# Generating Functions of Switched Linear Systems: Analysis, Computation, and Stability Applications

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***Abstract*—In this paper, a unified framework is proposed to study the exponential stability of discrete-time switched linear systems, and more generally, the exponential growth rates of their trajectories, under three types of switching rules: arbitrary switching, optimal switching, and random switching. It is shown that the maximum exponential growth rates of system trajectories over all initial states under these three switching rules are completely characterized by the radii of convergence of three suitably defined families of functions called the strong, the weak, and the mean generating functions, respectively. In particular, necessary and sufficient conditions for the exponential stability of the switched linear systems are derived based on these radii of convergence. Various properties of the generating functions are established and their relations are discussed. Algorithms for computing the generating functions and their radii of convergence are also developed and illustrated through examples.**

***Index Terms*—Switched systems, stability, optimal control.**

## I. Introduction

SWITCHED linear systems (SLSs) as a natural extension of linear systems are finding increasing applications in a diverse range of engineering systems [1]. A fundamental problem in the study of SLSs is to determine their stability. See [2], [3] for some recent reviews of the vast amount of existing results on this subject. These results can be roughly classified into two main categories: absolute (or uniform) stability where the switchings can be arbitrary; and stability under restricted switching rules such as switching rate constraints [4] and state-dependent switchings [5], [6]. A predominant approach to the study of stability in both cases is through the construction of common or multiple Lyapunov functions [5], [7], [8]. Other approaches include Lie algebraic conditions [9]–[11] and the LMI methods [12]–[14], etc.

The purpose of this paper is to characterize not only the stability of SLSs, but also the maximum exponential rates at which their trajectories can grow starting from all possible initial states under three switching rules: arbitrary switching, optimal switching, and random switching. Such rates give quantitative measures on the degree of the SLSs' exponential stability/stabilizability. Most existing results in this direction focus on the arbitrary switching case. The maximum exponential growth rate of system trajectories in this case is called the *joint spectral radius (JSR)* of subsystem matrices, and has been studied extensively (see, e.g., [15]–[19]). In comparison, the maximum exponential growth rate under optimal switching remains much less studied. The smallest such rate under all open-loop switching policies is given by the *joint spectral subradius (JSS)* of subsystem matrices [16], [17], [20], [21]; whereas in this paper, we study the smallest such rate under the more general closed-loop, state-dependent switching policies (see Section IV-B for an example showing their difference). Finally, under random switching, the SLSs become instances of random dynamical systems, for which the concept of *Lyapunov exponents* [22], [23] can be used to characterize the expected exponential growth rate of their trajectories.

It is well known that finding the maximum exponential growth rates of SLSs are difficult problems. For example, approximating the JSR within arbitrary precision has been proved to be an NP-hard problem [24]; and determining whether the JSR is less than or equal to one is algorithmically undecidable [20]. Computing the JSS is even more difficult [20], [24]. Despite these negative results, many approximation algorithms have been proposed in the literature, e.g. [25]–[29] for the JSR and [17], [21], [29] for the JSS, some with prescribed accuracy.

In this paper, we propose a unified method to characterize the maximum exponential growth rates of the SLSs' trajectories under the above three switching rules. The method is based on the novel concept of *generating functions* of SLSs, which are suitably defined power series with coefficients determined from the trajectories of the SLSs. The importance of these generating functions is twofold: (i) their radii of convergence characterize precisely the maximum exponential growth rates of the system trajectories; and (ii) they possess many amenable properties that make their efficient computation possible. Thus, generating functions provide both a theoretical framework and the computational tools for characterizing the maximum exponential growth rates of interest. In particular, they provide valid tests for the exponential stability/stabilizability of SLSs.

Compared with the existing methods, the proposed approach studies the stability of SLSs from the perspective of their optimal control: the generating functions are the value functions of certain optimal control problems for the SLSs with a varying discount factor, and automatically become Lyapunov functions for stable SLSs. This perspective enables us to uncover some common properties of the exponential growth rates of the SLSs under the different switching rules (see, e.g., Propositions 3, 9 and 14). Moreover, it makes our approach easily extendable to more general classes of systems, such

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as conewise linear inclusions [6], switched positive systems [30], [31], and controlled SLSs. A similar perspective has been adopted in [15], and by the variational approach [32], [33], which studies the stability of SLSs under arbitrary switching by finding their most divergent trajectories.

