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Mathematical Models for the Triaxial Attitude Control Testbed

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SUMMARY

The Triaxial Attitude Control Testbed has been developed as part of a research program at the University of Michigan on multibody rotational dynamics and control. In this paper, equations of motion are derived and presented in various forms. Actuation mechanisms are incorporated into the models; these include fan actuators, reaction wheel actuators and proof mass actuators that are £xed to the triaxial base body. The models also allow incorporation of unactuated auxiliary bodies that are constrained to move relative to the triaxial base body. The models expose the dynamic coupling between the rotational motion of the triaxial base body, the relative or shape motion of the unactuated auxiliary degrees of freedom, and dynamics associated with actuation mechanisms. Many different model simpli£cations and approximations are developed. Control models for the triaxial attitude control testbed are formulated that re¤ect speci£c assumptions.

1. INTRODUCTION

The Triaxial Attitude Control Testbed (TACT) is a laboratory-based testbed that has been developed to study a broad range of rotational dynamics and control problems. Spacecraft attitude dynamics and control [9, 8, 19] provides a motivation for study of this testbed. A photograph of the TACT in the Attitude Dynamics and Control Laboratory of the University of Michigan is shown in Figure 1; it is also described in detail in [1]. The TACT is based on a spherical air bearing. An aluminum sphere of 11 inch diameter ¤oats on a thin £lm of air. The air is supplied by a compressor and exits holes located in the surface of a supporting cup. Air at 70 psi is supplied to the cup from the compressor by means of a hose that passes through the center of the vertical support.

A one-piece 32 inch stainless steel shaft passes through the center of the sphere and is rigidly attached to the sphere. The steel shaft supports two 24-inch circular mounting plates. This shaft is designed to withstand stresses that might otherwise distort the sphere. All mounting plates are made from 1/4-inch aluminum alloy with 1/4-20 holes tapped in a 1 inch grid. Two 14-inch aluminum extension shafts connect the circular mounting plates to 30-inch $\times$ 30-inch square mounting plates. The distance between the square plates is thus 5 feet. All shafts have hollow interior to allow wiring to pass through the sphere and to connect any two points on the mounting plates. Access holes of size 1-inch $\times$ 2-inch are cut into the plates and shafts to allow cable jacks and plugs to be passed between connection points. This rigid assembly consisting of the aluminum sphere, the steel shaft, the extension shafts, and the mounting planes is referred to as the triaxial base body. The total weight of the base body, without additional instrumentation or components, is 180 lb. At 70 psi air pressure, the air bearing can support an additional 180 lb.

The spherical air bearing and supporting structure for the air bearing allow unrestricted motion in base body yaw and roll. The plates and shafts are designed to allow base body pitch angles up to 45 degrees at all roll and yaw angles.

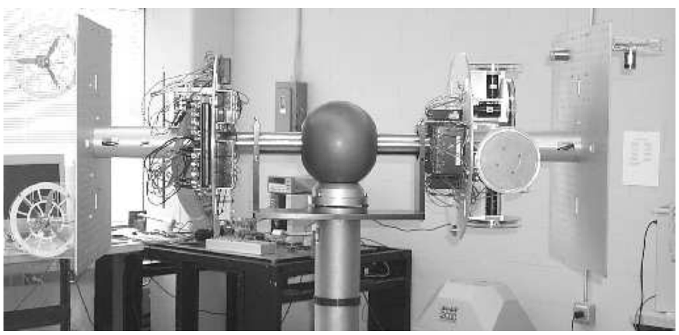

Fig. 1. Triaxial Attitude Control Testbed

The purpose of this paper is to develop different models that describe the attitude dynamics of the TACT. The models are very ¤exible in that they allow the inclusion of many multibody effects and interesting and novel actuation mechanisms. The models can be used to formulate a number of interesting dynamics and control problems.

The TACT is similar to a three-axis air bearing system used as a spacecraft simulator described in [9]. Estimation of inertial properties is the primary focus in [9]. This is to be contrasted with the ¤exibility of the TACT as an experimental facility, and the focus in this paper on dynamics and control issues.

2. TACT EQUATIONS OF MOTION

We £rst formulate a mathematical model for the TACT consisting of the rigid triaxial base body and several auxiliary bodies whose motion is constrained relative to the base body. We allow for external base body £xed moment actuators and actuators that act on the auxiliary bodies to change the shape, the latter referred to as shape change actuators.

The triaxial base body can rotate freely about all axes, but cannot translate since the pivot point is £xed inertially. A coordinate frame is £xed to the triaxial base body, with origin located at the pivot point. The attitude of a body £xed coordinate frame with respect to an inertially £xed coordinate frame de£nes the attitude of the triaxial base body. The generalized shape of the triaxial system is de£ned by the relative con£guration of $N > 0$ rigid auxiliary bodies connected to the base body; each auxiliary body can translate or rotate with respect to the base body according to a holonomic constraint relation. The relative motion of the auxiliary bodies, with respect to the base body £xed coordinate frame, is described by n generalized coordinates, referred to as shape coordinates. The con£guration space for the TACT is given by

$$Q = SO(3) \times Q_s,$$

where Q_s is a manifold of dimension n. The attitude of the base body is represented by a rotation matrix $R \in SO(3)$ although other representations are also utilized. The shape of the TACT is represented by shape coordinates $q = (q_1, \cdots, q_n) \in Q_s$ that determine the positions and/or the attitudes of the N auxiliary bodies with respect to the triaxial base body. If the shape coordinates are constant, then the TACT is effectively a rigid body. A schematic for the TACT where the auxiliary bodies are de£ned by a proof mass and a reaction wheel, is shown in Figure 2.

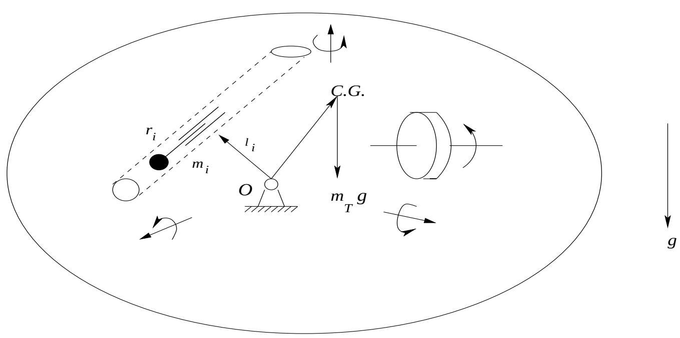

Fig. 2. Schematic con£guration of the TACT in a uniform gravitational £eld with a proof mass actuator and a reaction wheel

2.1. Lagrangian Expressions

The kinetic energy of the TACT depends on the angular velocity $\omega = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$ of the triaxial base body in base body coordinates, the velocity $v_i \in \mathbb{R}^3$ of the i-th auxiliary body center of mass in base body coordinates, and the angular velocity $\omega_i \in \mathbb{R}^3$ of the i-th auxiliary body in base body coordinates.

The kinetic energy of the TACT is given by

$$T = \frac{1}{2}\omega^T J_B \omega + \frac{1}{2}\sum_{i=1}^{N}\left\{m_i v_i^T v_i + \omega_i^T J_i(q)\omega_i\right\}, \tag{1}$$

where $J_B \in \mathbb{R}^{3\times 3}$ is the inertia tensor of the triaxial base body de£ned with respect to the base body-£xed coordinate frame, $m_i \in \mathbb{R}^+$ is the mass of the i-th auxiliary body, and $J_i(q) \in \mathbb{R}^{3\times 3}$ is the shape dependent inertia tensor of the i-th auxiliary body de£ned with respect to the base body-£xed coordinate frame, $i = 1, \cdots, N$.

Let $\rho_i(q)$ denote the relative position vector of the center of mass of the i-th auxiliary body with respect to the pivot point; $\rho_i(q)$ is a smooth function of the shape coordinates. We can express the translational velocity vector v_i of the i-th auxiliary body, expressed in base body coordinates, in terms of ω and $\dot{q}$ as

$$v_i \;=\; -\rho_i(q)\times\omega + \left[\frac{\partial\rho_i(q)}{\partial q}\right]\dot{q} \;=\; -\widehat{\rho}_i(q)\omega + \left[\frac{\partial\rho_i(q)}{\partial q}\right]\dot{q}\,, \quad i = 1, \cdots, N. \tag{2}$$

The above cross product operation can be represented in terms of a matrix product; that is for column vectors $a = [a_1, a_2, a_3]^T \in \mathbb{R}^3$ and $b = [b_1, b_2, b_3]^T \in \mathbb{R}^3$, the cross produce is $a \times b = \widehat{a}b = a^\wedge b$, where $\widehat{a}$ or $a^\wedge$ denotes a 3×3 skew-symmetric matrix:

$$\widehat{a} = a^\wedge = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

formed from the column vector a. Both cross product representations will be utilized in the subsequent development.

Based on an assumed holonomic constraint for the i-th auxiliary body, the angular velocity vector ω_i of the i-th auxiliary body can be expressed in terms of ω and $\dot{q}$ as

$$\omega_i = \omega + C_i(q)\dot{q}\,, \quad i = 1, \cdots, N\,, \tag{3}$$

where $C_i(q)$ de£nes a constraint function for the angular motion of the i-th auxiliary body with respect to the base body.

