Chapter 3

Asymptotic Stability of Multibody Attitude Systems*

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Abstract: A rigid base body, supported by a fixed pivot point, is free to rotate in three dimensions. Multiple elastic subsystems are rigidly mounted on the rigid body; the elastic degrees of freedom are constrained relative to the rigid base body. A mathematical model is developed for this multibody attitude system that exposes the dynamic coupling between the rotational degrees of freedom of the base body and the deformation or shape degrees of freedom of the elastic subsystems. The models are used to assess passive dissipation assumptions that guarantee asymptotic stability of an equilibrium solution. These results are motivated and inspired by a 1980 publication of R. K. Miller and A. N. Michel [6].

*This research has been supported in part by NSF under grant ECS-0140053.

3.1 Introduction

A photograph of the triaxial attitude control testbed (TACT) in the Attitude Dynamics and Control Laboratory at the University of Michigan is shown in Figure 3.1.1. Its physical properties are described in detail in [1], and a detailed derivation of mathematical models for the TACT is given in [2, 3]. The TACT is based on a spherical air bearing that provides a near-frictionless pivot for the base body. The stability problem treated in this paper is motivated by possible set-ups of the TACT.

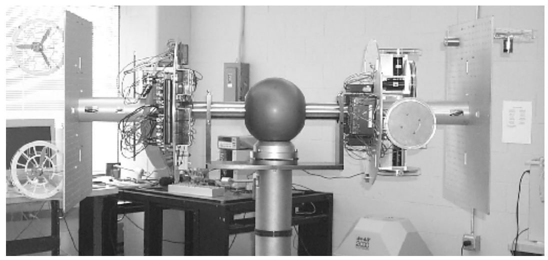

Figure 3.1.1: Triaxial Attitude Control Testbed

This paper presents results of a study of asymptotic stability properties of an abstraction of the TACT. This abstraction consists of a rigid base body that is free to rotate in three dimensions. Multiple elastic subsystems are rigidly mounted on the base body. The paper treats the uncontrolled motion of this multibody attitude system. Results are obtained that guarantee asymptotic stability of an equilibrium.

This paper is motivated by a 1980 publication of R. K. Miller and A. N. Michel in [6]. This Miller and Michel paper studied an elastic multibody system consisting of an interconnection of ideal mass elements and elastic springs. Lyapunov function arguments, based on the system Hamiltonian, were used to develop sufficient damping assumptions that guarantee asymptotic stability of the equilibrium. A key insight was the use of observability properties to guarantee asymptotic stability. The paper by Miller and Michel provided a clear and direct exposition of these issues. It was one of the earliest papers to make clear connections between properties of Hamiltonian systems and their control theoretical properties. During the last 23 years, these issues have been extensively studied. However, the Miller and Michel paper remains an important resource for researchers on dynamics and control of mechanical systems.

3.2 Equations of Motion

Consider the following class of multibody attitude systems: the base body rotates about a fixed pivot point; see Figure 3.2.1. A base body fixed coordinate frame is chosen with its origin located at the pivot point. We assume that the center of mass of the system is always at the pivot point, and thus does not depend on the shape. This is a restrictive assumption, but we demonstrate that it represents an interesting class of multibody attitude systems. Thus gravity does not affect the dynamics and its effects are irrelevant in the subsequent analysis.

The configuration manifold is given by $Q = SO(3) \times Q_s$, $\widehat{\omega} \in \mathfrak{so}(3)$ with $\omega \in \mathbb{R}^3$ representing the base body angular velocity expressed in the base body frame. We use $r \in Q_s$ to denote n-dimensional generalized shape coordinates or deformation of the elastic systems. Assuming that the kinetic and potential energy are invariant to SO(3)-action, the dynamics are only dependent on $(\omega, r, \dot{r})$. This leads to the reduced Lagrangian and the reduced equations of motion.

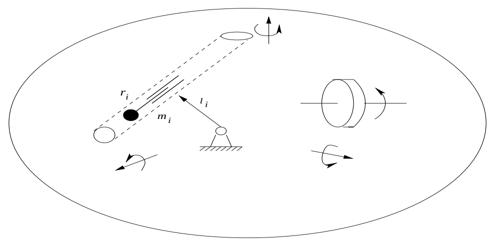

Figure 3.2.1: Schematic configuration of a multibody attitude system with a translational elastic degree of freedom and a rotational elastic degree of freedom.

The reduced kinetic energy is given by

$$T(\omega, r, \dot{r}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix}^T \underbrace{\begin{bmatrix} M_{11}(r) & M_{12}(r) \\ M_{21}(r) & M_{22}(r) \end{bmatrix}}_{M(r)} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix},$$

where $M(r)$ is symmetric and positive definite for all $r \in Q_s$. Let $V_s(r)$ denote the potential energy of the elastic subsystems that depends only on the shape coordinates. Throughout the subsequent analysis, we assume that $V_s(r)$ has a local minimum at the shape r_e, i.e., $\frac{\partial V_s(r_e)}{\partial r} = 0$ and $\frac{\partial^2 V_s(r_e)}{\partial r^2} > 0$.

We obtain the reduced Lagrangian on $\mathfrak{so}(3) \times TQ_s$ as

$$L(\omega, r, \dot{r}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix}^T \begin{bmatrix} M_{11}(r) & M_{12}(r) \\ M_{21}(r) & M_{22}(r) \end{bmatrix} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix} - V_s(r).$$

Therefore, the equations of motion for the multibody attitude system are in the form of Euler-Poincare equations

$$M(r) \begin{bmatrix} \dot{\omega} \\ \ddot{r} \end{bmatrix} = -\frac{dM(r)}{dt} \begin{bmatrix} \omega \\ \dot{r} \end{bmatrix} + \begin{bmatrix} \Pi \times \omega \\ \frac{\partial [T(\omega, r, \dot{r}) - V_s(r)]}{\partial r} \end{bmatrix}, \tag{3.2.1}$$

where $\Pi = \frac{\partial L}{\partial \omega} = M_{11}(r)\omega + M_{12}(r)\dot{r}$ is the conjugate angular momentum. It is easy to verify that the spatial angular momentum is conserved and $||\Pi||_2$ is a conserved quantity; see [8]. In addition, $(\omega, r, \dot{r}) = (0, r_e, 0)$ is an isolated equilibrium of (3.2.1).

We now add dissipation to (3.2.1). We consider linear passive damping at the pivot point of the base body attitude dynamics and in the shape dynamics so that equation (3.2.1) becomes

$$M(r)\begin{bmatrix} \dot{\omega} \\ \ddot{r} \end{bmatrix} = -\frac{dM(r)}{dt}\begin{bmatrix} \omega \\ \dot{r} \end{bmatrix} + \begin{bmatrix} \Pi \times \omega \\ \frac{\partial [T(\omega, r, \dot{r}) - V_s(r)]}{\partial r} \end{bmatrix} - \begin{bmatrix} C_a \omega \\ C_r \dot{r} \end{bmatrix}, \tag{3.2.2}$$

where $C_a \in \mathbb{R}^{3\times 3}, C_r \in \mathbb{R}^{n\times n}$ denote constant damping matrices that satisfy $C_a = C_a^T \geq 0$, $C_r = C_r^T \geq 0$, i.e., both are symmetric and positive semi-definite. As before, $(\omega, r, \dot{r}) = (0, r_e, 0)$ is an isolated equilibrium of (3.2.2).

3.3 Conservation of Energy and Lyapunov Stability

Define a noncanonical Hamiltonian based on the reduced Lagrangian $H(\omega, r, \dot{r}) = T(\omega, r, \dot{r}) + V_s(r) = L(\omega, r, \dot{r}) + 2V_s(r)$. This Hamiltonian can be viewed as the total energy of the system. We now verify that $\dot{H} \equiv 0$ along the solutions of the undamped dynamics in equation (3.2.1). In fact,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \omega}\dot{\omega} + \frac{\partial H}{\partial \dot{r}}\ddot{r} + \frac{\partial H}{\partial r}\dot{r} = \frac{\partial L}{\partial \omega}\dot{\omega} + \frac{\partial L}{\partial \dot{r}}\ddot{r} + \Big(\frac{\partial T}{\partial r} + \frac{\partial V_s}{\partial r}\Big)\dot{r}.$$

Note that

$$\begin{aligned} \frac{\partial L}{\partial \omega}\dot{\omega} + \frac{\partial L}{\partial \dot{r}}\ddot{r} &= \begin{pmatrix} \omega & \dot{r} \end{pmatrix}^T M(r)\begin{pmatrix} \dot{\omega} \\ \ddot{r} \end{pmatrix} = \begin{pmatrix} \omega & \dot{r} \end{pmatrix}^T \Big\{ -\dot{M}(r)\begin{pmatrix} \omega \\ \dot{r} \end{pmatrix} + \begin{bmatrix} \Pi \times \omega \\ \frac{\partial T}{\partial r} - \frac{\partial V_s}{\partial r} \end{bmatrix} \Big\} \\ &= -\Big(\frac{\partial T}{\partial r} + \frac{\partial V_s}{\partial r}\Big)\dot{r}, \end{aligned}$$

where we use equation (3.2.1) and

$$\begin{pmatrix} \omega & \dot{r} \end{pmatrix}^T \dot{M}(r)\begin{pmatrix} \omega \\ \dot{r} \end{pmatrix} = 2\frac{\partial T}{\partial r}\dot{r}.$$

Consequently, $\frac{dH}{dt} \equiv 0$. This result agrees with the fact that the total energy is conserved if there is no dissipation.