This paper is organized as follows. In Section II, the relevant stability notions of SLSs are introduced. In Section III (resp. Section IV), the strong (resp. weak) generating functions are defined, analyzed, and used to characterize the exponential stability of the SLSs under arbitrary (resp. optimal) switching. Their numerical computation algorithms are also presented. Section V discusses extensions to randomly switching linear systems. Finally, concluding remarks are given in Section VI.

## II. Stability of Switched Linear Systems

A discrete-time (autonomous) SLS is defined as follows: its state $x(t) \in \mathbb{R}^n$ evolves by switching among a set of linear dynamics indexed by the finite index set $\mathcal{M} := \{1, \dots, m\}$:

$$x(t+1) = A_{\sigma(t)} x(t), \quad t = 0, 1, \dots. \tag{1}$$

Here, $\sigma(t) \in \mathcal{M}$ for all $t$, or simply $\sigma$, is called the switching sequence; and $A_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{M}$, are the subsystem (dynamics) matrices. Starting from the initial state $x(0) = z$ and under the switching sequence $\sigma$, the trajectory of the SLS is denoted by $x(t; z, \sigma)$.

In this paper, unless otherwise stated, $\|\cdot\|$ denotes both the Euclidean norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$.

*Definition 1:* The SLS (1) is called

- *exponentially stable under arbitrary switching* (with the parameters $\kappa$ and $r$) if there exist $\kappa \geq 0$ and $r \in [0, 1)$ such that starting from any initial state $z$ and under any switching sequence $\sigma$, the trajectory $x(t; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, for all $t = 0, 1, \dots$.
- *exponentially stable under optimal switching* (with the parameters $\kappa$ and $r$) if there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that starting from any initial state $z$, there exists a switching sequence $\sigma$ for which the trajectory $x(t; z, \sigma)$ satisfies $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, for all $t = 0, 1, \dots$.

As for linear systems, we can also define stability (in the sense of Lyapunov) and asymptotic stability of SLSs under arbitrary (resp. optimal) switching. By homogeneity, local and global stability notions are equivalent for SLSs. Moreover, the asymptotic stability and the exponential stability of SLSs under arbitrary switching are equivalent [6], [34], [35]. We show next that this is also the case under optimal switching.

*Theorem 1:* Under optimal switching, the asymptotic stability and the exponential stability of the SLS (1) are equivalent.

*Proof:* It suffices to show that asymptotic stability implies exponential stability. Assume that the SLS (1) is asymptotically stable under optimal switching. Then, for any initial state $z$ on the unit sphere $\mathbb{S}^{n-1} := \{z \in \mathbb{R}^n \,|\, \|z\| = 1\}$, there is a switching sequence $\sigma_z$ such that $x(t; z, \sigma_z) \to 0$ as $t \to \infty$; hence $\|x(T_z; z, \sigma_z)\| \leq \frac{1}{4}$ for a time $T_z$ large enough. As $x(T_z; z, \sigma_z)$ is continuous in $z$ for fixed $\sigma_z$ and $T_z$, $\|x(T_z; y, \sigma_z)\| \leq \frac{1}{2}$ for $y$ in a neighborhood $U_z$ of $z$ in $\mathbb{S}^{n-1}$. The union of all such $U_z$ is an open cover of the compact set $\mathbb{S}^{n-1}$; hence $\mathbb{S}^{n-1} \subseteq \cup_{i=1}^{\ell} U_{z_i}$ for some $z_1, \dots, z_\ell \in \mathbb{S}^{n-1}$ with $\ell < \infty$. Starting from any initial $x(0) = z \in \mathbb{S}^{n-1}$, we have $z \in U_{z_i}$ for some $1 \leq i \leq \ell$. By our construction, $x(\tau_1) := x(T_{z_i}; z, \sigma_{z_i})$ with $\tau_1 := T_{z_i}$ satisfies $\|x(\tau_1)\| \leq \frac{1}{2}$. Assume without loss of generality that $x(\tau_1) \neq 0$. Then $x(\tau_1)/\|x(\tau_1)\| \in U_{z_j}$ for some $1 \leq j \leq \ell$, and as a result, $x(\tau_2) := x(T_{z_j}; x(\tau_1), \sigma_{z_j})$ with $\tau_2 := \tau_1 + T_{z_j}$ satisfies $\|x(\tau_2)\| \leq \frac{1}{2}\|x(\tau_1)\|$. By induction, we obtain a switching sequence $\sigma_z$ by concatenating $\sigma_{z_i}, \sigma_{z_j}, \dots$ and a sequence of times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ at most $\tau_* := \max_i T_{z_i}$ apart such that the resulting trajectory $x(t; z, \sigma_z)$ satisfies $\|x(\tau_{k+1}; z, \sigma_z)\| \leq \frac{1}{2}\|x(\tau_k; z, \sigma_z)\|$ for all $k \geq 0$. Let $\kappa := \sum_{j=0}^{\tau_*} \left(\max_{i\in\mathcal{M}} \|A_i\|\right)^j$. Then $\|x(t; z, \sigma_z)\| \leq \kappa (0.5)^{t/\tau_* - 1}\|z\|$ for all $t$. Thus, the SLS (1) is exponentially stable under optimal switching. ■