By substituting v_i and ω_i from (2) and (3) into the kinetic energy expression (1) and simplifying, we obtain an expression in terms of the base body angular velocity, the shape velocity, and the shape coordinates

$$T_1(q, \omega, \dot{q}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix}^T \begin{bmatrix} J(q) & B(q) \\ B^T(q) & M(q) \end{bmatrix} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix}, \tag{4}$$

where

$$J(q) = J_B + \sum_{i=1}^{N} \left\{ m_i \widehat{\rho}_i^T(q) \widehat{\rho}_i(q) + J_i(q) \right\},$$

$$M(q) = \sum_{i=1}^{N} \left\{ m_i \left[\frac{\partial \rho_i(q)}{\partial q} \right]^T \left[\frac{\partial \rho_i(q)}{\partial q} \right] + C_i^T(q) J_i(q) C_i(q) \right\},$$

$$B(q) = \sum_{i=1}^{N} \left\{ m_i \widehat{\rho}_i(q) \left[\frac{\partial \rho_i(q)}{\partial q} \right] + J_i(q) C_i(q) \right\}.$$

A uniform gravity £eld is assumed. In the inertial coordinate frame, the origin is assumed to be located at the pivot point and the standard orthonormal basis elements $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ are assumed to be selected so that e_1 and e_2 lie in the horizontal plane and e_3 points in the direction of gravity. Thus, the potential energy can be expressed in terms of the rotation matrix $R \in SO(3)$ and the shape coordinates $q = (q_1, \cdots, q_n)$ as

$$V(R, r) = -m_B g e_3^T R \rho_c - \sum_{i=1}^{N} m_i g e_3^T R \rho_i(q) + V_s(q), \tag{5}$$

where $m_B \in \mathbb{R}^+$ is the mass of the triaxial base body, $g \in \mathbb{R}^+$ is the gravitational constant, and $\rho_c \in \mathbb{R}^3$ is a constant vector that denotes the relative position vector of the center of mass of the triaxial base body with respect to the pivot point in base body coordinates.

The £rst term in eqn.(5) represents the gravitational potential of the triaxial base body; the second set of terms represents the shape related gravitational potential of the auxiliary bodies. Note that the gravitational potentials depend on the base body attitude only through the vector $R^T e_3$; this observation is important and it is exploited in the subsequent development. The last term in eqn.(5) represents any elastic potential energy that depends solely on the shape coordinates. The shape dependent center of mass of the TACT is denoted by $\rho_s(q)$ de£ned as

$$\rho_s(q) = \frac{1}{m_T}\left[m_B\rho_c + \sum_{i=1}^{N} m_i\rho_i(q)\right] , \tag{6}$$

where $m_T = m_B + \sum_{i=1}^{N} m_i$ is the total mass of the TACT. Thus the potential energy function can be expressed as

$$V(R, r) = -m_T g e_3^T R\rho_s(q) + V_s(q). \tag{7}$$

Hence, the Lagrangian of the TACT can be expressed in terms of the base body attitude, angular velocity, shape velocity, and shape coordinates as:

$$\begin{aligned} L_1(R, \omega, q, \dot{q}) &= T_1(\omega, q, \dot{q}) - V(R, q) \\ &= \frac{1}{2}\begin{pmatrix}\omega \\ \dot{q}\end{pmatrix}^T \begin{bmatrix} J(q) & B(q) \\ B^T(q) & M(q)\end{bmatrix}\begin{pmatrix}\omega \\ \dot{q}\end{pmatrix} + m_T g\rho_s^T(q)R^T e_3 - V_s(q). \end{aligned} \tag{8}$$

This Lagrangian expression is useful in our subsequent development. The formulation of the equations of motion consists of two parts: (1) kinematics equations for the base body and (2) dynamics equations for the base body and shape. We now discuss these equations in details.

2.2. Kinematics Equations

The kinematics equations for the triaxial base body are given by the standard relationship

$$\dot{R} = R\widehat{\omega}. \tag{9}$$

The above kinematics equation is expressed in terms of a rotation matrix that describes the attitude of the TACT base body. The attitude of the base body can be represented in a variety of ways, e.g. using quaternions, exponential coordinates, or Euler angles.

Here, a particular choice of Euler angles (the 3-2-1 choice) is made that leads to convenient equations of motion that clearly expose the gravitational symmetry of the TACT. The attitude of the base body can be represented by Euler angles corresponding to successive elementary rotations de£ned by yaw angle ψ, pitch angle θ, and roll angle ϕ. The rotation matrix R, viewed as a matrix transformation of vectors from body coordinates to inertial coordinates, can be written in terms of the Euler angles

[19]. The rotational kinematics for the base body are given by

$$\dot{\phi} = \omega_x + \sin\phi \tan\theta \omega_y + \cos\phi \tan\theta \omega_z, \tag{10}$$

$$\dot{\theta} = \cos\phi \omega_y - \sin\phi \omega_z, \tag{11}$$

$$\dot{\psi} = \sin\phi \sec\theta \omega_y + \cos\phi \sec\theta \omega_z, \tag{12}$$

where $\omega = (\omega_x, \omega_y, \omega_z)$ is the base body angular velocity. These equations are valid in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ which is consistent with the physical limits in pitch angle of the TACT. The angular velocity components of the base body can be written in the inverse kinematics form as

$$\omega_x = \dot{\phi} - \sin\theta \dot{\psi}, \tag{13}$$

$$\omega_y = \cos\phi \dot{\theta} + \sin\phi \cos\theta \dot{\psi}, \tag{14}$$

$$\omega_z = -\sin\phi \dot{\theta} + \cos\phi \cos\theta \dot{\psi}. \tag{15}$$

It is important to note that the above kinematics equations (10)-(12) and (13)-(15) do not depend explicitly on the yaw angle ψ. This fact makes clear that the direction of the gravity vector, represented in body coordinates, plays a critical role in the equations of motion. This suggests the de£nition

$$\Gamma = (\Gamma_x, \Gamma_y, \Gamma_z)^T = R^T e_3. \tag{16}$$

The vector Γ is a unit vector, $\|\Gamma\| = 1$, that points in the direction de£ned by gravity, expressed in the base body coordinate frame with origin at the TACT pivot point. We refer to Γ as the reduced attitude vector of the base body. In terms of the 3-2-1 Euler angles Γ can be expressed as in terms of the pitch and roll angles are

$$\Gamma(\theta, \phi) = \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix}. \tag{17}$$

It is easy to see that Γ satis£es the following differential equation:

$$\dot{\Gamma} = \Gamma \times \omega. \tag{18}$$

The rotation matrix R can be expressed in terms of the yaw angle ψ and the vector Γ as

$$R = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\Gamma_y^2 + \Gamma_z^2} & \frac{\Gamma_x \Gamma_y}{\sqrt{\Gamma_y^2+\Gamma_z^2}} & -\frac{\Gamma_x \Gamma_z}{\sqrt{\Gamma_y^2+\Gamma_z^2}} \\ 0 & \frac{\Gamma_z}{\sqrt{\Gamma_y^2+\Gamma_z^2}} & -\frac{\Gamma_y}{\sqrt{\Gamma_y^2+\Gamma_z^2}} \\ \Gamma_x & \Gamma_y & \Gamma_z \end{bmatrix},$$

so long as the reduced attitude vector satis£es $\Gamma_y^2 + \Gamma_z^2 > 0$. This condition is consistent with the restriction on the pitch angle mentioned previously, and it is satis£ed in the physical operating range of the TACT.

2.3. Dynamics Equations

The dynamics equations for the TACT are derived under the general assumptions stated in the previous section. The base body dynamics are given by the Euler-Lagrange-Poincare equations:

$$\frac{d}{dt}\Big(\frac{\partial L_1}{\partial \omega}\Big) = \frac{\partial L_1}{\partial \omega} \times \omega + \frac{\partial L_1}{\partial \Gamma} \times \Gamma + \tau_B, \tag{19}$$

where τ_B is the generalized (non-gravitational) moment acting on the triaxial base body, expressed in the base body frame. For simplicity, no external (non-gravitational) moment is assumed to act on the auxiliary bodies.

The equations of motion for the shape dynamics are given by the Euler-Lagrange equations:

$$\frac{d}{dt}\Big(\frac{\partial L_1}{\partial \dot{q}}\Big) - \frac{\partial L_1}{\partial q} = \tau_S, \tag{20}$$

where τ_S is the vector of generalized forces and moments that act to change the TACT shape dynamics.

Using the Lagrangian given in eqn. (8), computations show that the TACT equations of motion can be written as

$$\dot{R} = R\widehat{\omega} \tag{21}$$

$$\begin{bmatrix} J(q) & B(q) \\ B^T(q) & M(q) \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \ddot{q} \end{bmatrix}$$
$$= \begin{bmatrix} -\dot{J}(q)\omega - \dot{B}(q)\dot{q} + J(q)\omega \times \omega + B(q)\dot{q} \times \omega + m_T g \rho_s(q) \times \; R^T e_3 + \tau_B \\ -\dot{M}(q)\dot{q} - \dot{B}^T(q)\omega + \dfrac{\partial T_1(\omega, q, \dot{q})}{\partial q} + m_T g \left(\frac{\partial \rho_s(q)}{\partial q}\right)^T R^T e_3 - \dfrac{\partial V_s(q)}{\partial q} + \tau_S \end{bmatrix} \tag{22}$$

where the matrix function $\begin{bmatrix} J(q) & B(q) \\ B^T(q) & M(q) \end{bmatrix}$ is symmetric and positive de£nite for all $q \in Q_s$.

The above equations of motion are expressed in terms of the base body angular velocity ω, and the time rate of change of the shape coordinate $\dot{q}$. However, we can also express the equations of motion in terms of the conjugate momenta Π and Π_s.

