{\alpha}}u^1_{\tilde{\alpha}} = B_{\cdot\tilde{\alpha}}u^2_{\tilde{\alpha}}$. But this is clear from condition (A) with the choice of $(\alpha, \beta, \gamma) = (\alpha', \beta', \gamma' \cup (\{1, \ldots, m\} \backslash \tilde{\alpha}))$. To prove (b), let $(\alpha', \beta', \gamma')$ be a triple of index sets partitioning $\{1, \ldots, m\}$ with $\beta' \neq \emptyset$ and consider the system

$$
\begin{aligned}
0 &= [\, \widetilde{C}x + \widetilde{D}\widetilde{u}\,]_{\alpha'}, \\
0 \leq \widetilde{u}_{\beta'} &\perp [\, \widetilde{C}x + \widetilde{D}\widetilde{u}\,]_{\beta'} \geq 0, \\
0 &= \widetilde{u}_{\gamma'}.
\end{aligned} \tag{25}
$$

Letting $\widetilde{w} \equiv \widetilde{C}x + \widetilde{D}\widetilde{u}$ and "pivoting" on $(D_{\tilde{\alpha}\tilde{\alpha}})^{-1}$, we obtain $w \equiv Cx + Du$, where the relation between the pairs (w, u) and $(\widetilde{w}, \widetilde{u})$ is as follows:

$$
\begin{aligned}
w &\equiv \begin{pmatrix} w_{\tilde{\alpha}} \\ w_{\bar{\alpha}} \end{pmatrix}, \quad u \equiv \begin{pmatrix} u_{\tilde{\alpha}} \\ u_{\bar{\alpha}} \end{pmatrix}, \\
\widetilde{w} &= \begin{pmatrix} u_{\tilde{\alpha}} \\ w_{\bar{\alpha}} \end{pmatrix}, \quad \widetilde{u} = \begin{pmatrix} w_{\tilde{\alpha}} \\ u_{\bar{\alpha}} \end{pmatrix},
\end{aligned} \tag{26}
$$

where $\bar{\alpha}$ is the complement of $\tilde{\alpha}$ in $\{1, \ldots, m\}$. Therefore, system (25) is equivalent to system (23) for some suitable triple (α, β, γ) that is derived from $(\alpha', \beta', \gamma')$. Therefore, the existence of a solution to (25) follows from condition (A) on the original triple (B, C, D). Suppose that $\widetilde{u}^1$ and $\widetilde{u}^2$ are any two solutions satisfying (25). Corresponding to $\widetilde{u}^i$, for $i = 1, 2$, let $\widetilde{w}^i \equiv \widetilde{C}x + \widetilde{D}\widetilde{u}^i$ and (w^i, u^i) be defined accordingly. It follows that $Bu^1 = Bu^2$. We have

$$
\begin{aligned}
\widetilde{B}\widetilde{u}^i &= B_{\cdot\tilde{\alpha}}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}\widetilde{u}^i_{\tilde{\alpha}} + [\, B_{\cdot\bar{\alpha}} - B_{\cdot\alpha}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}D_{\tilde{\alpha}\bar{\alpha}}\,]\widetilde{u}^i_{\bar{\alpha}} \\
&= B_{\cdot\tilde{\alpha}}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}w^i_{\tilde{\alpha}} + [\, B_{\cdot\bar{\alpha}} - B_{\cdot\tilde{\alpha}}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}D_{\tilde{\alpha}\bar{\alpha}}\,]u^i_{\bar{\alpha}} \\
&= B_{\cdot\tilde{\alpha}}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}[\, Cx + Du^i\,]_{\tilde{\alpha}} + [\, B_{\cdot\bar{\alpha}} - B_{\cdot\tilde{\alpha}}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}D_{\tilde{\alpha}\bar{\alpha}}\,]u^i_{\bar{\alpha}} \\
&= B_{\cdot\tilde{\alpha}}(\, D_{\tilde{\alpha}\tilde{\alpha}}\,)^{-1}C_{\tilde{\alpha}\cdot}x + Bu^i;
\end{aligned}
$$

hence, $\widetilde{B}\widetilde{u}^1 = \widetilde{B}\widetilde{u}^2$. This proves (b). Finally (c) is obvious. □