In the case of linear damping, $\dot{H}$ along the solutions of (3.2.2) is given by $\dot{H} = -\omega^T C_a \omega - \dot{r}^T C_r \dot{r} \leq 0$. Therefore, $\dot{H}$ is negative semi-definite. Moreover, since the Hamiltonian is a positive definite function of $(\omega, r, \dot{r})$ in any small neighborhood of the equilibrium satisfying $H(0, r_e, 0) = 0$, it is a Lyapunov function. Thus by the property that $\dot{H} \leq 0$ along the solutions of (3.2.2), we conclude that the equilibrium $(\omega, r, \dot{r}) = (0, r_e, 0)$ is stable in the sense of Lyapunov.

Since both C_a and C_r are symmetric and positive semi-definite, it is easy to show that the set where $\dot{H} = 0$ is the set

$$S = \{(\omega, \dot{r}) | C_a \omega = 0, C_r \dot{r} = 0\}.$$

This result will be used in the subsequent development.

3.4 Asymptotic Stability Analysis

We now develop conditions on the damping matrices C_a and C_r and on the inertia matrix $M(r)$ for (local) asymptotic stability of the equilibrium $(\omega, r, \dot{r}) = (0, r_e, 0)$. The reader should be cautioned that stability has been established for the equilibrium. Thus, throughout the remaining sections, the equilibrium is always stable in the sense of Lyapunov, even when it is not asymptotically stable. The conditions obtained in the subsequent sections are only for asymptotic stability of the equilibrium.

While general conditions follow from LaSalle's invariance principle [5], which is related to nonlinear observability [4, 7], it is non-trivial to obtain concrete and easily verified conditions, due to the nonlinear dynamics and the noncanonical form of the Hamiltonian. We first present a trivial sufficient condition and a necessary condition as follows.

Proposition 3.4.1. *Assume $C_a > 0$, $C_r > 0$. Then the equilibrium is asymptotically stable.*

Proof. The set where $\dot{H} = 0$ is equivalent to $(\omega, \dot{r}) = (0, 0)$. Substituting this result into the equations of motion, we obtain $\frac{\partial V_s(r)}{\partial r} = 0$, which holds only at r_e in any small neighborhood of r_e. Consequently, the equilibrium is asymptotically stable by the invariance principle. □

Proposition 3.4.2. *The equilibrium is asymptotically stable only if $C_a \neq 0$.*

Proof. The attitude equation in (3.2.2) can be written as $\dot{\Pi} = \Pi \times \omega - C_a \omega$, where Π is the conjugate angular momentum. If $C_a = 0$, then $||\Pi(t)||_2 = ||\Pi(0)||_2$ for all $t \geq 0$ since $\frac{d||\Pi(t)||_2^2}{dt} = 2\Pi^T \dot{\Pi} = 0$. Hence for any initial $(\omega, \dot{r})$ in any small neighborhood of the equilibrium such that $\Pi(0) \neq 0$, $||\Pi(t)||_2 \neq 0$ for all $t \geq 0$. Moreover, since the equilibrium is stable, $M(r)$ is bounded along the solutions. This implies that $(\omega, \dot{r})$ does not converge to zero as $t \to \infty$. □

It is a challenge to obtain more concrete conditions. For simplicity, we focus on two cases in the subsequent development: the first case assumes C_a to be positive definite, and the second case assumes C_r to be positive definite. The first case makes use of the observability rank condition to obtain results for asymptotic stability. In the second case, both linear and nonlinear observability conditions fail to verify observability; thus a new approach is developed. In both cases, we emphasize how the coupling between the attitude dynamics and the shape dynamics can lead to asymptotic stability without full damping.

3.4.1 Asymptotic Stability Analysis: $C_a > 0,\ C_r \geq 0$

Throughout this section, we assume C_a to be positive definite. In this case, the set where $\dot{H} = 0$ is the set $S = \{(\omega, \dot{r}) | \omega = 0,\ C_r \dot{r} = 0\}$. Let $\mathcal{M}$ be the largest invariant set in S; solutions in $\mathcal{M}$ must satisfy the following equations:

$$\frac{d\Big(M_{12}(r)\dot{r}\Big)}{dt} = 0, \tag{3.4.1}$$

$$\frac{d\Big(M_{22}(r)\dot{r}\Big)}{dt} = \frac{1}{2}\Big(\dot{r}^T \frac{\partial M_{22}(r)}{\partial r}\dot{r}\Big) - \frac{\partial V_s(r)}{\partial r}, \tag{3.4.2}$$

$$C_r \dot{r} = 0, \tag{3.4.3}$$

where

$$\Big(\dot{r}^T \frac{\partial M_{22}(r)}{\partial r}\dot{r}\Big) = \Big(\dot{r}^T \frac{\partial M_{22}(r)}{\partial r_1}\dot{r},\ \cdots,\ \dot{r}^T \frac{\partial M_{22}(r)}{\partial r_n}\dot{r}\Big)^T \in \mathbb{R}^n.$$

Clearly, $(r, \dot{r}) = (r_e, 0)$ is a trivial solution to these equations.

The general conditions for asymptotic stability may be obtained using the concept of "local distinguishability" in [6]. We present a sufficient condition using the concept of nonlinear observability [4, 7]. Note that (3.4.2) can be written as

$$\begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{r} \\ M_{22}^{-1}(r)\Big\{ -\dot{M}_{22}(r)\dot{r} + \frac{1}{2}\Big(\dot{r}^T \frac{\partial M_{22}(r)}{\partial r}\dot{r}\Big) - \frac{\partial V_s(r)}{\partial r}\Big\} \end{bmatrix}}_{f(r,\dot{r})}, \tag{3.4.4}$$

and we define an output function according to equations (3.4.1) and (3.4.3)

$$y = \underbrace{\begin{bmatrix} \dot{M}_{12}(r)\dot{r} + M_{12}(r)M_{22}^{-1}(r)\Big\{ -\dot{M}_{22}(r)\dot{r} + \frac{1}{2}\Big(\dot{r}^T \frac{\partial M_{22}(r)}{\partial r}\dot{r}\Big) - \frac{\partial V_s(r)}{\partial r}\Big\} \\ C_r \dot{r} \end{bmatrix}}_{h(r,\dot{r})}, \tag{3.4.5}$$

where $h = (h_1, \cdots, h_{2n})$ and each h_i is a scalar function of $(r, \dot{r})$. Denote the observability co-distribution [4, 7] by dO defined as:

$$dO(r, \dot{r}) = \text{span}\Big\{dL_f^k h_i(r, \dot{r}),\ \ i = 1, \cdots, 2n,\ \ k = 0, 1, \cdots\Big\}.$$

Proposition 3.4.3. *Assume $C_a > 0$. The equilibrium is asymptotically stable if* $\operatorname{rank}\{dO\} = 2n$ *at* $(r, \dot{r}) = (r_e, 0)$.

Proof. If $\operatorname{rank}\{dO\} = 2n$ holds at $(r, \dot{r}) = (r_e, 0)$, then the system described by (3.4.4)–(3.4.5) is locally observable at $(r_e, 0)$ [4, 7]. This means that there exists a neighborhood $\mathcal{V}$ of $(r_e, 0)$ such that for any initial condition in $\mathcal{V}$, the output $y(t) = 0,\ t \geq 0$ if and only if the initial condition is equal to $(r_e, 0)$. This shows that the only solution in $\mathcal{V}$ that lies in the invariant set $\mathcal{M}$ is the trivial solution $r(t) = r_e,\ t \geq 0$. Hence, the equilibrium is asymptotically stable. □

Checking the observability rank condition using the observability co-distribution may require complicated computations; a restrictive but computationally tractable condition is based on linearization of (3.4.4)–(3.4.5) at the equilibrium. Because observability of the linearized system implies observability of the nonlinear system, we have the following result.

Corollary 3.4.1. *Assume $C_a > 0$. Let*

$$K_s = \frac{\partial^2 V_s(r_e)}{\partial r^2}.$$

Then the equilibrium is asymptotically stable if the following pair is observable

$$\left(\begin{bmatrix} -M_{12}(r_e)M_{22}^{-1}(r_e)K_s, & C_r \end{bmatrix}, \begin{bmatrix} 0 & I_n \\ -M_{22}^{-1}(r_e)K_s & 0 \end{bmatrix} \right).$$

Remark 3.4.1. *A restrictive but easily verified sufficient condition is that the intersection of the kernel of $M_{12}(r_e)$ and the kernel of C_r only contains the zero element. Note that if $M_{12}(r_e)$ has full column rank (which implies that the number of shape degrees of freedom cannot be more than three), then the condition*

$$\mathit{Ker}(M_{12}(r_e)) \cap \mathit{Ker}(C_a) = \{0\}$$

holds regardless of C_r, even when $C_r = 0$. This observation shows that asymptotic stability can be achieved via coupling between the attitude dynamics and the shape dynamics without full damping in the shape dynamics.