We now introduce the conjugate momenta vectors Π and Π_s, expressed in base body coordinates, according to the Legendre transformation

$$\Pi = \frac{\partial T_1(\omega, q, \dot{q})}{\partial \omega} = J(q)\omega + B(q)\dot{q}, \tag{23}$$

$$\Pi_s = \frac{\partial T_1(\omega, q, \dot{q})}{\partial \dot{q}} = B^T(q)\omega + M(q)\dot{q}. \tag{24}$$

We can solve eqn.(23) and eqn.(24) for ω and $\dot{q}$ as

$$\begin{bmatrix} \omega \\ \dot{q} \end{bmatrix} = \begin{bmatrix} J^{-1}(q) - A(q)M_s^{-1}(q)A^T(q) & -A(q)M_s^{-1}(q) \\ -M_s^{-1}(q)A^T(q) & M_s^{-1}(q) \end{bmatrix} \begin{bmatrix} \Pi \\ \Pi_s \end{bmatrix}, \tag{25}$$

where $A(q)$ is the mechanical connection de£ned as $A(q) = J^{-1}(q)B(q)$ and $M_s(q) = M(q) - A^T(q)J(q)A(q)$ [14]. The matrix functions $J(q)$ and $M_s(q)$ are symmetric and positive de£nite for all $q \in Q_s$.

The kinetic energy expression can be written in terms of the conjugate momenta as

$$T_2(q, \Pi, \Pi_s) = \frac{1}{2} \begin{pmatrix} \Pi \\ \Pi_s \end{pmatrix}^T \begin{pmatrix} J^{-1}(q) - A(q)M_s^{-1}(q)A^T(q) & -A(q)M_s^{-1}(q) \\ -M_s^{-1}(q)A^T(q) & M_s^{-1}(q) \end{pmatrix} \begin{pmatrix} \Pi \\ \Pi_s \end{pmatrix}. \tag{26}$$

Equations (23) and (24) can be substituted into eqns.(19) and (20) using the Lagrangian (8) to obtain a simpli£ed form of the equations of motion, namely

$$\dot{R} = R\widehat{\omega}, \tag{27}$$

$$\begin{bmatrix} \dot{\Pi} \\ \dot{\Pi}_s \end{bmatrix} = \begin{bmatrix} \Pi \times \omega + m_T g \rho_s(q) \times R^T e_3 \\ \frac{\partial T_2(\Pi, q, \Pi_s)}{\partial q} + m_T g \left(\frac{\partial \rho_s(q)}{\partial q}\right)^T R^T e_3 - \frac{\partial V_s(q)}{\partial q} \end{bmatrix} + \begin{bmatrix} \tau_B \\ \tau_S \end{bmatrix}, \tag{28}$$

where

$$\omega = \left(J^{-1}(q) - A(q)M_s^{-1}(q)A^T(q)\right)\Pi - A(q)M_s^{-1}(q)\Pi_s.$$

Equations (27) and (28) have a relatively simple form and provide a useful alternative to eqns.(21) and (22). Note that equations (21), (22) and (27), (28) can be written in terms of other attitude representations. Three sets of equations can be expressed in terms of the Euler angles (in place of the rotation matrix). It is more convenient to expressed these equations in terms of the reduced attitude vectors (in place of the rotation matrix). The resulting kinematics and dynamics equations are considerably simpli£ed.

2.4. Conservation Law

It is clear that the kinetic energy expressions do not depend on the base body attitude, that is, the kinetic energy is invariant with respect to the group of all rotations of the base body. In addition, the gravitational potential energy expressions do not depend on the yaw angle; that is, the gravitational potential energy is invariant with respect to the group of rotations of the base body about a vertical axis. Thus the Lagrangian is also invariant with respect to the group of rotations of the base body about a vertical axis. This symmetry property implies from Noether's theorem that if the base body moment vector $\tau_B = 0$ there is a conservation law.

It is easy to see that if $\tau_B = 0$, there is no external moment acting on the base body about the vertical axis; thus the vertical component of the spatial angular momentum is conserved. This can be veri£ed from the equations of motion. Since the vertical component of the spatial angular momentum is given by

$$\mu_3 = \Gamma^T \Pi, \tag{29}$$

it follows that

$$\dot{\mu}_3 = \dot{\Gamma}^T \Pi + \Gamma^T \dot{\Pi} = \Gamma^T \widehat{\omega} \Pi - \Gamma^T \widehat{\omega} \Pi + \Gamma^T \widehat{\Gamma} \rho_s(q) = 0, \tag{30}$$

so that μ_3 is constant. If the conjugate momentum vector $\Pi = 0$ initially, then Π and Γ must be always orthogonal. This conservation law plays a crucial role in the controllability properties of the TACT. There are subtle issues addressed in [16].

3. EQUILIBRIA

The simplest dynamics of the TACT are de£ned by equilibria. In this section, we identify such solutions. Assume that TACT inputs τ_B and τ_S are constants denoted by τ_{Be} and τ_{Se}. Equilibrium solutions are de£ned by zero conjugate momenta, constant attitude of the base body, and constant shape. Thus $\Pi = 0$, $\omega = 0$, $\dot{R} = 0$, $\dot{q} = 0$. The conditions for equilibrium at constant shape q_e and constant attitude R_e are

$$m_T g \rho_s(q_e) \times R_e^T e_3 + \tau_{Be} = 0, \tag{31}$$

$$m_T g \left[\frac{\partial \rho_s(q)}{\partial q} |_{q=q_e} \right]^T R_e^T e_3 - \frac{\partial V_s(q)}{\partial q} |_{q=q_e} + \tau_{Se} = 0. \tag{32}$$

Now consider the special case that the base body £xed moment input $\tau_{Be} = 0$ in the remainder of this section. The £rst condition above means that the TACT is in equilibrium only if the system center of mass vector $\rho_s(q_e)$ is co-linear with the gravity vector $R_e^T e_3$. If $\rho_s(q_e) = 0$, then the equilibrium attitude of the base body R_e

is arbitrary. If $\rho_s(q_e) \neq 0$, this condition can be written as $R_e^T e_3 = \lambda \frac{\rho_s(q_e)}{|\rho_s(q_e)|}$, where the parameter λ can have the value $+1$ or -1, which we write as $\lambda = \pm 1$. Equation (32) guarantees that the shape coordinates are held constant by the shape input τ_{Se}.

It is convenient to distinguish two classes of equilibria in the case that $\tau_{Be} = 0$. The £rst class corresponds to the case that $\rho_s(q_e) = 0$. This guarantees that the TACT center of mass is located exactly at the pivot point if the TACT shape is q_e. In such case the TACT can be in equilibria at any attitude of the triaxial base body. We subsequently refer to such equilibria as *balanced equilibria*.

The second class corresponds to equilibria for which $\rho_s(q_e) \neq 0$. Such an equilibrium necessarily depends on both the attitude of the triaxial base body and the shape. In physical terms, such equilibria are characterized by particular triaxial base body attitudes and shape coordinates for which the center of mass of the system lies on the vertical line through the pivot point; the center of mass vector can either point in the direction of the gravity vector or opposite to the gravity vector. The £rst case corresponds to the condition $\lambda = +1$ while the second case corresponds to the condition $\lambda = -1$.

Assume the TACT is in equilibrium when the attitude of the body is R_e and the shape is q_e. Then it is also in equilibrium when the attitude of the base body satis£es

$$R = R_e e^{\psi \widehat{\Gamma}_e},$$

where $\Gamma_e = R_e^T e_3$ and ψ is an arbitrary constant. This means that if R_e de£nes an equilibrium base body attitude corresponding to a £xed shape q_e, then the TACT is also in equilibrium for any base body attitude that corresponds to a simple rotation about the gravity vector. This conclusion follows from Rodrigues's formula; since $\Gamma_e = R_e^T e_3$, we obtain $R^T e_3 = \left(e^{\psi \widehat{\Gamma}_e}\right)^T \Gamma_e = \left[I_3 + \sin\psi \widehat{\Gamma}_e + (1-\cos\psi)\widehat{\Gamma}_e^2\right]^T \Gamma_e = \Gamma_e$.

4. LINEARIZATION OF TACT MODELS

The above nonlinear models for the TACT can be simpli£ed in a number of different ways. We now derive linear models, expressed in a linear control form, that describe small perturbations of the dynamics from an equilibrium condition. Such an approach is classical, and it provides excellent insight into the TACT dynamics locally near the equilibrium; in many cases, it also provides valuable models for control system design. Several such linear approximation models are provided in this section. The models include both base body moment inputs and shape inputs for maximum generality.

In subsequent sections, several speci£c actuation assumptions are made that provide further simpli£cations.

4.1. Linearized Equations of Motion

We £rst consider the nonlinear TACT model given by eqns.(21-22). These equations are expressed in terms of the base body rotation matrix R, the base body angular velocity vector ω, the shape coordinates vector q and the shape velocity vector $\dot{q}$; the inputs are the base body moment vector τ_B and the shape input vector τ_S.