To motivate the following discussion, consider an unobservable state $x^* = x(t_*)$. In this case, $(x(t), u(t)) = (e^{A(t-t_*)}x^*, 0)$ is trivially an admissible solution trajectory to (1) for $t > t_*$. Moreover, under assumption (A), the trajectory $x(t) = e^{A(t-t_*)}x^*$ is unique for $t > t_*$. Nevertheless, if D is not an R_0-matrix, i.e., if the homogeneous LCP $(0, D)$ has a nonzero solution, then it is very difficult, if not impossible, to ascertain the Zeno properties of $(x(t), u(t))$ jointly. The reason is very simple: the LCP $(0, D)$, which must be satisfied by the u-trajectory in this case, is totally unaffected by the x-trajectory. Consequently, if one expects an unobservable state x^* to be (right) non-Zeno, one must restrict oneself to the class of matrices D for which the LCP $(0, D)$ has a polyhedral solution set; this is the principal motivation to introduce the class of weakly column sufficient matrices (Definition 1).

The following result extends the key expansion Lemma 14 and is applicable to an arbitrary tuple (A, B, C, D) satisfying condition (A).

Lemma 18. *Suppose that* (A, B, C, D) *satisfy condition* (A). *Let* $x^* = x(t_*)$ *be a given state of the solution trajectory* $(x(t), u(t))$ *such that* $CA^jx^* = 0$ *for all* $j = 0, 1, \ldots, k-1$ *for some integer* $k \geq 0$. *The following two statements hold:*

(a) *For each* $t > t_*$ *sufficiently near* t_*, *there exists* $u^{t+} \in \mathrm{SOL}(CA^k x^*, D)$ *such that*

$$x(t) = \sum_{j=0}^{k+1} \frac{(\,t - t_*)^j}{j\,!}\, A^j x^* + \frac{(\,t - t_*\,)^{k+1}}{(\,k+1\,)\,!}\, Bu^{t+} + O(|t - t_*|^{k+2}),$$

$$u(t) = \frac{(\,t - t_*\,)^k}{k\,!}\, u^{t+} + O(|t - t_*|^{k+1}).$$

(b) *If* k *is even, then for each* $t < t_*$ *sufficiently near* t_*, *there exists* $u^{t+} \in \mathrm{SOL}(CA^k x^*, D)$ *such that the expansion in part* (a) *remains valid. If* k *is odd, then for each* $t < t_*$, *there exists* $u^{t-} \in \mathrm{SOL}(-CA^k x^*, D)$ *such that*

$$x(t) = \sum_{j=0}^{k+1} \frac{(\,t - t_*)^j}{j\,!}\, A^j x^* - \frac{(\,t - t_*\,)^{k+1}}{(\,k+1\,)\,!}\, Bu^{t-} + O(|t - t_*|^{k+2}),$$

$$u(t) = \frac{|\,t - t_*\,|^k}{k\,!}\, u^{t-} + O(|t - t_*|^{k+1}).$$

Proof. We prove only statement (a). Proceeding as in the proof of Lemma 14, we define $z(t)$ by (15) and note that (16) and (17) must hold. By the same result due of Gowda that we used in the proof of Proposition 16, we can deduce the existence of a constant $\eta > 0$ such that

$$\|\,Bu(t)\,\| \;\le\; \eta\,[\,\|\,z(t)\,\| + |\,t - t_*\,|^k\,]$$

for all t. Consequently, it follows that (18) holds. The vector $v(t) \equiv k!u(t)/|t - t_*|^k$ satisfies, for all $t > t_*$,

$$0 \;\le\; v(t) \;\perp\; k\,!\,C\,\frac{z(t)}{|\,t - t_*\,|^k} + CA^k x^* + Dv(t) \;\ge\; 0.$$