Remark 3.4.2. *Suppose M_{12} and M_{22} are constant for all $r \in Q_s$ and the elastic potential energy $V_s(r) = \frac{1}{2}(r - r_e)^T K_s (r - r_e)$, where K_s is a positive definite constant matrix. Then the linear observability condition in Corollary 3.4.1 is also a necessary condition for asymptotic stability.*

3.4.2 Asymptotic Stability Analysis: $C_r > 0$, $C_a \geq 0$

Throughout this section, C_r is assumed to be positive definite, i.e., the shape dynamics are fully damped. In this case, the set where $\dot{H} = 0$ is the set $S = \{(\omega, \dot{r}) | C_a \omega = 0,\ \dot{r} = 0\}$, i.e., $\omega(t)$ is in the kernel of C_a while $r(t) = r_c$ for some constant r_c in a small neighborhood of r_e. Let $\mathcal{M}$ be the largest invariant set of solutions of (3.2.2) that lie in S. The solutions $\omega(t)$ that lie in $\mathcal{M}$ must satisfy the following equations:

$$
\begin{aligned}
M_{11}(r_c)\dot{\omega} &= \Big[M_{11}(r_c)\omega\Big] \times \omega, && (3.4.6)\\
M_{21}(r_c)\dot{\omega} &= \frac{1}{2}\Big(\omega^T \frac{\partial M_{11}(r_c)}{\partial r}\omega\Big) - \frac{\partial V_s(r_c)}{\partial r}, && (3.4.7)\\
C_a\omega &= 0, && (3.4.8)
\end{aligned}
$$

where

$$\Big(\omega^T \frac{\partial M_{11}(r_c)}{\partial r}\omega\Big) = \Big(\omega^T \frac{\partial M_{11}(r_c)}{\partial r_1}\omega, \cdots, \omega^T \frac{\partial M_{11}(r_c)}{\partial r_n}\omega\Big)^T \in \mathbb{R}^n.$$

Using a similar argument as in the proof of Proposition 3.4.2, we see that the equilibrium is asymptotically stable if and only if the solution $\omega(t)$ in $\mathcal{M}$ is identically zero. To see this, note that the magnitude of $\Pi(t) = M_{11}(r_c)\omega(t)$ is conserved along the solutions in $\mathcal{M}$. Thus if $\omega(t) \neq 0$ at some $t \geq 0$, then $||\Pi(t)||_2$ cannot approach zero, which contradicts asymptotic stability. The sufficiency follows from the fact that if $\omega(t) = 0, t \geq 0$ lies in $\mathcal{M}$, then r_c must equal r_e.

The observability rank condition [4, 7] fails to show (nonlinear) observability at the equilibrium when C_a is strictly positive semi-definite. In fact, let (3.4.6) describe the solution ω, and define an output function according to (3.4.7)–(3.4.8). It can be shown that the rank of the observability co-distribution evaluated at the equilibrium $\omega = 0$ is equal to rank(C_a). Thus if C_a is strictly positive semi-definite, the observability rank condition, which is only sufficient, does not guarantee local observability or asymptotic stability. In the following, we provide an alternative approach that leads to necessary and sufficient conditions for asymptotic stability when C_a is strictly positive semi-definite.

A. Properties of solutions in the invariant set

We first study solutions of (3.4.6)–(3.4.8) by representing it by the following time-series expansion

$$\omega(t) = b_0 + \sum_{k=1}^{\infty} b_k \frac{t^k}{k!},$$

where $b_k \in \mathbb{R}^3, k = 0, 1, \cdots$, are constant and each b_k can be viewed as the k-th order time derivative of $\omega(t)$ at $t = 0$. Substituting this series expansion into (3.4.6)–(3.4.7) and equating the orders of time, we obtain the following (nonlinear) algebraic equations

$$\frac{M_{11}(r_c)}{k!} b_{k+1} = \sum_{j=0}^{k} \frac{1}{j!(k-j)!} M_{11}(r_c) b_j \times b_{k-j}, \quad k = 0, 1, 2, \cdots \qquad (3.4.9)$$

and

$$M_{21}(r_c)b_1 = \frac{1}{2}\Big(b_0^T\frac{\partial M_{11}(r_c)}{\partial r}b_0\Big) - \frac{\partial V_s(r_c)}{\partial r}; \tag{3.4.10}$$

$$\frac{M_{21}(r_c)}{k!}b_{k+1} = \frac{1}{2}\sum_{j=0}^{k}\frac{1}{j!(k-j)!}\Big(b_j^T\frac{\partial M_{11}(r_c)}{\partial r}b_{k-j}\Big), \ \ k \geq 1. \tag{3.4.11}$$

To be more specific, we list these algebraic equations up to order three as follows:

$$\begin{aligned}
M_{11}(r_c)b_1 &= M_{11}(r_c)b_0 \times b_0,\\
M_{11}(r_c)b_2 &= M_{11}(r_c)b_0 \times b_1 + M_{11}(r_c)b_1 \times b_0,\\
\frac{1}{2!}M_{11}(r_c)b_3 &= \frac{1}{2!}M_{11}(r_c)b_0 \times b_2 + M_{11}(r_c)b_1 \times b_1 + \frac{1}{2!}M_{11}(r_c)b_2 \times b_0,
\end{aligned}$$

and

$$\begin{aligned}
M_{21}(r_c)b_1 &= \frac{1}{2}\Big(b_0^T\frac{\partial M_{11}(r_c)}{\partial r}b_0\Big) - \frac{\partial V_s(r_c)}{\partial r},\\
M_{21}(r_c)b_2 &= \Big(b_0^T\frac{\partial M_{11}(r_c)}{\partial r}b_1\Big),\\
\frac{1}{2!}M_{21}(r_c)b_3 &= \frac{1}{2}\Big(2b_0^T\frac{\partial M_{11}(r_c)}{\partial r}b_2 + b_1^T\frac{\partial M_{11}(r_c)}{\partial r}b_1\Big).
\end{aligned}$$

In addition, it is clear that (3.4.8) holds, i.e.,

$$C_a\omega(t) = 0 \text{ for } t \geq 0,$$

if and only if

$$C_a b_k = 0, \quad\quad k = 0, 1, 2, \cdots . \tag{3.4.12}$$

We conclude that $\omega(t)$ is a solution of (3.4.6)–(3.4.8) if and only if $b_k, k \geq 1$, generated from (3.4.9) for a given b_0 satisfy equations (3.4.10)–(3.4.12) at some r_c.

One can solve $b_k, k \geq 1$, recursively via the algebraic equation (3.4.9) for a given b_0; this defines a sequence $\{b_k, k \geq 0\}$. Thus (3.4.9) can be viewed as a generating equation, while (3.4.10)–(3.4.12) can be viewed as constraint equations for $b_k, k \geq 0$. Solving the algebraic equation (3.4.9) for the solution $\omega(t)$ is much easier than solving the original nonlinear differential (3.4.6). Moreover, this makes it possible that asymptotic stability conditions are described by simple algebraic equations, which are easy to check via computational tools.

The following properties can be easily verified for b_k generated from (3.4.9):

P1. If $b_0 = 0$, then $b_k = 0$ for all $k \geq 1$ and $\omega(t) = 0, \ t \geq 0$;

P2. If $b_1 = 0$, then $b_k = 0$ for all $k \geq 2$ for any b_0, and thus $\omega(t) = b_0$ for all $t \geq 0$;

P3. For any $j \geq 1$, if $b_j = b_{j+1} = \cdots = b_{2j-1} = 0$, then $b_k = 0$ for all $k \geq 2j$ regardless what $b_0, b_1, \cdots, b_{j-1}$ are; in this case $\omega(t) = \sum_{i=0}^{j-1} b_i \frac{t^i}{i!}$, $t \geq 0$.

We show two results that will be used for the subsequent analysis.

Lemma 3.4.1. *Let $b_k, k \geq 1$, be generated via (3.4.9) for b_0. Then the identity $\sum_{j=0}^{k} C_j^k b_j^T M_{11}(r_c) b_{k-j} = 0$ holds for $k = 1, 2, \cdots$, where $C_j^k = \frac{k!}{j!(k-j)!}$.*

Proof. Note that

$$\omega^T(t) M_{11}(r_c) \dot{\omega}(t) = \omega^T(t) \Big(M_{11}(r_c) \omega(t) \times \omega(t) \Big) = 0 \text{ for all } t \geq 0$$

and

$$\frac{d\big(\omega^T(t) M_{11}(r_c) \omega(t)\big)}{dt} = 2\omega^T(t) M_{11}(r_c) \dot{\omega}(t) = 0, \text{ for all } t \geq 0.$$

Hence, $\omega^T(t) M_{11}(r_c) \omega(t)$ is constant for all $t \geq 0$. Moreover because

$$\begin{aligned} \omega^T(t) M_{11}(r_c) \omega(t) &= \Big(\sum_{j=0}^{\infty} b_j \frac{t^j}{j!} \Big)^T M_{11}(r_c) \Big(\sum_{i=0}^{\infty} b_i \frac{t^i}{i!} \Big), \\ &= \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \frac{1}{j!(k-j)!} b_j^T M_{11}(r_c) b_{k-j} \Big) t^k = \text{constant}, \end{aligned}$$

for all $t \geq 0$, we have

$$\sum_{j=0}^{k} \frac{1}{j!(k-j)!} b_j^T M_{11}(r_c) b_{k-j} = 0 \text{ for all } k \geq 1.$$

As a result,

$$\sum_{j=0}^{k} C_j^k b_j^T M_{11}(r_c) b_{k-j} = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} b_j^T M_{11}(r_c) b_{k-j} = 0, \ k \geq 1.$$

Lemma 3.4.2. *Let $b_k, k \geq 1$, be generated via (3.4.9) from b_0. Suppose $b_2 \in$ span(b_0, b_1), then $b_k \in$ span(b_0, b_1) for all $k \geq 0$.*

Proof. We show this by induction. Clearly, $b_0, b_1, b_2 \in \text{span}(b_0, b_1)$, following from the given conditions. Assume $b_j \in \text{span}(b_0, b_1)$ for $j = 0, 1, 2, \cdots, k$, where $k \geq 2$. Therefore, b_j can be expressed as $b_j = e_j^0 b_0 + e_j^1 b_1$ for real numbers e_j^0, e_j^1.