Assume that the TACT is in equilibrium corresponding to constant base body generalized force $\tau_B = \tau_{Be}$ and constant shape force $\tau_S = \tau_{Se}$, where the equilibrium is characterized by the constant base body attitude R_e and constant shape q_e. Consequently, the equilibrium eqns.(31-32) are satis£ed. We now consider small perturbations of the TACT dynamics from the equilibrium. To the £rst order, we have

$$R = R_e e^{\widehat{\Delta\Theta}}, \quad \omega = \Delta\omega, \quad q = q_e + \Delta q, \quad \dot{q} = \Delta\dot{q},$$

and

$$\tau_B = \tau_{Be} + \Delta\tau_B, \quad \tau_S = \tau_{Se} + \Delta\tau_S,$$

where Δz denotes a perturbation for a variable z, and exponential coordinates $\Delta\Theta \in \mathbb{R}^3$ are used to express the base body attitude perturbations for simplicity. The exponential coordinates are, to £rst order, identical to perturbations of the Euler angles $\Delta\Theta = (\Delta\phi, \Delta\theta, \Delta\psi)$.

It is easy to show that $\Delta\dot{\Theta} = \Delta\omega$. The linearized TACT model can be expressed in terms of the base body attitude perturbations and the shape perturbations as follows:

$$\begin{bmatrix} J(q_e) & B(q_e) \\ B^T(q_e) & M(q_e) \end{bmatrix} \begin{bmatrix} \Delta\ddot{\Theta} \\ \Delta\ddot{q} \end{bmatrix} = \bar{A} \begin{bmatrix} \Delta\Theta \\ \Delta q \end{bmatrix} + \begin{bmatrix} \Delta\tau_B \\ \Delta\tau_S \end{bmatrix}, \tag{33}$$

where

$$\bar{A} = \begin{bmatrix} m_T g \widehat{\rho}_s(q_e)(R_e^T e_3)^\wedge & -m_T g (R_e^T e_3)^\wedge \frac{\partial \rho_s(q)}{\partial q}|_{q=q_e} \\ m_T g \left(\frac{\partial \rho_s(q)}{\partial q}\right)^T |_{q=q_e} (R_e^T e_3)^\wedge & m_T g \frac{\partial}{\partial q}\left[\left(\frac{\partial \rho_s(q)}{\partial q}\right)^T R_e^T e_3\right]|_{q=q_e} - \frac{\partial^2 V_s(q)}{\partial q^2}|_{q=q_e} \end{bmatrix}.$$

The coef£cients of the linearized equations of motion depend on the equilibrium base body attitude R_e and the equilibrium shape q_e.

Following the terminology in [17] for linear second order vector systems, the effective "mass" matrix is symmetric and positive de£nite. If $\tau_{Be} = 0$, it can be shown that

$$m_T g \widehat{\rho}_s(q_e)(R_e^T e_3)^\wedge = m_T g \lambda |\rho_s(q_e)| \left[(R_e^T e_3)^\wedge\right]^2$$
$$= -m_T g \lambda |\rho_s(q_e)| \left[(R_e^T e_3)^\wedge\right]^T \left[(R_e^T e_3)^\wedge\right]$$

is symmetric, where as before $\lambda = \pm 1$, depending on the TACT equilibrium selected. Hence, the effective "stiffness" matrix is symmetric but not necessarily de£nite. These constant matrices can be computed directly from the fundamental model data that describes the TACT and the particular equilibrium. Note that the 1-1 block of the "stiffness" matrix vanishes for the case of a balanced equilibrium.

The above equations re¤ect the underlying Euler-Lagrange-Poincare form of the equations of motion of the TACT [13]. In particular, if $\tau_{Be} = 0$, it is easy to show that the open loop eigenvalues de£ned by eqn.(33) have a symmetric pattern about the real axis and about the imaginary axis in the complex plane. This property may be exploited to achieve control design objectives. If $\tau_{Be} \neq 0$, this symmetric eigenvalue pattern, apparently, may not hold.

4.2. Linearized Equations of Motion Expressed in Terms of Conjugate Momenta

We now consider the nonlinear TACT model given by eqn.(27 -28). The equations are expressed in terms of the base body rotation matrix R, the conjugate momenta vectors Π and Π_s and the shape coordinates vector q; the inputs are the base body moment vector τ_B and the shape input vector τ_S.

As before, we consider small perturbations of the TACT dynamics from an equilibrium corresponding to constant $R = R_e$, $q = q_e$, $\tau_B = \tau_{Be}$, and $\tau_S = \tau_{Se}$. To the £rst order, we have

$$R = R_e e^{\widehat{\Delta\Theta}}, \quad \Pi = \Delta\Pi, \quad q = q_e + \Delta q, \quad \dot{\Pi}_s = \Delta\dot{\Pi}_s,$$

and

$$\tau_B = \tau_{Be} + \Delta\tau_B, \quad \tau_S = \tau_{Se} + \Delta\tau_S.$$

The linearized equations of motion can be shown to be given by

$$\begin{bmatrix} \Delta\dot{\Theta} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} J^{-1}(q_e) - A(q_e)M_s^{-1}(q_e)A^T(q_e) & -A(q_e)M_s^{-1}(q_e) \\ -M_s^{-1}(q_e)A^T(q_e) & M_s^{-1}(q_e) \end{bmatrix} \begin{bmatrix} \Delta\Pi \\ \Delta\Pi_s \end{bmatrix}, \quad (34)$$

and

$$\begin{bmatrix} \Delta\dot{\Pi} \\ \Delta\dot{\Pi}_s \end{bmatrix} = \tilde{A} \begin{bmatrix} \Delta\Theta \\ \Delta q \end{bmatrix} + \begin{bmatrix} \Delta\tau_B \\ \Delta\tau_S \end{bmatrix}, \quad (35)$$

where

$$\tilde{A} = \begin{bmatrix} m_T g \widehat{\rho}_s(q_e)(R_e^T e_3)^\wedge & -m_T g (R_e^T e_3)^\wedge \frac{\partial \rho_s(q)}{\partial q}|_{q=q_e} \\ m_T g \left(\frac{\partial \rho_s(q)}{\partial q}\right)^T|_{q=q_e}(R_e^T e_3)^\wedge & m_T g \frac{\partial}{\partial q}\left[\left(\frac{\partial \rho_s(q)}{\partial q}\right)^T R_e^T e_3\right]|_{q=q_e} - \frac{\partial^2 V_s(q)}{\partial q^2}|_{q=q_e} \end{bmatrix}.$$

These linearized equations of motion are in a linear Hamiltonian form [13]. Note that if $\tau_{Be} = 0$, then the open loop eigenvalues exhibit the symmetric pattern identi£ed previously.

5. TACT ACTUATED BY THREE FANS

In this section, we formulate a model of a rigid TACT actuated by three motor driven fans (or reaction jets) £xed to the base body. The fan actuators provide control forces, and hence control moments, on the TACT base body that can be used to achieve base body attitude control objectives. In this case, the TACT is a rigid body; that is there are no auxiliary bodies and no shape dynamics.

5.1. Equations of Motion

We derive the equations of motion following the development in the previous sections. Let ρ_c denote the constant position vector of the TACT base body center of mass with respect to the pivot point expressed in the base body coordinate frame. Following the notation introduced previously, the Lagrangian is

$$L(\Gamma, \omega) = \frac{1}{2}\omega^T J_B \omega + m_B g \rho_c^T \Gamma.$$

The inertia matrix J_B and the mass m_B denote the inertia and mass of the triaxial base body and the fans that are £xed to it. The inertia matrix J_B is de£ned with respect to the base body coordinate frame. In the above Lagrangian, we ignore the gyroscopic effects of the fan blades.

The equations of motion are

$$\dot{R} = R\widehat{\omega}, \tag{36}$$

$$J_B \dot{\omega} = J_B \omega \times \omega + m_B g \rho_c \times \Gamma + \tau_B. \tag{37}$$

The moment τ_B is produced by the forces generated from the three fan actuators,

$$\tau_B = \sum_{i=1}^{3} \rho_i \times u_i \nu_i.$$

Here ρ_i denotes the £xed vector describing the location at which the i-th fan actuator is £xed to the base body and ν_i denotes the £xed unit vector describing the axis of the i-th fan actuator, all de£ned with respect to the base body coordinates. Thus u_i denotes the scalar force produced by the i-th fan actuator. We can write

$$\tau_B = Gu,$$

where $G = [\widehat{\rho}_1\nu_1 \;\; \widehat{\rho}_2\nu_2 \;\; \widehat{\rho}_3\nu_3]$ is a constant input in¤uence matrix and $u = (u_1 \;\; u_2 \;\; u_3)$ is the vector of fan forces. We assume that the matrix G is nonsingular.

Thus the equations of motion of the TACT controlled by three fan actuators are given by

$$\dot{R} = R\widehat{\omega}, \tag{38}$$

$$J_B\dot{\omega} = J_B\omega \times \omega + m_B g\rho_c \times \Gamma + Gu. \tag{39}$$

The conditions for equilibrium of the TACT, for constant fan force vector u_e, are

$$m_B g\rho_c \times \Gamma_e + Gu_e = 0.$$

In the special case that the fan forces $u_e = 0$, the condition for equilibrium is that the attitude of the base body Γ_e is co-linear with ρ_c. If $\rho_c \neq 0$, this implies that $\Gamma_e = \lambda\frac{\rho_c}{|\rho_c|}$, where $\lambda = \pm 1$. If $\lambda = +1$, then the equilibrium corresponds to the the TACT center of mass pointing in the direction of the gravity vector; if $\lambda = -1$, then the equilibrium corresponds to the the TACT center of mass vector pointing opposite to the direction of the gravity vector. The special case that $\rho_c = 0$ corresponds to a balanced equilibrium, where the center of mass of the TACT is at the pivot point. In this case, the TACT is in equilibrium for any base body attitude.