Since $\|z(t)\|$ is of order $|t - t_*|^{k+1}$, by the semistability of the LCP $(CA^k x^*, D)$, it follows that for every $t > t_*$ sufficiently near t_*, there exists $u^{t+} \in \mathrm{SOL}(CA^k x^*, D)$ such that $\|v(t) - u^{t+}\|$ is of order $O(|t - t_*|)$. The expansion for $u(t)$ in part (a) thus follows readily. Substituting this expansion into the differential equation

$$\begin{aligned}\dot{z}(t) &= Az(t) + \frac{(\,t - t_*\,)^k}{k\,!}\, A^{k+1} x^* + Bu(t) \\ &= Az(t) + \frac{(\,t - t_*\,)^k}{k\,!}\,[\,A^{k+1} x^* + Bu^{t+}\,] + O(|\,t - t_*\,|^{k+1})\end{aligned}$$

using the fact that Bu^{t+} is independent of t (because $B\mathrm{SOL}(CA^k x^*, D)$ is a singleton), and integrating, we can deduce the desired expansion for $x(t)$. The details are not repeated. ∎

On the basis of the concept of an identifiable LCP, we introduce the following.

Definition 19. *A state* x^* *is said to be* identifiable *with respect to the triple* (A, C, D) *if, for each subset* α *of* $\{1, \ldots, m\}$, *if* x^* *is observable with degree* k *with respect to the pair* $(C_{\alpha\cdot}, A)$, *then the LCPs* $(\pm C_{\alpha\cdot} A^k x^*, D_{\alpha\alpha})$ *are identifiable. The state* x^* *is said to be* totally identifiable *with respect to the tuple* (A, B, C, D) *if* x^* *is* *identifiable with respect to all triples* $(\widehat{A}, \widehat{C}, \widehat{D})$, *where*

$$\widehat{A} \equiv A - B_{\cdot\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot},$$
$$\widehat{C} \equiv C_{\bar\alpha\cdot} - D_{\bar\alpha\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot},$$
$$\widehat{D} \equiv D_{\bar\alpha\bar\alpha} - D_{\bar\alpha\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar\alpha},$$

and α, *with complement* $\bar\alpha$, *ranges over all subsets of* $\{1,\dots,m\}$ *for which* $D_{\alpha\alpha}$ *is nonsingular.* □

The next lemma asserts that the above identifiability property is inherited by the principal (sub)transforms of a given tuple.

LEMMA 20. *Suppose that* x^* *is totally identifiable with respect to* (A,B,C,D). *Then* x^* *is also totally identifiable with respect to the following tuples:*

(a) $(A, B_{\cdot\alpha}, C_{\alpha\cdot}, D_{\alpha\alpha})$ *for all subsets* α *of* $\{1,\dots,m\}$;

(b) *the principal subtuples* $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ *associated with all legitimate principal pivot transforms of* (A,B,C,D), *where*

$$\widehat{A} \equiv A - B_{\cdot\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot}, \qquad \widehat{B} \equiv B_{\cdot\bar\alpha} - B_{\cdot\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar\alpha},$$
$$\widehat{C} \equiv C_{\bar\alpha\cdot} - D_{\bar\alpha\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\cdot}, \quad \widehat{D} \equiv D_{\bar\alpha\bar\alpha} - D_{\bar\alpha\alpha}(D_{\alpha\alpha})^{-1}D_{\alpha\bar\alpha},$$

and α, *with complement* $\bar\alpha$, *ranges over all subsets of* $\{1,\dots,m\}$ *for which* $D_{\alpha\alpha}$ *is nonsingular.*

Moreover, Px^* *is totally identifiable with respect to the triple* $(PAP^{-1}, PB, CP^{-1}, D)$ *for any nonsingular matrix* P.