Recalling $C_j^k = \frac{k!}{j!(k-j)!}$ and using $\sum_{j=0}^k C_j^k e_j^0 e_{k-j}^1 = \sum_{j=0}^k C_j^k e_j^1 e_{k-j}^0$, we have

$$
\begin{aligned}
&M_{11}(r_c)b_{k+1} \\
&= \sum_{j=0}^{k} C_j^k M_{11}(r_c) b_j \times b_{k-j}, \\
&= \sum_{j=0}^{k} C_j^k M_{11}(r_c) \Big(e_j^0 b_0 + e_j^1 b_1\Big) \times \Big(e_{k-j}^0 b_0 + e_{k-j}^1 b_1\Big), \\
&= \sum_{j=0}^{k} C_j^k \Big[e_j^0 e_{k-j}^0 M_{11}(r_c) b_0 \times b_0 + \frac{e_j^0 e_{k-j}^1 + e_j^1 e_{k-j}^0}{2} \Big(M_{11}(r_c) b_0 \times b_1 \\
&\qquad + M_{11}(r_c) b_1 \times b_0\Big) + e_j^1 e_{k-j}^1 M_{11}(r_c) b_1 \times b_1\Big], \\
&= \sum_{j=0}^{k} C_j^k \Big[e_j^0 e_{k-j}^0 M_{11}(r_c) b_1 + \frac{e_j^0 e_{k-j}^1 + e_j^1 e_{k-j}^0}{2} M_{11}(r_c) b_2 \\
&\qquad - (b_1^T b_0) e_j^1 e_{k-j}^1 M_{11}(r_c) b_0\Big],
\end{aligned}
$$

where we use the identities for b_1, b_2 and the formula $M_{11}(r_c)b_1 \times b_1 = \Big(M_{11}(r_c)b_0 \times b_0\Big) \times b_1 = (b_1^T M_{11}(r_c)b_0)b_0 - (b_1^T b_0)M_{11}(r_c)b_0 = -(b_1^T b_0)M_{11}(r_c)b_0$. Consequently, we obtain

$$
b_{k+1} = \sum_{j=0}^{k} C_j^k \Big[e_j^0 e_{k-j}^0 b_1 + \frac{e_j^0 e_{k-j}^1 + e_j^1 e_{k-j}^0}{2} b_2 - (b_1^T b_0) e_j^1 e_{k-j}^1 b_0\Big]. \tag{3.4.13}
$$

Hence, $b_{k+1} \in \text{span}(b_0, b_1)$ because $b_2 \in \text{span}(b_0, b_1)$. □

The following result expresses asymptotic stability conditions in terms of the sequence b_k.

Proposition 3.4.4. *Assume $C_r > 0$. In any small neighborhood $\mathcal{U}$ of $(b, r) = (0, r_e) \in \mathbb{R}^{3+n}$, if for any $(b_0, r_c) \in \mathcal{U}$ with $b_0 \neq 0$, there exists some b_k generated via (3.4.9) from b_0 that violates one of the constraint (3.4.10)–(3.4.12), then the equilibrium is asymptotically stable.*

Proof. Suppose the given conditions hold. Then in any small neighborhood of $b = 0 \in \mathbb{R}^3$, no sequence $\{b_k\}$ generated from nonzero b_0 is a solution in the invariant set $\mathcal{M}$. Consider a sequence $\{b_k\}$ with $b_0 = 0$. By P1, all $b_k = 0, k \geq 1$ and they satisfy the constraint (3.4.11)–(3.4.12) trivially. Thus $b_k = 0, k \geq 0$, which implies $\omega(t) = 0,\ t \geq 0$, is the only solution in $\mathcal{M}$. This also implies that r_c must equal r_e using the properties of $V_s(r)$. Hence, the equilibrium is asymptotically stable. □

B. Conditions for asymptotic stability

According to Proposition 3.4.2, the dimension of $\mathrm{Ker}(C_a)$, the kernel of C_a (or the null space of C_a), must be either one or two for asymptotic stability. We consider these cases individually.

We refer to the following conditions as A1: in any small neighborhood $\mathcal{U}$ of $(b, r) = (0, r_e) \in \mathbb{R}^{3+n}$, there exists $(b_0, r_c) \in \mathcal{U}$ with $b_0 \neq 0$ satisfying:

A1.1 $C_a b_0 = 0$;

A1.2 $M_{11}(r_c) b_0 \times b_0 = 0$;

A1.3 $\frac{1}{2}\left(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_0\right) = \frac{\partial V_s(r_c)}{\partial r}$.

Before proceeding to the case $\dim\{\mathrm{Ker}(C_a)\} = 1$, we point out a necessary condition for asymptotic stability when C_a is strictly positive semi-definite.

Lemma 3.4.3. *Assume $C_r > 0$ and A1 holds. Then the equilibrium is* not *asymptotically stable.*

Proof. It follows from P2: the conditions in A1 imply that in any small neighborhood of the equilibrium, there is $\omega(t) = b_0 \neq 0$ for all $t \geq 0$ in the invariant set $\mathcal{M}$. This contradicts asymptotic stability. □

Proposition 3.4.5. *Assume $C_r > 0$ and dim$\{$Ker$(C_a)\} = 1$. Then the equilibrium is asymptotically stable if and only if A1 fails.*

Proof. It is obvious that the necessity follows from Lemma 3.4.3. Before proving the sufficiency, we first show for $b_0 \neq 0$, (b_0, b_1) are linearly independent if $b_1 \neq 0$. Suppose it is not. Thus b_1 must be in the form of $b_1 = \alpha b_0$ for a nonzero real α. Moreover, recall that b_1 satisfies

$$M_{11}(r_c) b_1 = M_{11}(r_c) b_0 \times b_0.$$

Therefore,

$$b_0^T M_{11}(r_c) b_1 = \alpha b_0^T M_{11}(r_c) b_0 = b_0^T \left(M_{11}(r_c) b_0 \times b_0\right) = 0.$$

Since $M_{11}(r)$ is always positive definite, we have $\alpha = 0$, which is a contradiction.

Now we show the sufficiency using the above result and Proposition 3.4.4. It is clear if A1.1 fails, then Proposition 3.4.4 holds at $k = 0$, which means the equilibrium is asymptotically stable. Suppose A1.1 holds. Thus $\mathrm{Ker}(C_a) = \mathrm{span}\{b_0\}$ by the dimensional assumption for $\mathrm{Ker}(C_a)$. This implies $b_1 \in \mathrm{Ker}(C_a) = \mathrm{span}\{b_0\}$ which holds only if $b_1 = 0$ or equivalently $M_{11}(r_c) b_0 \times b_0 = 0$. This means the failure of A1.2 implies asymptotic stability. Using the constraint (3.4.10), we also see the failure of A1.3 implies asymptotic stability even if A1.1 and A1.2 holds. This completes the proof. □

Next, we study the case where $\dim\{\mathrm{Ker}(C_a)\} = 2$. We refer to the following conditions as A2: in any small neighborhood $\mathcal{U}$ of $(b, r) = (0, r_e) \in \mathbb{R}^{3+n}$, there exists $(b_0, r_c) \in \mathcal{U}$ with $b_0 \neq 0$ satisfying:

A2.1 $b_0, b_1, b_2 \in \mathrm{Ker}(C_a)$, i.e., $C_a b_0 = C_a b_1 = C_a b_2 = 0$;

A2.2 $M_{21}(r_c)b_1 = \frac{1}{2}\left(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_0\right) - \frac{\partial V_s(r_c)}{\partial r}$, $M_{21}(r_c)b_2 = \left(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_1\right)$, and $M_{21}(r_c)b_3 = \left(2b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_2 + b_1^T \frac{\partial M_{11}(r_c)}{\partial r} b_1\right)$;

A2.3 $\alpha_0\alpha_1 M_{21}(r_c)b_0 + (2\alpha_0 + \alpha_1^2)M_{21}(r_c)b_1 + 2\alpha_0 \frac{\partial V_s(r_c)}{\partial r} = 0$ when $b_1 \neq 0$, where

$$\alpha_0 = -\frac{b_1^T M_{11}(r_c) b_1}{b_0^T M_{11}(r_c) b_0} \text{ and } \alpha_1 = -\frac{b_0^T b_1}{\alpha_0},$$

where b_1, b_2, b_3 are generated via (3.4.9) from the nonzero b_0.

Proposition 3.4.6. *Assume $C_r > 0$ and $\dim\{\mathrm{Ker}(C_a)\} = 2$. Then the equilibrium is asymptotically stable if and only if A2 fails.*

Proof. We first show the necessity. Suppose A2 holds. It is clear that if b_1 that satisfies A1 is equal to zero, then the equilibrium is not asymptotically stable by Lemma **??**. We now focus on the case where $b_1 \neq 0$. The basic idea for the proof in this case is as follows: we show that all $b_k, k \geq 1$ generated via (3.4.9) from the nonzero b_0 satisfy the constraint (3.4.10)–(3.4.12) under the given assumptions; and this implies that in any small neighborhood of the equilibrium, there exists a nonzero solution ω that lies in the invariant set $\mathcal{M}$.

As shown in the proof of Proposition 3.4.5, b_0 and b_1 are linearly independent if $b_1 \neq 0$ ($b_0 \neq 0$ follows from the assumption). Thus $\mathrm{span}(b_0, b_1) = \mathrm{Ker}(C_a)$ since both lie in the kernel of C_a. Note that $C_a b_2 = 0$ implies $b_2 \in \mathrm{span}(b_0, b_1)$. Therefore using Lemma 3.4.2, all $b_k \in \mathrm{span}(b_0, b_1) = \mathrm{Ker}(C_a)$ for $k = 0, 1, 2, \cdots$. This shows that all b_k satisfy (3.4.12).