The above nonlinear model for the TACT is now simpli£ed by considering small perturbations from an equilibrium condition. Assume $\rho_c \neq 0$ and consider the TACT equilibrium corresponding to $u_e = 0$ so that the reduced equilibrium attitude is $\Gamma_e = \lambda\frac{\rho_c}{|\rho_c|}$ for $\lambda = \pm 1$. The linearized equations of motion for the TACT , in scalar form, are given by

$$J_B\Delta\ddot{\Theta} = -m_B g\frac{\lambda}{|\rho_c|}\widehat{\rho}_c^T\widehat{\rho}_c\Delta\Theta + G\Delta u,$$

where $\Delta\Theta$ is the vector of exponential coordinate perturbations from the equilibrium attitude.

Now consider the balanced case and assume that $\rho_c = 0$ and consider TACT equilibrium corresponding to $u_e = 0$. The TACT equilibrium attitude is arbitrary. The linearized equations of motion for the TACT, in vector form, are given by

$$J_B \Delta\ddot{\Theta} = G\Delta u.$$

5.2. A Special Case

We now consider the special case that the fan actuators are located on the principal axes with nonzero offsets given by

$$\rho_1 = \begin{bmatrix} b_x \\ 0 \\ 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 \\ b_y \\ 0 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} 0 \\ 0 \\ b_z \end{bmatrix}, \quad b_x \neq 0, \quad b_y \neq 0, \quad b_z \neq 0,$$

with their axes aligned so that $\nu_1 = e_2, \nu_2 = e_3, \nu_3 = e_1$.

We also assume that the base body axes are principal axes of the base body and the center of mass lies on the body £xed z-axis; that is $\rho_c = (0, 0, \rho_{cz})$. If $\rho_{cz} \neq 0$, there are two distinct equilibrium attitudes corresponding to $\Gamma_e = (0, 0, \lambda)$, where $\lambda = \pm 1$. If $\rho_{cz} = 0$, the TACT can be in equilibrium at any attitude.

Let diag(J_{xx}, J_{yy}, J_{zz}) denote the inertia matrix assuming the origin of the principal axes is at the center of mass of the base body; then it follows from the parallel axis theorem that

$$J_B = \text{diag}(J_{xx} + m_B\rho_{cz}^2, J_{yy} + m_B\rho_{cz}^2, J_{zz}).$$

Assume $\rho_{cz} \neq 0$. If we denote $\Delta\Theta = (\Delta\phi, \Delta\theta, \Delta\psi)$ and $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_2)$ then the linearized equations can be expressed by three scalar equations as

$$(J_{xx} + m_B\rho_{cz}^2)\Delta\ddot{\phi} = -m_B g\lambda|\rho_{cz}|\Delta\phi + b_y\Delta u_2, \tag{40}$$

$$(J_{yy} + m_B\rho_{cz}^2)\Delta\ddot{\theta} = -m_B g\lambda|\rho_{cz}|\Delta\theta + b_z\Delta u_3, \tag{41}$$

$$J_{zz}\Delta\ddot{\psi} = b_x\Delta u_1. \tag{42}$$

It is clear that, to £rst order, the roll, pitch and yaw dynamics are not coupled. In other words, the roll, pitch, and yaw dynamics are coupled only through higher order effects. The linearized roll and pitch dynamics depend on the equilibrium parameter λ. If $\lambda = +1$, the uncontrolled linearized roll and pitch dynamics are oscillatory corresponding to imaginary eigenvalues. If $\lambda = -1$, the uncontrolled linearized roll and pitch dynamics are unstable corresponding to a positive and a negative eigenvalue of equal magnitude. The uncontrolled yaw dynamics are always de£ned by a double eigenvalue at the origin, re¤ecting the fact that gravitational effects do not in¤uence the yaw dynamics at least to £rst order.

Assume $\rho_{cz} = 0$. The TACT is balanced and can be in equilibrium in any base body attitude. The linearized equation of motion for the TACT are given by

$$J_{xx}\Delta\ddot{\phi} = b_y \Delta u_2, \tag{43}$$

$$J_{yy}\Delta\ddot{\theta} = b_z \Delta u_3, \tag{44}$$

$$J_{zz}\Delta\ddot{\psi} = b_x \Delta u_1. \tag{45}$$

The uncontrolled linearized roll, pitch and yaw dynamics are decoupled and de£ned by a double eigenvalue at the origin.

It is clear from the linearized equations of motion that the three fan actuators can be used to control the complete base body attitude in all cases. This is a standard control problem that can be treated using classical control design procedures.

6. TACT ACTUATED BY THREE REACTION WHEELS

In this section, we formulate a model of the TACT consisting of a rigid base body actuated by three reaction wheels. Reaction wheels are assumed to have an axial symmetric mass distribution with respect to their rotation axes and their rotation axes are £xed with respect to the base body. We also assume that the shape potential energy function $V_s(q) = 0$ and there is no external moment on the system, i.e. $\tau_B = 0$.

6.1. Equations of Motion

Let ρ_i denote the constant position vector of the center of mass of the i-th reaction wheel in the base body frame; ν_i is a unit vector along the spin axis of the i-th reaction wheel, $i = 1, 2, 3$, and let ρ_c denote the constant position vector of the base body center of mass, all expressed in the base body coordinates.

We derive the equations of motion following the development in the previous sections. Here, the shape coordinates are $q = (\phi_1, \phi_2, \phi_3)$, the rotation angles of the reaction wheels 1, 2, 3, respectively. The Lagrangian of the TACT is given by

$$L = \frac{1}{2}\omega^T J_B \omega + \frac{1}{2}\sum_{i=1}^{3}\left\{m_p v_i^T v_i + \omega_i^T J_i \omega_i\right\} + m_T g \rho_s^T \Gamma.$$

Here J_B is the inertia matrix of the base body expressed in base body coordinate frame, m_B is the base body mass, and m_r is the common mass of the reaction wheels. The total TACT mass is $m_T = m_B + 3m_r$. The center of the i-th reaction wheel with respect to the pivot point is given by $\rho_i - \rho_c, i = 1,2,3$. The constant position vector of the TACT center of mass, in the base body coordinate frame, is $\rho_s = \frac{1}{m_T}(m_B\rho_c + m_r \sum_{i=1}^{3} \rho_i)$.

Denote the constant moments of inertia of each wheel by J_s along its spinning axis and by J_r along its radial axis. All three reaction wheels are assumed to have identical physical properties. De£ne a body-£xed orthogonal coordinate frame for each reaction wheel with origin at the center of the wheel so that its £rst axis is along the spin axis of the wheel and the other two axes de£ne the plane within which the reaction wheel rotates. Assume the rotation matrix from the attitude frame to the base body frame is $R_i, i = 1,2,3$. It can be shown that the inertia matrix J_i of the i-th wheel, expressed in the base body coordinate frame, is given by

$$J_i = R_i^T \left\{ \text{diag}(J_s, J_r, J_r) - m_r(\rho_c - \rho_i)^\wedge(\rho_c - \rho_i)^\wedge \right\} R_i, \quad i = 1,2,3.$$

The translational velocity vector of the i-th wheel is $v_i = \omega \times \rho_i, i = 1,2,3$, and the angular velocity of the i-th wheel is $\omega_i = w + \dot{\phi}_i \nu_i$, where $\nu_i = R_i e_1, i = 1,2,3$. Here ω is the angular velocity of the base body expressed in the base body coordinate frame.

The Lagrangian can be written as

$$L(\Gamma, \omega, \dot{q}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix}^T \begin{bmatrix} J & B \\ B^T & M \end{bmatrix} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix} + m_T g \rho_s^T \Gamma \,,$$

where

$$J = J_B + \sum_{i=1}^{3} \left\{ m_p \widehat{\rho}_i^T \widehat{\rho}_i + J_i \right\}, \quad M = \text{diag}(\nu_1^T J_1 \nu_1^T, \nu_2^T J_2 \nu_2, \nu_3^T J_3 \nu_3),$$

and $B = \begin{bmatrix} J_1\nu_1 & J_2\nu_2 & J_3\nu_3 \end{bmatrix}$. The mechanical connection is given by $A = J^{-1}B$. Note that these matrices are constant and do not depend on the shape coordinates.

The equations of motion for the TACT actuated by reaction wheels are given by

$$\dot{R} = R\widehat{\omega}, \tag{46}$$

$$\begin{bmatrix} J & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} J\omega \times \omega + B\dot{q} \times \omega + m_T g \rho_s \times \Gamma \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}. \tag{47}$$

In eqn.(47) $u = (u_1, u_2, u_3)$, and u_i denotes the scalar moment about the spin axis of the i-th reaction wheel provided by the motor driving the i-th reaction wheel.

Assuming that $u = 0$, the vector condition for equilibrium of the TACT actuated by reaction wheels is

$$\rho_s \times \Gamma_e = 0.$$

If $\rho_s \neq 0$, this implies that the reduced equilibrium attitude of the base body Γ_e is co-linear with the center of mass vector ρ_s, that is $\Gamma_e = \lambda \frac{\rho_s}{|\rho_s|}$ where $\lambda = \pm 1$. If $\lambda = +1$, then the equilibrium corresponds to the TACT center of mass vector pointing in the direction of the gravity vector; if $\lambda = -1$, then the equilibrium corresponds to the TACT center of mass vector opposite to the direction of the gravity vector. The case that $\rho_s = 0$ corresponds to a balanced equilibrium, where the center of mass of the TACT is at the pivot point. In this case, the TACT is in equilibrium at any base body attitude. Note that the conditions for equilibria are independent of the shape coordinates.