Proof. Statement (a) is obvious. The validity of statement (b) is based on the observation that a tuple $(\widehat{\widehat{A}}, \widehat{\widehat{C}}, \widehat{\widehat{D}})$, where

$$\widehat{\widehat{A}} \equiv \widehat{A} - \widehat{B}_{\cdot\hat\alpha}(\widehat{D}_{\hat\alpha\hat\alpha})^{-1}\widehat{C}_{\hat\alpha\cdot},$$
$$\widehat{\widehat{C}} \equiv \widehat{C}_{\bar{\hat\alpha}\cdot} - D_{\bar{\hat\alpha}\hat\alpha}(D_{\hat\alpha\hat\alpha})^{-1}\widehat{C}_{\hat\alpha\cdot},$$
$$\widehat{\widehat{D}} \equiv \widehat{D}_{\bar{\hat\alpha}\bar{\hat\alpha}} - \widehat{D}_{\bar{\hat\alpha}\hat\alpha}(\widehat{D}_{\hat\alpha\hat\alpha})^{-1}D_{\hat\alpha\bar{\hat\alpha}},$$

and $\hat\alpha$, with complement $\bar{\hat\alpha}$, is a subset of $\bar\alpha$ for which $\widehat{D}_{\hat\alpha\hat\alpha}$ is nonsingular, can be shown to be a principal subtuple associated with the $(\alpha \cup \hat\alpha)$-principal pivot transform of (A,B,C,D). Hence (b) holds. The last assertion follows easily from the identity $CP^{-1}(PAP^{-1})^k = CA^kP^{-1}$. □

Our extended Zeno result for an LCS without the P-property is the following. The statement of the theorem assumes that x^* is neither the initial nor the terminal state of the solution trajectory so that we do not need to pay attention to the one-sidedness of these special states.

THEOREM 21. *Let* D *be a weakly column sufficient matrix. Suppose that condition* (A) *holds for the tuple* (A,B,C,D). *The following two statements hold for any state* $x^* = x(t_*)$ *and any* u*-trajectory:*

(a) *If* x^* *is unobservable with respect to* (C,A), *then* x^* *is weakly non-Zeno.*

(b) *If* x^* *is totally identifiable with respect to the tuple* (A,B,C,D), *then* x^* *is weakly non-Zeno.*

Proof. We follow the proof of Theorem 9. Suppose that the initial state x^0 is unobservable with respect to (C,A). In this case, the unique x-trajectory is $x(t) = e^{At}x^0$

and we have $Cx(t) = 0$ for all t. Hence $u(t) \in \mathrm{SOL}(0, D)$ for all t. The weak column sufficiency of D then completes the proof. So we assume that x^0 is observable. The proof of Lemma 12 shows that all subsequent states are observable. This establishes part (a). We use induction on m to prove that if x^* is totally identifiable with respect to (A, B, C, D), then x^* is weakly right non-Zeno; the proof of weakly left non-Zenoness is similar and therefore omitted. Since the LCP $(CA^k x^*, D)$ is identifiable, where k is the observability degree of x^* with respect to the pair (C, A), either one of the two index sets $\mathcal{I}_u$ or $\mathcal{I}_w$ is nonempty. Without loss of generality, assume that the former is so. By Proposition 4, there exists a scalar $\sigma > 0$ such that $u_i \geq \sigma$ for all $u \in \mathrm{SOL}(CA^k x^*, D)$ and all $i \in \mathcal{I}_u$. Consequently, by the expansion of $u(t)$ near t_* as described in part (a) of Lemma 18 it follows that $u_i(t) > 0$ for all $i \in \mathcal{I}_u$ and all $t > t_*$ sufficiently near t_*. Moreover, $D_{\mathcal{I}_u \mathcal{I}_u}$ is nonsingular by the identifiability assumption. Consequently, the trajectory $(x(t), u(t))$ must satisfy the principal subtransform (9) with $\alpha \equiv \mathcal{I}_u$ for all $t > t_*$ sufficiently near t_*. The induction hypothesis then completes the proof. □

5. A special bimodal system. As an illustration of another application of the expansion Lemma 18, we consider a special bimodal system which has $D \equiv ff^T$, $B \equiv bf^T$, and $C \equiv fc^T$ for some m-vector f and n-vectors b and c. To avoid trivialities, we assume that f has no zero components. It is easy to see that condition (A) holds for the triple $(B, C, D) \equiv (bf^T, fc^T, ff^T)$. Notice that the LCS (1) with this triple remains an MIMO (multiple input, multiple output) system; nevertheless, it is a bimodal system because of the lemma below.