We now show that all b_k satisfy (3.4.10)–(3.4.11). The conditions in A2 have shown this holds up to $k = 2$; we show it for the remaining k's. In the following, we express each b_k as $b_k = e_k^0 b_0 + e_k^1 b_1$ for real e_k^0 and e_k^1 as in the proof of Lemma 3.4.2. Thus using (3.4.13), we obtain the coefficients e_{k+1}^0 and e_{k+1}^1 for $b_{k+1}, k \geq 0$ as

$$e_{k+1}^0 = \alpha_0 \sum_{j=0}^{k} C_j^k \Big[(e_j^0 + \alpha_1 e_j^1)e_{k-j}^1\Big], \tag{3.4.14}$$

$$e_{k+1}^1 = \sum_{j=0}^{k} C_j^k \Big[(e_j^0 + \alpha_1 e_j^1)e_{k-j}^0\Big]. \tag{3.4.15}$$

To simplify the notation, we denote e_2^0 and e_2^1 by α_0 and α_1, respectively (i.e., $b_2 = \alpha_0 b_0 + \alpha_1 b_1$). It is easy to verify via the equations for b_2 and b_3 that

$$\alpha_0 = -\frac{b_1^T M_{11} b_1}{b_0^T M_{11} b_0} < 0,$$

$$-b_1^T b_0 = \alpha_0 \alpha_1,$$

and

$$b_3 = 3\alpha_0\alpha_1 b_0 + (2\alpha_0 + \alpha_1^2) b_1.$$

Using these results, we obtain the following identity from A2:

$$\begin{aligned}
&\left[M_{21}(r_c) b_0, \quad M_{21}(r_c) b_1, \quad \tfrac{\partial V_s(r_c)}{\partial r} \right] \begin{bmatrix} 0 & \alpha_0 & 3\alpha_0\alpha_1 \\ 2 & \alpha_1 & (2\alpha_0 + \alpha_1^2) \\ 2 & 0 & 0 \end{bmatrix} \\
&\quad = \left[\left(b_0^T \tfrac{\partial M_{11}(r_c)}{\partial r} b_0 \right), \quad \left(b_0^T \tfrac{\partial M_{11}(r_c)}{\partial r} b_1 \right), \quad \left(b_1^T \tfrac{\partial M_{11}(r_c)}{\partial r} b_1 \right) \right] \begin{bmatrix} 1 & 0 & 2\alpha_0 \\ 0 & 1 & 2\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Therefore

$$\left(b_1^T \frac{\partial M_{11}(r_c)}{\partial r} b_1 \right) = \alpha_0\alpha_1 M_{21}(r_c) b_0 - (2\alpha_0 + \alpha_1^2) M_{21}(r_c) b_1 - 4\alpha_0 \frac{\partial V_s(r_c)}{\partial r}.$$

Moreover, for all $k \geq 1$,

$$\begin{aligned}
\sum_{j=0}^{k} C_j^k b_j^T M_{11}(r_c) b_{k-j} &= \sum_{j=0}^{k} C_j^k \left(e_j^0 b_0 + e_j^1 b_1 \right)^T M_{11}(r_c) \left(e_{k-j}^0 b_0 + e_{k-j}^1 b_1 \right) \\
&= \sum_{j=0}^{k} C_j^k \left(e_j^0 e_{k-j}^0 b_0^T M_{11}(r_c) b_0 + e_j^1 e_{k-j}^1 b_1^T M_{11}(r_c) b_1 \right).
\end{aligned}$$

Using Lemma 3.4.1 and the identity

$$\alpha_0 = -\frac{b_1^T M_{11}(r_c) b_1}{b_0^T M_{11}(r_c) b_0},$$

it can be further shown

$$\sum_{j=0}^{k} C_j^k \left[e_j^0 e_{k-j}^0 - \alpha_0 e_j^1 e_{k-j}^1 \right] = 0, \quad k \geq 1. \tag{3.4.16}$$

With these results, we have for $k \geq 3$,

$$\sum_{j=0}^{k} C_j^k \Big[b_j^T \frac{\partial M_{11}(r_c)}{\partial r} b_{k-j} \Big] = \sum_{j=0}^{k} C_j^k \Big[\big(e_j^0 b_0 + e_j^1 b_1\big)^T \frac{\partial M_{11}(r_c)}{\partial r} \big(e_{k-j}^0 b_0 + e_{k-j}^1 b_1\big) \Big]$$

$$\begin{aligned}
= & \sum_{j=0}^{k} C_j^k \Big[e_j^0 e_{k-j}^0 \Big(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_0 \Big) + \big(e_j^0 e_{k-j}^1 + e_j^1 e_{k-j}^0\big) \Big(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_1 \Big) \\
& \quad + e_j^1 e_{k-j}^1 \Big(b_1^T \frac{\partial M_{11}(r_c)}{\partial r} b_1 \Big) \Big] \\
= & \; 2 \sum_{j=0}^{k} C_j^k \Big[e_j^0 e_{k-j}^0 M_{21}(r_c) b_1 + \frac{e_j^0 e_{k-j}^1 + e_j^1 e_{k-j}^0}{2} M_{21}(r_c) b_2 \\
& + e_j^1 e_{k-j}^1 \alpha_0 \alpha_1 M_{21}(r_c) b_0 \Big] + 2 \sum_{j=0}^{k} C_j^k \Big[e_j^0 e_{k-j}^0 - \alpha_0 e_j^1 e_{k-j}^1 \Big] \frac{\partial V_s(r_c)}{\partial r} \\
& - \sum_{j=0}^{k} C_j^k e_j^1 e_{k-j}^1 \Big[\alpha_0 \alpha_1 M_{21}(r_c) b_0 + (2\alpha_0 + \alpha_1^2) M_{21}(r_c) b_1 + 2\alpha_0 \frac{\partial V_s(r_c)}{\partial r} \Big] \\
= & \; 2 M_{21}(r_c) b_{k+1} - \sum_{j=0}^{k} C_j^k e_j^1 e_{k-j}^1 \Big[\alpha_0 \alpha_1 M_{21}(r_c) b_0 + (2\alpha_0 + \alpha_1^2) M_{21}(r_c) b_1 \\
& + 2\alpha_0 \frac{\partial V_s(r_c)}{\partial r} \Big] = 2 M_{21}(r_c) b_{k+1},
\end{aligned}$$

following Lemma 3.4.2, (3.4.14), (3.4.15), (3.4.16), and A2.3. Thus $b_k, k \geq 3$, satisfy (3.4.11). Finally, we conclude that in any small neighborhood of the equilibrium, there exists a nonzero solution ω that lies in the invariant set $\mathcal{M}$. This contradicts the assumption of asymptotic stability.

We now show the sufficiency. As in the proof of Proposition 3.4.5, the failure of A2.1 and A2.2 implies asymptotic stability, following from Proposition 3.4.4 at $k = 2$. Now suppose A2.1 and A2.2 hold but A2.3 fails, i.e., $\alpha_0 \alpha_1 M_{21}(r_c) b_0 + (2\alpha_0 + \alpha_1^2) M_{21}(r_c) b_1 + 2\alpha_0 \frac{\partial V_s(r_c)}{\partial r} \neq 0$ for some nonzero b_0 and b_1 at some r_c, where $\alpha_0 = -\frac{b_1^T M_{11}(r_c) b_1}{b_0^T M_{11}(r_c) b_0}$ and $\alpha_0 \alpha_1 = -b_0^T b_1$. This implies $\alpha_0 < 0$. Following the proof for the necessity and using $\sum_{j=0}^{3} C_j^k e_j^1 e_{k-j}^1 = 6\alpha_1$, $\sum_{j=0}^{4} C_j^k e_j^1 e_{k-j}^1 = 2(8\alpha_0 + 7\alpha_1^2)$, we see that the identities $M_{21}(r_c) b_4 = \frac{1}{2} \sum_{j=0}^{3} C_j^k \Big[b_j^T \frac{\partial M_{11}(r_c)}{\partial r} b_{k-j} \Big]$, $M_{21}(r_c) b_5 = \frac{1}{2} \sum_{j=0}^{4} C_j^k \Big[b_j^T \frac{\partial M_{11}(r_c)}{\partial r} b_{k-j} \Big]$ cannot both hold. Thus by Proposition 3.4.4, asymptotic stability follows. □

We now collect some conditions that are useful in the subsequent examples based on the above results. As shown in Lemma 3.4.3 and in Propositions 3.4.5 and 3.4.6, a key step for asymptotic stability is to check if there exists a nonzero $b_0 \in \text{Ker}(C_a)$ such that $M_{11}(r_c) b_0 \times b_0 = 0$ at any r_c in a small neighborhood of r_e. Recall that this condition has been referred to as A1.2. If A1.2 fails, asymptotic stability is immediately established; otherwise, we need to check other conditions. In the following, we give simplified steps to check this condition.