The above nonlinear model for the TACT is now simpli£ed by considering small perturbations from an equilibrium condition denoted by base body attitude R_e. If $\rho_{cz} \neq 0$, the linearized equations of motion for the TACT, in vector form, are given by

$$\begin{bmatrix} J & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \Delta\ddot{\Theta} \\ \Delta\ddot{q} \end{bmatrix} = \begin{bmatrix} -m_T g \frac{\lambda}{|\rho_s|} \widehat{\rho}_s^T \widehat{\rho}_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\Theta \\ \Delta q \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta u \end{bmatrix}, \tag{48}$$

where $\Delta\Theta$ is the vector of exponential coordinate perturbations from the equilibrium attitude, and Δq is the vector of shape perturbations. Note that the linearized dynamics are independent of the reaction wheel angles Δq.

The linearized TACT equations of motion can be simpli£ed and expressed as

$$\left[J - AMA^T\right] \Delta\ddot{\Theta} = -m_T g \frac{\lambda}{|\rho_s|} \widehat{\rho}_s^T \widehat{\rho}_s \Delta\Theta - BM^{-1} \Delta u,$$

which can be compared with the model for the TACT controlled by three body £xed moments given in the previous section.

If $\rho_s = 0$, the TACT can be in equilibrium for any attitude. The linearized equations of motion for the TACT, in vector form, are given by

$$\begin{bmatrix} J & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \Delta\ddot{\Theta} \\ \Delta\ddot{q} \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta u \end{bmatrix}, \tag{49}$$

or equivalently

$$\left[J - AMA^T\right] \Delta\ddot{\Theta} = -BM^{-1} \Delta u.$$

6.2. A Special Case

Now suppose the center of mass of the base body, in the base body coordinate frame, is located at $\rho_c = (0, 0, \rho_{cz})$, and the base body coordinate frame, if translated to the base body center of mass, de£nes principal axes of the base body with principal moments of inertia (J_{xx}, J_{yy}, J_{zz}). Assume the reaction wheels are located on the principal axes and aligned so that each spin axis is along that principal axis.

Assuming $\rho_{cz} \neq 0$, the TACT is in equilibrium if $\Gamma_e = \lambda e_3$ for $\lambda = \pm 1$, assuming $u_e = 0$. Furthermore, we assume the mass of the reaction wheels are small compared with that of the base body. Then the linearized equations of motion can be expressed in terms of the attitude perturbations $\Delta\Theta = (\Delta\phi, \Delta\theta, \Delta\psi)$ and the wheel moments $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ as

$$(J_{xx} + m_B\rho_{cz}^2 + J_s + 2J_r)\Delta\ddot{\phi} = -m_B g\lambda|\rho_{cz}|\Delta\phi - \Delta u_1, \tag{50}$$

$$(J_{yy} + m_B\rho_{cz}^2 + J_s + 2J_r)\Delta\ddot{\theta} = -m_B g\lambda|\rho_{cz}|\Delta\theta - \Delta u_2, \tag{51}$$

$$(J_{zz} + J_s + 2J_r)\Delta\ddot{\psi} = -\Delta u_3. \tag{52}$$

It is clear that, to £rst order, the roll, pitch and yaw dynamics are not coupled. In other words, the roll, pitch, and yaw dynamics are coupled only through higher order effects. The linearized roll and pitch dynamics depend on the value of the parameter λ. If $\lambda = +1$, the uncontrolled linearized roll and pitch dynamics are oscillatory corresponding to imaginary eigenvalues. If $\lambda = -1$, the uncontrolled linearized roll and pitch dynamics are unstable corresponding to a positive and a negative eigenvalue of equal magnitude. The uncontrolled yaw dynamics are always de£ned by a double eigenvalue at the origin, re¤ecting the fact that gravitational effects do not in¤uence the yaw dynamics at least to £rst order.

If $\rho_{cz} = 0$, then the TACT is balanced and the linearized equations of motion are

$$(J_{xx} + J_s + 2J_r)\Delta\ddot{\phi} = -\Delta u_1, \tag{53}$$

$$(J_{yy} + J_s + 2J_r)\Delta\ddot{\theta} = -\Delta u_2, \tag{54}$$

$$(J_{zz} + J_s + 2J_r)\Delta\ddot{\psi} = -\Delta u_3. \tag{55}$$

The uncontrolled linearized roll, pitch, and yaw dynamics are de£ned by a double eigenvalue at the origin.

It is also clear from the linearized equations of motion that in all cases the three reaction wheels can be used to control the complete base body attitude. This is a standard control problem that can be treated using classical control design procedures.

7. TACT ACTUATED BY THREE PROOF MASS ACTUATORS

In this section, we formulate a model of the TACT actuated by three proof mass actuators. The positions and velocities of the proof masses generate gravitational moments on the base body and also dynamic coupling between the base body and the shape. Therefore, these devices can be used as actuators for the base body. We also assume that the shape potential energy function $V_s(q)$ arises from linear elastic springs that connect each proof mass to the base body, and there is no external torque on the base body, i.e. $\tau_B = 0$.

7.1. Equations of Motion

Each proof mass actuator consists of an ideal mass particle that can be translated along the linear axis of the actuator by a motor. The proof masses are assumed to have identical masses m_p. The shape coordinates $q = (q_1, q_2, q_3)$ denote the positions of the proof masses relative to their respective axes. Let $\tilde{\rho}_1$, $\tilde{\rho}_2$, and $\tilde{\rho}_3$ denote the constant position vectors of the locations of the three proof masses with respect to the pivot point expressed in the body coordinate frame, assuming the proof masses are £xed at zero shape; this is assumed to correspond to zero potential energy in the proof mass restoring springs. Let $\nu_i, i = 1, 2, 3,$ denote unit vectors that de£ne the axes of the proof mass actuators. For a general shape, the position vectors of the three proof masses with respect to the body coordinate frame are

$$\rho_i(q) = \tilde{\rho}_i + q_i \nu_i, \quad i = 1, 2, 3.$$

The kinetic energy is given by

$$T = \frac{1}{2}\omega^T J_B \omega + \frac{m_p}{2} \sum_{i=1}^{3} \left\{ v_i^T v_i \right\},$$

where v_i is the velocity vector of the i-th proof mass:

$$v_i = \omega \times \rho_i(q) + \dot{q}_i \nu_i, \quad i = 1, 2, 3.$$

Therefore,

$$T(q, \omega, \dot{q}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix}^T \begin{bmatrix} J(q) & B \\ B^T & M \end{bmatrix} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix},$$

where

$$J(q) = J_B + m_p \sum_{i=1}^{3} \left\{ \widehat{\rho}_i^T(q) \widehat{\rho}_i(q) \right\},$$

$$M = m_p \text{diag}(\nu_1^T \nu_1, \nu_2^T \nu_2, \nu_3^T \nu_3),$$

$$B(q) = m_p \big[\tilde{\rho}_1 \times \nu_1 \;\; \tilde{\rho}_2 \times \nu_2 \;\; \tilde{\rho}_3 \times \nu_3 \big].$$

The inertia matrix J_B is de£ned with respect to the base body coordinate frame whose origin is not at the base body center of mass when $\rho_c \neq 0$. Note that the matrices M and B are constant and do not depend on the shape. The mechanical connection is $A(q) = J^{-1}(q)B$.

The gravitational potential energy is given by

$$V(\Gamma, q) = -(m_B g \tilde{\rho}_c + m_p g \sum_{i=1}^{3} \rho_i(q))^T \Gamma = -m_T g (\rho_c + Pq)^T \Gamma,$$

where $\tilde{\rho}_c$ is the location of the center of mass of the base body, $m_T = m_B + 3m_p$ is the total mass of the base body and proof masses, $\rho_c = \frac{1}{m_T}(m_B \tilde{\rho}_c + m_p \sum_{i=1}^{3} \tilde{\rho}_i)$ is the location of the center of mass of the base body and proof masses, where the proof masses are at their zero positions, and $P = \frac{m_p}{m_T}[\nu_1 \; \nu_2 \; \nu_3]$. The elastic potential energy is given by

$$V_s(q) = \frac{1}{2} \sum_{i=1}^{3} k_i q_i^2,$$

where $k_i > 0, \; i = 1, 2, 3$, are spring constants, and $K = \text{diag}(k_1, k_2, k_3)$.

The Lagrangian is

$$L(\Gamma, \omega, q, \dot{q}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix}^T \begin{bmatrix} J(q) & B \\ B^T & M \end{bmatrix} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix} + m_T g (\rho_c + Pq)^T \Gamma - \frac{1}{2} q^T K q.$$

The equations of motion of the TACT with proof mass actuators are given by

$$\dot{R} = R\widehat{\omega}, \tag{56}$$

$$\begin{bmatrix} J(q) & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} -\dot{J}(q)\omega + J(q)\omega \times \omega + B\dot{q} \times \omega + m_T g (\rho_c + Pq) \times \Gamma \\ \frac{\partial T(\omega, q, \dot{q})}{\partial q} + m_T g P^T \Gamma - Kq \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}. \tag{57}$$

In eqn.(57) $u = (u_1, u_2, u_3)$, and u_i denotes the scalar axial force on the i-th proof mass generated by its motor.