LEMMA 22. *The LCP*

$$0 \leq u \perp fc^T x + ff^T u \geq 0$$

has a solution for all $x \in \Re^n$*; moreover, for any such solution* u, $f^T u = 0$ *if* $fc^T x \geq 0$, *and* $c^T x + f^T u = 0$ *otherwise. Consequently,*

$$\mathrm{SOL}(fc^T x, ff^T) = \begin{cases} \{\, u \geq 0 : f^T u = 0 \,\} & \text{if } fc^T x \geq 0, \\ \{\, u \geq 0 : c^T x + f^T u = 0 \,\} & \text{otherwise.} \end{cases}$$

Proof. If $fc^T x \geq 0$, then $u = 0$ is a solution of the LCP. Since $f^T u$ is a constant on the solution set of this LCP, it follows that $f^T u = 0$ for all such solutions in this case. If $fc^T x \not\geq 0$, then $f^T u \neq 0$ for all solutions of the LCP. For any such solution u, we have

$$0 = (f^T u)(c^T x) + (f^T u)^2,$$

which yields $c^T x + f^T u = 0$ as claimed. The representation of $\mathrm{SOL}(fc^T x, ff^T)$ is easy to establish. □

In view of the above lemma, it follows that the LCS (1) is of the following bimodal kind:

$$\dot{x} = \begin{cases} Ax & \text{if } fc^T x \geq 0, \\ (A - bc^T)x & \text{otherwise.} \end{cases}$$

Since $f \neq 0$, it is clear that x^* is an observable state of the pair (C, A) if and only if the scalar $c^T A^k x^* \neq 0$ for some integer $k \geq 0$. If $c^T x^* = \cdots = c^T A^{k-1} x^* = 0 \neq c^T A^k x^*$, Lemma 18 implies that, for $t > t_*$,

$$c^T x(t) = \frac{(t - t_*)^k}{k!} c^T A^k x^* + O(|t - t_*|^{k+1}).$$

Since $c^T A^k x^*$ is a nonzero scalar, it follows that $c^T x(t)$ is nonzero and of one sign for all $t > t_*$ sufficiently near t_*. Since f is a constant vector, it follows that either $fc^T x(t) \geq 0$ for all $t > t_*$ sufficiently near t_*, which implies $f^T u(t) = 0$ for all $u(t) \in \mathrm{SOL}(fc^T x(t), ff^T)$, or $c^T x(t) + f^T u(t) = 0$ for all such t. In the latter case, it follows that x^* is a weakly right non-Zeno state with respect to the trajectory $(x(t), u(t))$. In the former case, f must be either a positive or a negative vector depending on whether $c^T A^k x^* > 0$ or $c^T A^k x^* < 0$; in either case, we must have $u(t) = 0$ for all $t > t_*$ sufficiently near t_* because $f^T u(t) = 0$ and $u(t) \geq 0$. This is enough to show that x^* is a weakly right non-Zeno state. A similar argument will establish that x^* is also a weakly left non-Zeno state. We have, therefore, proved the following result for an observable state. The proof of the result for an unobservable state is the same as before and is not repeated.

THEOREM 23. *Let* $(B, C, D) \equiv (bf^T, fc^T, ff^T)$*, where* f *has no zero component. The LCS* (1) *has no Zeno states of the second kind.* □

6. Concluding remarks. In this paper, via a basic expansion of the solution trajectory near a given time, we have shown that an LCS with the P-property has no Zeno states of the first kind, that the totally identifiable states of an LCS with the weakly column sufficient property are weakly non-Zeno, and that a certain bimodal LCS has no Zeno states of the second kind. Subsequently to the completion to this work, we have extended the results in several directions, in particular, to a special LCS of the "positive semidefinite plus" type [12] and to a strongly regular nonlinear complementarity system [22]. An interesting extension that we have *not* yet resolved is the case where $D = 0$ and CB is positive definite (but not symmetric). Such an LCS is not necessarily passifiable. Lastly, in the paper [22], we use the results established herein to study the "local observability" of an LCS.

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