Proposition 3.4.7. *Assume $C_r > 0$. For each r_c in a neighborhood of r_e, let $U = (u_1^T, u_2^T, u_3^T)^T$, $u_i^T \in \mathbb{R}^3$ be a unitary matrix that diagonalizes $M_{11}(r_c)$. Then the following statements are true:*

1. *Assume all eigenvalues of $M_{11}(r_c)$ are equal. Then A1.2 never fails at r_c: it holds for any $b_0 \in \mathbb{R}^3$;*
2. *Assume exactly two eigenvalues of $M_{11}(r_c)$ are equal. Then*
   - 2.1 *In the case dim$\{$Ker$(C_a)\} = 1$, A1.2 fails at r_c if and only if Ker$(C_a) \cap \{$span$(u_3^T) \cup$span$(u_1^T, u_2^T)\} = \{0\}$, where u_3^T corresponds to the distinct eigenvalue;*
   - 2.2 *In the case dim$\{$Ker$(C_a)\} = 2$, A1.2 never fails at r_c, i.e., there always exists a nonzero $b_0 \in$ Ker(C_a) such that A1.2 holds at r_c;*
3. *Assume all eigenvalues of $M_{11}(r_c)$ are distinct. Then A1.2 fails at r_c if and only if $u_i^T \notin$ Ker(C_a), $i = 1, 2, 3$.*

Proof. Since $M_{11}(r)$ is symmetric and positive definite for all r, it can be diagonalized by a unitary matrix U at any fixed r, i.e., $M_{11} = U^T \Lambda U$ at r_c, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is diagonal. Moreover, we can always assume $\det(U) = +1$ (since if $\det(U) = -1$, we can choose the unitary matrix that simultaneously diagonalizes $M_{11}(r_c)$ and C_a as $-U$). This implies U can be regarded as a rotation matrix in SO(3). With these results, we express

$$M_{11}(r_c)b_0 \times b_0 = \left(U^T \Lambda U b_0\right) \times b_0 = U^T \left[\left(\Lambda U b_0\right) \times \left(U b_0\right)\right],$$

where we use $U(v \times w) = Uv \times Uw$ for any $v, w \in \mathbb{R}^3$ and $U \in$ SO(3). Let $h = (h_1, h_2, h_3)^T = U b_0 \in \mathbb{R}^3$, we further have

$$M_{11}(r_c)b_0 \times b_0 = U^T \begin{bmatrix} (\lambda_2 - \lambda_3) & & \\ & (\lambda_3 - \lambda_1) & \\ & & (\lambda_1 - \lambda_2) \end{bmatrix} \begin{bmatrix} h_2 h_3 \\ h_3 h_1 \\ h_1 h_2 \end{bmatrix}. \quad (3.4.17)$$

With this result, we see that Statement 1 holds. Now we show Statement 2. Without loss of generality, we assume $\lambda_1 = \lambda_2 \neq \lambda_3$. We see from (3.4.17) that A1.2 holds for a nonzero b_0 if and only if $h_3 = 0$ or $h_1 = h_2 = 0$ but not both. Since $b_0 = U^T h$, this condition is equivalent to $b_0 \in \text{span}(u_1^T, u_2^T)$ or $b_0 \in \text{span}(u_3^T)$. Hence Statement 2.1 holds. When dim$\{\text{Ker}(C_a)\} = 2$, we claim that there must be some nonzero $b_0 \in \text{Ker}(C_a)$ that lies in span(u_1^T, u_2^T). To see this, note that if the claim was not true, then the plane of Ker(C_a) must be parallel to the plane of span(u_1^T, u_2^T) but this contradicts the fact that the two planes intersect at the origin. Hence, A1.2 always holds in this case. This shows that Statement 2.2 is true. Finally, we prove Statement 3. In this case, using (3.4.17), we see that A1.2 holds for a nonzero b_0 if and only if two of (h_1, h_2, h_3) are zero, which is equivalent to $b_0 \in \text{span}(u_1^T) \cup \text{span}(u_2^T) \cup \text{span}(u_3^T)$. Thus Statement 3 holds. □

The following result, which only requires knowledge of $M_{11}(r)$ at r_e, follows from Statement 3, Proposition 3.4.5, and the continuity of $M(r)$.

Corollary 3.4.2. *Assume $C_r > 0$. If all eigenvalues of $M_{11}(r_e)$ are distinct, then there exists a positive semi-definite C_a with dim$\{Ker(C_a)\} = 1$ such that the equilibrium is asymptotically stable.*

Another result is a direct consequence of Statement 1.

Corollary 3.4.3. *Assume $C_r > 0$, and in any small neighborhood $\mathcal{U}$ of $(b, r) = (0, r_e)$, there exists $(b_0, r_c) \in \mathcal{U}$ with $b_0 \neq 0$ and $b_0 \in Ker(C_a)$ such that all the eigenvalues of $M_{11}(r_c)$ are equal and that $\frac{1}{2}\left(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_0\right) = \frac{\partial V_s(r_c)}{\partial r}$, then the equilibrium is* not *asymptotically stable.*

Corollary 3.4.4. *Assume $C_r > 0$ and in any small neighborhood $\mathcal{V}$ of r_e, there exists a $r_c \in \mathcal{V}$ such that $M_{11}(r_c)$ and C_a are simultaneously diagonalizable by a unitary matrix and that $\frac{1}{2}\left(b^T \frac{\partial M_{11}(r_c)}{\partial r} b\right) = \frac{\partial V_s(r_c)}{\partial r}$ holds for any $b \in Ker(C_a) \cap \mathcal{W}$, where $\mathcal{W}$ is any small neighborhood of $0 \in \mathbb{R}^3$. Then the equilibrium is asymptotically stable if and only if C_a is positive definite.*

Proof. The sufficiency is obvious; we show the necessity. Let U be the unitary matrix that simultaneously diagonalizes $M_{11}(r_c)$ and C_a, i.e., $M_{11}(r_c) = U^T \Lambda U$ and $C_a = U^T \Delta U$, where Λ and Δ are diagonal matrices and $U \in$ SO(3) as shown in the pervious proposition. Moreover, let $h = (h_1, h_2, h_3)^T = U b_0$, $M_{11}(r_c) b_0 \times b_0$ can be expressed in the form in (3.4.17). Suppose C_a is not positive definite. Then at least one of the diagonal elements of Δ is zero; without loss of generality, we assume the first element to be zero. Choose $b_0 = \epsilon U^T e_1$, where $\epsilon \neq 0$ and $e_1 = (1, 0, 0)^T$, thus $C_a b_0 = \epsilon U^T \Delta e_1 = 0$ and $h = (h_1, h_2, h_3)^T = U b_0 = \epsilon e_1$, thus $h_2 = h_3 = 0$. This implies $M_{11}(r_c) b_0 \times b_0 = 0$. The second condition means that $\frac{1}{2}\left(b_0^T \frac{\partial M_{11}(r_c)}{\partial r} b_0\right) = \frac{\partial V_s(r_c)}{\partial r}$ holds for the b_0 defined above in any small neighborhood of $(b, r) = (0, r_e)$ by choosing $|\epsilon|$ sufficiently small. Hence, in any small neighborhood of $(b, r) = (0, r_e)$, all the conditions in A1 are satisfied. Therefore by Lemma 3.4.3, we claim the equilibrium is not asymptotically stable. □

3.5 Examples

In this section, we apply the general results in the previous sections to two classes of multibody attitude systems: one class includes an elastic rotational degrees of freedom, the other class includes an elastic translational degrees of freedom.

3.5.1 A Multibody Attitude System with Elastic Rotational Degrees of Freedom

A. System description and equations of motion

The elastic subsystem consists of n elastic rotational components that are attached to the base body through rotational linear elastic springs. We further assume the center of mass of the system, which is constant and independent of shape, is always at the pivot point. Thus gravity has no influence on the dynamics.

All the n components are assumed to have an axial symmetric mass distribution with respect to their rotation axes which are fixed with respect to the base body. Moreover, they are assumed to have identical physical properties. Let m_r be the identical mass of the rotational components. Denote the constant moments of inertia of each rotational component by J_s along its spin axis and by J_r along its transverse axes. Define a body-fixed orthogonal coordinate frame for each component with origin at the center of the component so that its first axis is along the spin axis of the component. Let the rotation matrix from the i-th component frame to the base body frame be R_i. It can be shown that the inertia matrix J_i of the i-th rotational component, expressed in the base body coordinate frame, is given by $J_i = R_i \mathrm{diag}(J_s, J_r, J_r) R_i^T$. Choose a base body coordinate frame whose origin is located at the pivot point. Let ρ_i be the constant position vector of the center of mass of the i-th component, and J_B be the inertia matrix of the base body, both expressed in the base body coordinates. The shape coordinates are $r = (\phi_1, \cdots, \phi_n)$, the rotation angles of the components.