Let $\rho_s(q) = \rho_c + Pq$ denote the shape dependent center of the mass of the TACT. The conditions for equilibrium of the TACT actuated by proof mass actuators, assuming constant motor forces u_e on the proof masses, are

$$\rho_s(q_e) \times \Gamma_e = 0,$$
$$m_T g P^T \Gamma_e - K q_e + u_e = 0.$$

If $\rho_s(q_e) \neq 0$, this implies that the reduced equilibrium attitude of the base body Γ_e is co-linear with the center of mass vector $\rho_s(q_e)$, that is $\Gamma_e = \lambda \frac{\rho_s(q_e)}{|\rho_s(q_e)|}$ for $\lambda = \pm 1$. If $\lambda = +1$, then the equilibrium corresponds to the TACT center of mass vector pointing in the direction of gravity; if $\lambda = -1$, then the equilibrium corresponds to the TACT center of mass vector opposite to the direction of gravity. The special case that $\rho_s(q_e) = 0$ corresponds to a shape for which the equilibrium is balanced; that is the center of mass of the TACT is at the pivot point for shape q_e. In this case, the TACT is in equilibrium at any base body attitude.

The above nonlinear model for the TACT is now simpli£ed by considering small perturbations from an equilibrium condition corresponding to a base body attitude and proof mass positions satisfying $\Gamma_e = \lambda \frac{\rho_s(q_e)}{|\rho_s(q_e)|}$ for $\lambda = \pm 1$. The linearized equations of motion for TACT with proof mass actuators are given in vector form by

$$\begin{bmatrix} J(q_e) & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \Delta\ddot{\Theta} \\ \Delta\ddot{q} \end{bmatrix}$$
$$= \begin{bmatrix} -m_T g \frac{\lambda}{|\rho_s(q_e)|} \widehat{\rho}_s^T(q_e) \widehat{\rho}_s(q_e) & -m_T g \frac{\lambda}{|\rho_s(q_e)|} \widehat{\rho}_s^T(q_e) P \\ m_T g \frac{\lambda}{|\rho_s(q_e)|} P^T \widehat{\rho}_s(q_e) & -K \end{bmatrix} \begin{bmatrix} \Delta\Theta \\ \Delta q \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta u \end{bmatrix}, \quad (58)$$

where $\Delta\Theta$ is the vector of exponential coordinate perturbations from the equilibrium attitude, and Δq is the vector of shape perturbations.

7.2. A Special Case

Now suppose the center of mass of the base body, in the base body coordinate frame, is located at $\rho_c = (0, 0, \rho_{cz})$, and the base body coordinate frame, if translated to the base body center of mass, de£nes principal axes of the base body with principal moments of inertia (J_{xx}, J_{yy}, J_{zz}). Assume the axes of the proof mass actuators are aligned with the principal axes of the base body and the zero positions of the i-th proof mass corresponds to the TACT center of mass at the pivot point. Thus, $\tilde{\rho}_i = 0, \nu_i = e_i, i = 1, 2, 3.$

Assume that the equilibrium shape is $q_e = (0,0,0)$, and $\rho_s(0) = (0,0,\rho_{cz}) \neq 0$. The TACT is in equilibrium if $\Gamma_e = \lambda e_3$ for $\lambda = \pm 1$. The linearized equations of motion can be expressed in terms of the attitude perturbations $\Delta\Theta = (\Delta\phi, \Delta\theta, \Delta\psi)$ and shape perturbations $\Delta q = (\Delta q_1, \Delta q_2, \Delta q_3)$ as

$$(J_{xx} + m_B\rho_{cz}^2)\Delta\ddot{\phi} = -m_T g\lambda|\rho_{cz}|\Delta\phi + m_p g\Delta q_2, \tag{59}$$

$$m_p\Delta\ddot{q}_2 = m_p g\Delta\phi - k_2\Delta q_2 + \Delta u_2, \tag{60}$$

$$(J_{yy} + m_B\rho_{cz}^2)\Delta\ddot{\theta} = -m_T g\lambda|\rho_{cz}|\Delta\theta - m_p g\Delta q_1, \tag{61}$$

$$m_p\Delta\ddot{q}_1 = -m_p g\Delta\theta - k_1\Delta q_1 + \Delta u_1, \tag{62}$$

$$J_{zz}\Delta\ddot{\psi} = 0, \tag{63}$$

$$m_p\Delta\ddot{q}_3 = -k_3\Delta q_3 + \Delta u_3. \tag{64}$$

It is clear that, to £rst order, the roll dynamics are in¤uenced only by the proof mass that is mounted on the body £xed y-axis and translates along the body £xed y-axis. Similarly, the pitch dynamics are in¤uenced only by the proof mass that is mounted on the body £xed x-axis and translates along the body £xed x-axis. The pitch and roll dynamics, together with dynamics of the two proof masses mounted on the body £xed x-axis and y-axis, are decoupled from the yaw dynamics and the dynamics of the third proof mass that is mounted on the body £xed z-axis.

Control of the pitch and roll dynamics of the base body can be studied using only equations (59), (61), (62), and (60). It is also clear from equation (63) that, to £rst order, the yaw dynamics are not affected by any of the three proof masses, and the proof mass mounted on the body £xed z-axis does not in¤uence the base body attitude dynamics at least to the £rst order. This suggests that the proof mass actuator mounted on the body £xed z-axis is useless as an actuator to control the base body attitude. Control of the yaw dynamics is not possible using this linear model; control of the yaw dynamics may be possible based on the nonlinear models developed previously.

Now assume that the equilibrium shape is $q_e = (0,0,0)$ and $\rho_s(0) = (0,0,0)$. The TACT is in equilibrium for any attitude at zero shape. The linearized equations of motion in this case can be expressed as

$$J_{xx}\Delta\ddot{\phi} = m_p g \Delta q_2, \tag{65}$$

$$m_p \Delta\ddot{q}_2 = m_p g \Delta\phi - k_2 \Delta q_2 + \Delta u_2, \tag{66}$$

$$J_{yy}\Delta\ddot{\theta} = -m_p g \Delta q_1, \tag{67}$$

$$m_p \Delta\ddot{q}_1 = -m_p g \Delta\theta - k_1 \Delta q_1 + \Delta u_1, \tag{68}$$

$$J_{zz}\Delta\ddot{\psi} = 0, \tag{69}$$

$$m_p \Delta\ddot{q}_3 = -k_3 \Delta q_3 + \Delta u_3. \tag{70}$$

It is clear that the roll dynamics are the roll and pitch dynamics are linearly controllable using eqns.(65-66) and eqns.(67-68). But the linearized yaw dynamics are not controllable. Control of the yaw dynamics may be possible based on the nonlinear models developed previously. The proof mass mounted on the body £xed z-axis does not in¤uence the base body attitude dynamics, at least to the £rst order.

8. TACT WITH AN ELASTIC SUBSYSTEM FIXED TO THE BASE BODY ACTUATED BY THREE FANS

We now formulate a model of the TACT, assuming an elastic subsystem is £xed to the base body and the TACT base body is actuated by three fans. The elastic subsystem can be a multi-degree of freedom system de£ned in terms of auxiliary bodies that move with respect to the base body constrained by linear elastic restoring forces. The elastic subsystem is assumed to be unactuated in the sense that there are no actuation forces or moments that directly affect the shape dynamics.

8.1. Equations of Motion

The elastic subsystem consists of n idealized mass particles. These mass particles are attached to the base body through linear elastic springs. Let $K \in \mathbb{R}^{n\times n}$ de£ne the stiffness matrix of the elastic subsystem. It is assumed, without loss of generality, that K is a symmetric and positive semi-de£nite matrix. Each mass particle is assumed to translate along a particular direction that is £xed with respect to the base body, and this is referred to as the motion axis of the mass particle.

The unit vector ν_i de£nes the motion axis of the i-th mass particle and the shape coordinate q_i denotes the position of the i-th mass particle along its axis with respect to the base body. The direction vectors $\nu_i, i = 1, \cdots, n$, are expressed in the body coordinate frame and $q = (q_1, \cdots, q_n)$ represents the vector of shape coordinates.

Let $\tilde{\rho}_i$ denote the constant position vector of i-th mass particle $i = 1, \cdots, n$, with respect to the pivot point expressed in the body coordinate frame, assuming no elastic deformation at zero shape; this is assumed to correspond to zero elastic potential energy. For a general shape, the position vectors of the mass particles with respect to the body coordinate frame are

$$\rho_i(q) = \tilde{\rho}_i + q_i\nu_i, \quad i = 1, \cdots, n.$$

The kinetic energy is given by

$$T = \frac{1}{2}\omega^T J_B \omega + \frac{1}{2}\sum_{i=1}^{n}\left\{m_i v_i^T v_i\right\},$$

where J_B is the inertia matrix of the base body, expressed in the base body coordinate frame, and m_i is the mass of the i-th auxiliary body, $i = 1, \cdots, n$. The velocity vector of the i-th mass particle, expressed in the body coordinate frame, is

$$v_i = \omega \times \rho_i(q) + \dot{q}_i\nu_i, \quad i = 1, \cdots, n.$$

Therefore, the kinetic energy is

$$T(q, \omega, \dot{q}) = \frac{1}{2}\begin{pmatrix}\omega \\ \dot{q}\end{pmatrix}^T \begin{bmatrix} J(q) & B \\ B^T & M\end{bmatrix}\begin{pmatrix}\omega \\ \dot{q}\end{pmatrix},$$

where

$$\begin{aligned}
J(q) &= J_B + \sum_{i=1}^{n}\left\{m_i \widehat{\rho}_i^T(q)\widehat{\rho}_i(q)\right\},\\
M &= \mathrm{diag}(m_1, \cdots, m_n),\\
B &= \left[m_1(\tilde{\rho}_1 \times \nu_1) \;\; m_2(\tilde{\rho}_2 \times \nu_2) \;\; \cdots \;\; m_n(\tilde{\rho}_n \times \nu_n)\right].
\end{aligned}$$

Note that the matrices M and B are constant and do not depend on the shape q. The mechanical connection is $A(q) = J^{-1}(q)B$.