Let $V_s(r) = \frac{1}{2} r^T K_s r$ be the elastic potential energy, where K_s is symmetric and positive definite, and without loss generality, we assume that zero shape corresponds to zero elastic potential energy. The reduced Lagrangian is

$$L(\omega, r, \dot{r}) = \frac{1}{2} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \begin{pmatrix} \omega \\ \dot{r} \end{pmatrix} - \frac{1}{2} r^T K_s r,$$

where

$$M_{11} = J_B + \sum_{i=1}^{n} \big(m_r \widehat{\rho}_i^T \widehat{\rho}_i + J_i \big), \;\; M_{12} = J_s \big[R_1 e_1, \; \cdots, \; R_n e_1 \big], \;\; M_{22} = J_s I_n.$$

Note that all the inertia matrix components are constant and are independent of the shape. Moreover, $R_i e_1$ in M_{12} denotes the direction of the spin axis of the i-th component expressed in the base body frame. We call the rotational components "independent" if $R_i e_1, \; i = 1, \cdots, n$, are linearly independent. The equations of motion are given by

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \ddot{r} \end{bmatrix} = \begin{bmatrix} \Pi \times \omega \\ -K_s r \end{bmatrix} - \begin{bmatrix} C_a \omega \\ C_r \dot{r} \end{bmatrix}. \tag{3.5.1}$$

Clearly, $(\omega, r, \dot{r}) = (0, 0, 0)$ is an isolated stable equilibrium of (3.5.1). Furthermore, because the equilibrium is globally stable, we have:

Proposition 3.5.1. *The equilibrium $(\omega, r, \dot{r}) = (0, 0, 0)$ is globally asymptotically stable if and only if it is locally asymptotically stable.*

B. Asymptotic stability analysis: $C_a > 0,\ C_r \geq 0$.

It is easy to see that this case satisfies all the conditions in Remark 3.4.2. Thus we obtain:

Corollary 3.5.1. *Assume $C_a > 0$. The equilibrium is asymptotically stable if and only if the following pair is observable*

$$\left(\begin{bmatrix} -M_{12}M_{22}^{-1}K_s, & C_r \end{bmatrix}, \begin{bmatrix} 0 & I_n \\ -M_{22}^{-1}K_s & 0 \end{bmatrix} \right).$$

Corollary 3.5.2. *Assume $C_a > 0$. The following statements hold:*

1. *Suppose the number of rotational degrees of freedom is no more than three, i.e., $1 \leq n \leq 3$, and they are independent. Then the equilibrium is asymptotically stable regardless of the shape damping C_r.*

2. *Suppose there are more than three rotational degrees of freedom, of which three are independent. Then the equilibrium is asymptotically stable if the remaining $(n-3)$ degrees of freedom are fully damped.*

C. Asymptotic Stability Analysis: $C_r > 0,\ C_a \geq 0$

In this case, the shape is fully damped. The equations that characterize a solution $\omega(t)$ in the invariant set $\mathcal{M}$ are given by

$$M_{11}\dot{\omega} = M_{11}\omega \times \omega, \quad M_{21}\dot{\omega} = -K_s r_c, \quad C_a\omega = 0,$$

where r_c is some constant. The results in Section 4.2 can be greatly simplified and refined using the fact that M is constant. The following proposition describes asymptotic stability conditions for this case. Particularly, we obtain concrete necessary and sufficient conditions for the case $\dim\{\mathrm{Ker}(C_a)\} = 1$.

Proposition 3.5.2. *Assume $C_r > 0$. Let $U = (u_1^T, u_2^T, u_3^T)^T$ be a unitary matrix that diagonalizes M_{11}, where $u_i^T \in \mathbb{R}^3$. The following statements hold:*

1. *Assume all eigenvalues of M_{11} are equal. Then the equilibrium is asymptotically stable if and only if C_a is positive definite.*

2. *Assume exactly two eigenvalues of M_{11} are equal.*

2.1. *Suppose dim*$\{Ker(C_a)\} = 1$. *Then the equilibrium is asymptotically stable if and only if* $Ker(C_a) \cap \{span(u_3^T) \cup span(u_1^T, u_2^T)\} = \{0\}$, *where* u_3^T *corresponds to the distinct eigenvalue.*

2.2. *Suppose dim*$\{Ker(C_a)\} = 2$. *Then the equilibrium is* not *asymptotically stable.*

3. *Assume all eigenvalues of* M_{11} *are distinct.*

3.1. *Suppose dim*$\{Ker(C_a)\} = 1$. *Then the equilibrium is asymptotically stable if and only if* $u_i^T \notin Ker(C_a)$ *for* $i = 1, 2, 3$.

3.2. *Suppose dim*$\{Ker(C_a)\} = 2$. *Then the equilibrium is asymptotically stable only if* $u_i^T \notin Ker(C_a)$ *for* $i = 1, 2, 3$.

Proof. We first simplify conditions A1 and A2 used in Lemma 3.4.3 and in Propositions 3.4.5 and 3.4.6. Since M_{11} is constant, A1.3 is satisfied if and only if $r_c = 0$ and for all $b_0 \in \mathbb{R}^3$. Thus we remove this condition from consideration. Therefore, satisfaction of A1 is equivalent to existence of a nonzero $b_0 \in \mathrm{Ker}(C_a)$ satisfying $M_{11}b_0 \times b_0 = 0$, or equivalently the condition A1.2. Similarly, using the fact that M_{11} is constant, we see that satisfaction of A2 is equivalent to existence of a nonzero b_0 and r_c satisfying $(b_0, b_1, b_2) \in \mathrm{Ker}(C_a)$ and $M_{21}b_1 = -K_s r_c$, $M_{21}b_2 = 0$, $M_{21}b_3 = 0$, $\alpha_0\alpha_1 M_{21}b_0 + (2\alpha_0 + \alpha_1^2)M_{21}b_1 + 2\alpha_0 K_s r_c = 0$ when $b_1 \neq 0$, where α_0, α_1 are given in A2.3.

We use these simplified conditions and Proposition 3.4.7 to prove the statements. If all the eigenvalues of M_{11} are equal, then A1.2 holds for any $b_0 \neq 0$. This means A1 is satisfied for any $b_0 \neq 0$. By Lemma 3.4.3, the equilibrium is asymptotically stable if and only if C_a is positive definite. The Statement 2.1 follows from Statement 2.1 in Proposition 3.4.7, the simplified A1 condition and constant M_{11}. We now prove Statement 2.2. By Statement 2.2 in Proposition 3.4.7, we see that there always exists a nonzero $b_0 \in Ker(C_a)$ such that $b_1 = 0$ which implies $b_2 = b_3 = 0$ and $r_c = 0$. Thus A2 holds for such b_0. By Proposition 3.4.6, the equilibrium is *not* asymptotically stable. Statement 3.1 is due to Statement 3 in Proposition 3.4.7 and the fact that A1 is equivalent to A1.2; Statement 3.2 is a direct consequence of Statement 3 of Proposition 3.4.7: if one of u_i^T lies in $\mathrm{Ker}(C_a)$, then we can always find a nonzero $b_0 \in \mathrm{Ker}(C_a)$ such that $b_1 = 0$, which leads to $b_2 = b_3 = 0$ and $r_c = 0$. Hence A2 is satisfied and the equilibrium is not asymptotically stable. □

It can be seen from Proposition 3.4.7 that it is more difficult to achieve asymptotic stability when $\dim\{\mathrm{Ker}(C_a)\} = 2$. This agrees with physical intuition. Note that Statement 3.2 only gives a necessary condition for asymptotic stability; we now present a sufficient condition that makes use of coupling effects between the attitude dynamics and the shape dynamics.

Corollary 3.5.3. *Assume* $C_r > 0$ *and assume that* C_a *satisfies*

$$dim\{Ker(C_a)\} = 2 \text{ and } u_i^T \notin Ker(C_a) \text{ for } i = 1, 2, 3.$$

Moreover, suppose rank$\{M_{21}\} = 3$. *Then the equilibrium is asymptotically stable.*

Proof. We show in the following that $M_{21}b_2 \neq 0$ for any $b_0 \neq 0$ under the given conditions. This thus implies the failure of A2 and leads to asymptotic stability. Note that if $u_i^T \notin \mathrm{Ker}(C_a)$ for $i = 1, 2, 3$ holds, then for any nonzero $b_0 \in \mathrm{Ker}(C_a)$, b_1 cannot be zero. Suppose both b_1 and b_2 lie in $\mathrm{Ker}(C_a)$. (If they are not, we have asymptotic stability.) As shown in Proposition 3.4.6, $b_2 = \alpha_0 b_0 + \alpha_1 b_1$, where $\alpha_0 < 0$. Recall that (b_0, b_1) are linearly independent as shown in Proposition 3.4.5. Therefore, $b_2 \neq 0$. Consequently, if $\mathrm{rank}(M_{21}) = 3$, then $M_{21}b_2 \neq 0$ as desired. □

Corollary 3.5.4. *Assume* $C_r > 0$ *and suppose* M_{11} *and* C_a *are simultaneously diagonalizable by a unitary matrix. Then the equilibrium is asymptotically stable if and only if* C_a *is positive definite.* □

This result is a consequence of Proposition 3.4.6 and condition A1 and has some interesting implications. It shows that if both M_{11} and C_a are diagonal or all the eigenvalues of M_{11} are equal, both special cases of Corollary 3.5.4, then C_a must be positive definite for asymptotic stability.

We present an example to illustrate these observations.

Example. Consider a multibody attitude system with the following inertia and damping assumptions:

$$M_{11} = \begin{bmatrix} J_{11} & J_{12} & 0 \\ J_{12} & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} > 0, \quad J_{12} \neq 0, \quad C_a = \mathrm{diag}(c_1,\ c_2,\ c_3) \geq 0.$$

We consider two cases:

1. Assume $\dim\{\mathrm{Ker}(C_a)\} = 1$. If $c_1 = 0, c_2 > 0, c_3 > 0$ or if $c_1 > 0, c_2 = 0, c_3 > 0$, then the equilibrium is asymptotically stable. If $c_1 > 0, c_2 > 0, c_3 = 0$, then the equilibrium is *not* asymptotically stable. These conclusions follow from Proposition 3.5.2. In fact, consider the first case where $C_a = (0, c_2, c_3)$. Choose $b_0 = e_1 \in \mathrm{Ker}(C_a)$; we obtain $M_{11}b_0 \times b_0 = -J_{12}e_3 \neq 0$. This shows asymptotic stability.