The gravitational potential energy is given by

$$V(\Gamma, q) = -(m_B g\tilde{\rho}_c + g\sum_{i=1}^{n} m_i\rho_i(q))^T\Gamma = -m_T g(\rho_c + Pq)^T\Gamma,$$

where $\tilde{\rho}_c$ is the location of the center of mass of the base body, $m_T = m_B + \sum_{i=1}^{N} m_i$ is the total mass of the base body and all mass particles, $\rho_c = \frac{1}{m_T}(m_B\tilde{\rho}_c + \sum_{i=1}^{N} m_i\tilde{\rho}_i)$ is the location of the center of mass of the base body and mass particles, when the particles are at their zero positions, and $P = \frac{1}{m_T}[m_1\nu_1 \; m_2\nu_2 \cdots \; m_n\nu_n]$.

The elastic potential energy is given by

$$V_s(q) = \frac{1}{2} q^T K q.$$

The Lagrangian is

$$L(\Gamma, \omega, q, \dot{q}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix}^T \begin{bmatrix} J(q) & B \\ B^T & M \end{bmatrix} \begin{pmatrix} \omega \\ \dot{q} \end{pmatrix} + m_T g (\rho_c + Pq)^T \Gamma - \frac{1}{2} q^T K q.$$

The equations of motion of the TACT are given by

$$\dot{R} = R\widehat{\omega}, \tag{71}$$

$$\begin{bmatrix} J(q) & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} -\dot{J}(q)\omega + J(q)\omega \times \omega + B\dot{q} \times \omega + m_T g(\rho_c + Pq) \times \Gamma \\ \frac{\partial T(\omega, q, \dot{q})}{\partial q} + m_T g P^T \Gamma - Kq \end{bmatrix} + \begin{bmatrix} Gu \\ 0 \end{bmatrix}. \tag{72}$$

As previously $u = (u_1, u_2, u_3)$ is the vector of fan forces, and G is a constant input in¤uence matrix. We assume that the matrix G is nonsingular.

Let $\rho_s(q) = \rho_c + Pq$ denote the shape dependent center of mass of the TACT. Assume the fan actuator forces are constant $u = u_e$. The conditions for equilibrium of the TACT in this case are given by

$$q_e = m_T g K^{-1} P^T \Gamma_e, \tag{73}$$

$$G u_e = -m_T g [\rho_c + m_T g P K^{-1} P^T \Gamma_e] \times \Gamma_e. \tag{74}$$

Thus, if the TACT is to be in equilibrium in the reduced attitude Γ_e, the required constant shape is given by equation (73) and the required constant fan force vector is given by equation (74). In equilibrium, the center of mass of the TACT is $\rho_s(q_e) = \rho_c + m_T g P K^{-1} P^T \Gamma_e$.

The above nonlinear model for the TACT can be simpli£ed by considering small perturbations from an equilibrium condition. The linearized equations of motion for the TACT are given in vector form by

$$\begin{bmatrix} J(q_e) & B \\ B^T & M \end{bmatrix} \begin{bmatrix} \Delta\ddot{\Theta} \\ \Delta\ddot{q} \end{bmatrix} = \begin{bmatrix} m_T g \widehat{\rho}_s(q_e) \widehat{\Gamma}_e & -m_T g \widehat{\Gamma}_e P \\ m_T g P^T \widehat{\Gamma}_e & -K \end{bmatrix} \begin{bmatrix} \Delta\Theta \\ \Delta q \end{bmatrix} + \begin{bmatrix} G\Delta u \\ 0 \end{bmatrix}, \tag{75}$$

where $\Delta\Theta$ is the vector of exponential coordinate perturbations from the equilibrium attitude, and Δq is the vector of shape perturbations. In general, $m_T g \widehat{\rho}_s(q_e)\widehat{\Gamma}_e$ is not a symmetric matrix.

8.2. Two Special Cases

In order to gain more insight into the linearized equations, we consider two special cases where the elastic subsystem consists of a single unactuated mass particle. The two cases differs in terms of the alignment of the axis of the unactuated mass particle. Other simplifying assumptions are also made.

We assume that the base body axes are principal axes of the base body, and the center of mass of the base body lies on the base body £xed z-axis, that is $\rho_c = (0, 0, \rho_{cz})$. Therefore if diag$(J_{xx}, J_{yy}, J_{zz})$ denotes the inertia matrix, assuming the origin of the principal axes is at the center of mass of the base body, then

$$J_B = \text{diag}(J_{xx} + m_B\rho_{cz}^2, J_{yy} + m_B\rho_{cz}^2, J_{zz}).$$

We also assume that the fan actuators are mounted on the principal axes with nonzero offsets given by

$$\rho_1 = \begin{bmatrix} b_x \\ 0 \\ 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 \\ b_y \\ 0 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} 0 \\ 0 \\ b_z \end{bmatrix},$$

with their axes aligned so that

$$\nu_1 = e_2, \;\; \nu_2 = e_3, \;\; \nu_3 = e_1.$$

The £rst special case assumes that the unactuated mass particle is constrained to translate along the z-axis of the base body. The mass of the particle is m, and it is acted on by a linear spring with elastic stiffness constant $k > 0$. The elastic restoring force on the mass particle is zero when the particle's position relative to the base body £xed axis corresponds to shape $q = 0$.

Assume that $\rho_{cz} \neq 0$. In this case, the TACT is in equilibrium if

$$\Gamma_e = \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix}, \quad q_e = \frac{mg}{k}, \quad u_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

for $\lambda = \pm 1$. It can be shown that

$$P = \begin{bmatrix} 0 \\ 0 \\ \frac{m}{m_T} \end{bmatrix}, \quad \rho(q_e) = \begin{bmatrix} 0 \\ 0 \\ \frac{mg\lambda}{k} \end{bmatrix}, \quad \rho_s(q_e) = \begin{bmatrix} 0 \\ 0 \\ \rho_{cz} + \frac{m^2 g\lambda}{m_T k} \end{bmatrix},$$

so that the linearized equations of motion are

$$(J_{xx} + m_B\rho_{cz}^2 + \frac{m^3g^2}{k^2})\Delta\ddot{\phi} = -(m_T g\rho_{cz} + \frac{m^2g^2}{k})\Delta\phi + b_y\Delta u_2, \tag{76}$$

$$(J_{yy} + m_B\rho_{cz}^2 + \frac{m^3g^2}{k^2})\Delta\ddot{\theta} = -(m_T g\rho_{cz} + \frac{m^2g^2}{k})\Delta\theta + b_z\Delta u_3, \tag{77}$$

$$J_{zz}\Delta\ddot{\psi} = b_x\Delta u_1, \tag{78}$$

$$m\Delta\ddot{q} = -k\Delta q. \tag{79}$$

Clearly, the fan actuators can be used to control the base body attitude. But at least to the £rst order, the dynamics of the unactuated elastic subsystem can not be controlled by the fan actuators. Using the nonlinear equations of motion, it may be possible to control both the base body attitude and the elastic subsystem; this is an open research question.

The second special case assumes that the unactuated mass particle is constrained to translate along the x-axis of the base body. The mass of the particle is m, and the elastic stiffness constant is $k > 0$, as previously.

Assume that $\rho_{cz} \neq 0$. In this case, the TACT is in equilibrium if

$$\Gamma_e = \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix}, \quad q_e = 0, \quad u_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

for $\lambda = \pm 1$. It can be shown that

$$P = \begin{bmatrix} \frac{m}{m_T} \\ 0 \\ 0 \end{bmatrix}, \quad \rho(q_e) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \rho_s(q_e) = \begin{bmatrix} 0 \\ 0 \\ \rho_{cz} \end{bmatrix},$$

so that the linearized equations of motion are

$$(J_{xx} + m_B\rho_{cz}^2)\Delta\ddot{\phi} = -\lambda m_T g\rho_{cz}\Delta\phi + b_y\Delta u_2, \tag{80}$$

$$(J_{yy} + m_B\rho_{cz}^2)\Delta\ddot{\theta} = -\lambda m_T g\rho_{cz}\Delta\theta + b_z\Delta u_3, \tag{81}$$

$$J_{zz}\Delta\ddot{\psi} = b_x\Delta u_1, \tag{82}$$

$$m\Delta\ddot{q} = -\lambda m_T g\Delta\theta - k\Delta q. \tag{83}$$

Note that the dynamics of the unactuated elastic subsystem are in¤uenced by the controlled pitch dynamics. On the basis of these linearized equations, the fan actuators can control both the base body attitude and the elastic subsystem.

9. CONCLUSIONS

In this paper, we have developed equations of motion for the TACT, an experimental testbed that can be used for research on a wide variety of rotational dynamics and control problems. Many models and model approximations have been presented. These models can be used to study the dynamics and control of the TACT. They can be used for simulation and computational purposes, and they can be used to validate and explain TACT experiments.

Both TACT nonlinear equations of motion and TACT linear approximate equations of motion have been developed. Some dynamics and control issues can be studied using the linear approximations. But many physical setups have been identi£ed for which the linear models are not suitable. For such cases, the dynamics and control problems are fundamentally nonlinear. It is these latter cases that are of most interest in our continuing research on the TACT.

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