2. Assume $\dim\{\mathrm{Ker}(C_a)\} = 2$. If $c_1 = c_2 = 0, c_3 > 0$, then the equilibrium is not asymptotically stable. To see this, let $V \in \mathbb{R}^{2\times 2}$ be a unitary matrix that diagonalizes the upper 2×2 block of M_{11}. Thus the following unitary matrix

   $$U = \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}$$

   simultaneously diagonalizes M_{11} and C_a. Therefore, the equilibrium is not asymptotically stable. If $c_1 > 0, c_2 = c_3 = 0$ or $c_2 > 0, c_1 = c_3 = 0$, then the equilibrium is not asymptotically stable because we can choose $b_0 = e_3 \in \mathrm{Ker}(C_a)$ such that $M_{11}b_0 \times b_0 = 0$.

3.5.2 A Multibody Attitude System with Elastic Translational Degrees of Freedom

A. System description and equations of motion

The elastic subsystems consist of n idealized mass particles that are attached to the base body through linear elastic springs. We assume that the particles are constrainted to translate in a symmetric mode such that their motions do not change the center of mass of the system, which is assumed to be at the pivot point. Thus gravity does not influence the dynamics.

We choose a base body coordinate frame with its origin at the pivot point. Let J_B denote the inertia matrix of the base body expressed in this frame. Each mass particle is assumed to translate along a particular direction fixed with respect to the base body that is denoted by the unit vector ν_i. The shape coordinate r_i denotes the position along this direction with respect to the base body frame. Thus the shape coordinates are $r = (r_1, \cdots, r_n)$. Let ρ_{i0} denote the constant offset position vector from the origin to zero shape (i.e., $r_i = 0$) of i-th mass particle, assuming no elastic deformation at the zero shape; this is assumed to correspond to zero elastic potential energy. The position vectors of the mass particles expressed in the body coordinate frame are $\rho_i(r) = \rho_{i0} + r_i\nu_i,\ i = 1, \cdots, n$ and the mass of the i-th mass particle is denoted by m_i.

Let the elastic potential energy be $V_s(r) = \frac{1}{2}r^T K_s r$, where K_s defines the stiffness matrix of the elastic subsystem and K_s is assumed to be symmetric and positive definite. The reduced Lagrangian is

$$L(\omega, r, \dot{r}) = \frac{1}{2}\begin{pmatrix}\omega\\ \dot{r}\end{pmatrix}^T \begin{bmatrix}M_{11}(r) & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{pmatrix}\omega\\ \dot{r}\end{pmatrix} - \frac{1}{2}r^T K_s r.$$

where $M_{11}(r) = J_B + \sum_{i=1}^n m_i\widehat{\rho}_i^T(r)\widehat{\rho}_i(r)$, $M_{12} = \big[m_1(\rho_{10}\times\nu_1),\ \cdots,\ m_n(\rho_{n0}\times \nu_n)\big]$, and $M_{22} = \mathrm{diag}(m_1, \cdots, m_n)$. Note that the matrices M_{12} and M_{22} are constant and do not depend on the shape. The equations of motion for the damped multibody attitude system are given by

$$\begin{bmatrix}M_{11}(r) & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix}\dot{\omega}\\ \ddot{r}\end{bmatrix} = \begin{bmatrix}-\dot{M}_{11}(r)\omega + \Pi\times\omega\\ \frac{\partial T(\omega,r,\dot{r})}{\partial r} - K_s r\end{bmatrix} - \begin{bmatrix}C_a\omega\\ C_r\dot{r}\end{bmatrix}, \qquad (3.5.2)$$

where $\Pi = M_{11}(r)\omega + M_{12}\dot{r}$ is the conjugate angular momentum. Clearly $(\omega, r, \dot{r}) = (0, 0, 0)$ is an isolated stable equilibrium of (3.5.2).

B. Asymptotic stability analysis: $C_a > 0,\ C_r \geq 0$

It is easy to see that all the conditions in Remark 3.4.2 are satisfied in this case since M_{12} and M_{22} are constant and $V_s(r) = \frac{1}{2}r^T K_s r$. Hence, Corollaries 3.5.1 and 3.5.2 are applicable to this case.

An interesting special case is that all the offset position vectors $\rho_{i0} = 0$. In this case, M_{12} is identically zero. By Corollary 3.5.5, we conclude that the equilibrium is asymptotically stable if and only if the pair $(C_r, M_{22}^{-1}K_s)$ is observable. Note that this condition is much more restrictive than the general condition for $M_{12} \neq 0$. Since M_{12} characterizes the coupling between the attitude dynamics and the shape dynamics, this observation demonstrates that this coupling can provide an important mechanism for asymptotic stability.

C. Asymptotic stability analysis: $C_r > 0,\ C_a \geq 0$

In this case, the shape dynamics are fully damped. The equations that characterize a solution in the invariant set are given by

$$M_{11}(r_c)\dot{\omega} = M_{11}(r_c)\omega \times \omega, \quad M_{12}^T\dot{\omega} = -K_s r_c, \quad C_a\omega = 0,$$

for some constant r_c.

We now obtain asymptotic stability conditions using the structure of $M_{11}(r)$. Note that $M_{11}(r)$ can be written as

$$M_{11}(r) = J_B + \sum_{i=1}^{n} m_i \widehat{\rho}_{i0}^T \widehat{\rho}_{i0} + \sum_{i=1}^{n} m_i \Big(\widehat{\rho}_{i0}^T \widehat{\nu}_i + \widehat{\nu}_i^T \widehat{\rho}_{i0} \Big) r_i + \sum_{i=1}^{n} m_i \widehat{\nu}_i^T \widehat{\nu}_i r_i^2,$$

and we can write

$$\begin{aligned}&\left(b^T \frac{\partial M_{11}(r)}{\partial r} b\right) \\ &= \ 2 \underbrace{\begin{bmatrix} m_1 \left(\rho_{10} \times b\right)^T \left(\nu_1 \times b\right) \\ \vdots \\ m_n \left(\rho_{n0} \times b\right)^T \left(\nu_n \times b\right) \end{bmatrix}}_{E(b)} + 2 \underbrace{\mathrm{diag}\Big(m_1 ||\nu_1 \times b||_2^2,\ \cdots,\ m_n ||\nu_n \times b||_2^2 \Big)}_{G(b)} r.\end{aligned}$$

Thus condition A1.3 is equivalent to $E(b) = -[K_s - G(b)]r$. Since K_s is positive definite and is invertible, for sufficiently small $||b||$, $G(b)$ is sufficiently small so that $K_s - G(b)$ is invertible. Thus we see that by choosing $r_c = -[K_s - G(b_0)]^{-1}E(b_0)$, A1.3 is satisfied for any sufficiently small b_0. Using this observation, we obtain the following result similar to Corollary 3.4.2.

Corollary 3.5.5. *Assume $C_r > 0$ and suppose all eigenvalues of $M_{11}(0) = J_B + \sum_{i=1}^n m_i \widehat{\rho}_{i0}^T \widehat{\rho}_{i0}$ are distinct. Let $U = (u_1^T, u_2^T, u_3^T)^T$ be a unitary matrix that diagonalizes $M_{11}(0)$, where $u_i^T \in \mathbb{R}^3$. Then the equilibrium is asymptotically stable only if $u_i^T \notin$ Ker(C_a), $i = 1, 2, 3$. Moreover, if dim$\{$Ker$(C_a)\} = 1$, then this condition is also sufficient and such C_a always exists.*

We consider a special case using Propositions 3.4.7 and 3.5.2.

Corollary 3.5.6. *Assume $C_r > 0$. Suppose the offset position vectors $\rho_{i0} = 0,\ i = 1, \cdots, n$. Then the following statements hold:*

1. *Assume all eigenvalues of J_B are equal. Then the equilibrium is asymptotically stable if and only if C_a is positive definite.*

2. *Assume exactly two eigenvalues of J_B are equal. Then there exist a C_a with dim$\{Ker(C_a)\} = 1$ such that the equilibrium is asymptotically stable; the equilibrium is* not *asymptotically stable if dim$\{Ker(C_a)\} = 2$.*

3. *Assume all eigenvalues of J_B are distinct. Then there exists a C_a satisfying dim$\{Ker(C_a)\} = 1$ such that the equilibrium is asymptotically stable.*

Proof. Note that $\rho_{i0} = 0,\ i = 1, \cdots, n$, implies $M_{21} = 0$, $M_{11}(0) = J_B$ and $\left(b^T \frac{\partial M_{11}(r)}{\partial r} b\right) = 2G(b)r$. Thus A1.3 and the first condition in A2.2 are satisfied if and only if $r_c = 0$ for any sufficiently small b_0. Hence, the satisfaction of A1 is equivalent to satisfaction of A1.2. The statements thus follow from Proposition 3.4.7. □

3.6 Conclusions

In this paper we have studied asymptotic stability of a class of multibody attitude systems using passive damping. Emphases have been given to the nonlinear dynamics in a noncanonical Hamiltonian form and to the coupling effects between the attitude dynamics and the shape dynamics that can enable asymptotic stability. Under the stated assumptions, the equilibrium is always stable in the sense of Lyapunov. A number of results that guarantee that the equilibrium is asymptotically stable have been obtained. These results have been presented in both a general form, and specific results have been presented for typical multibody attitude examples. Linear and nonlinear observability rank conditions lead to asymptotic stability results when the attitude dynamics are fully damped; but they fail for the case where the attitude dynamics are partially damped. Our results also suggest the importance of coupling effects in achieving asymptotic stability.

A key assumption made in this paper is that there are no gravity effects in the multibody attitude system. If the center of mass of the system does not remain at the pivot, then gravity effects must be considered. This fundamentally changes the equilibrium structure of the multibody attitude system and its stability properties.

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