*Remark 1:* The above proof implies that to prove the exponential stability of the SLS (1) under optimal switching, it suffices to show that for any $z \in \mathbb{R}^n$, $x(t; z, \sigma_z) \to 0$ as $t \to \infty$ for at least one $\sigma_z$. This fact will be used in Section IV.

Another switching rule we consider is random switching. Let $p := \{p_i\}_{i\in\mathcal{M}}$ be a probability distribution with $p_i \geq 0$ and $\sum_{i\in\mathcal{M}} p_i = 1$. The SLS (1) under the (stationary) random switching probability $p$ has the dynamics

$$\mathbf{x}(t+1) = \mathbf{A}(t)\mathbf{x}(t), \quad t = 0, 1, \dots. \tag{2}$$

Here, at each time $t$, $\mathbf{A}(t)$ is drawn independently randomly from $\{A_i\}_{i\in\mathcal{M}}$ with the probability $\mathbf{P}\{\mathbf{A}(t) = A_i\} = p_i$. Denote by $\mathbf{x}(t; z, p)$ the stochastic trajectory of the system (2) from a deterministic initial state $x(0) = z$, and denote by $\mathbf{P}$ and $\mathbf{E}$ the probability and expectation operators, respectively.

*Definition 2:* The SLS (2) under the random switching probability $p$ is called

- *mean square exponentially stable* (with the parameters $\kappa$ and $r$) if there exist $\kappa \geq 0$ and $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, $\mathbf{E}[\|\mathbf{x}(t; z, p)\|^2] \leq \kappa r^t \|z\|^2$, for all $t = 0, 1, \dots$.

Similarly, the SLS (2) is called *mean square asymptotically stable* if $\mathbf{E}[\|\mathbf{x}(t; z, p)\|^2] \to 0$ as $t \to \infty$ for all $z \in \mathbb{R}^n$, and *almost sure asymptotically stable* if $\mathbf{P}\{\lim_{t\to\infty} \|x(t; z, p)\| = 0\} = 1$ for all $z \in \mathbb{R}^n$. From results on *random jumped linear systems* [36, Theorem 4.1.1], mean square asymptotic stability and mean square exponential stability of the SLS (2) are equivalent; and each of them implies almost sure asymptotic stability. We shall focus on mean square exponential stability.

## III. Strong Generating Functions

Central to the stability analysis of SLSs is the task of determining the exponential rate at which $\|x(t; z, \sigma)\|$ grows as $t \to \infty$ for trajectories $x(t; z, \sigma)$ of the SLSs. The following lemma, adopted from [37, Corollary 1.1.10], hints at an indirect way of characterizing this growth rate.

*Lemma 1:* Given a scalar sequence $\{w_t\}_{t=0,1,\dots}$, suppose the power series $\sum_{t=0}^{\infty} w_t \lambda^t$ has the radius of convergence $R$. Then for any $r > \frac{1}{R}$, there exists a constant $C_r$ such that $|w_t| \leq C_r r^t$ for all $t = 0, 1, \dots$.

As a result, for any trajectory $x(t; z, \sigma)$, an (asymptotically) tight bound on the exponential growth rate of $\|x(t; z, \sigma)\|^2$ as $t \to \infty$ is given by the reciprocal of the radius of convergence of the power series $\sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^2$.

Motivated by this, we define the *strong generating function* $G(\cdot,\cdot):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+\cup\{\infty\}$ of the SLS (1) as

$$G(\lambda,z):=\sup_\sigma\sum_{t=0}^\infty \lambda^t\|x(t;z,\sigma)\|^2,\quad \forall\lambda\geq 0,\ z\in\mathbb{R}^n,\quad (3)$$

where the supremum is taken over all switching sequences $\sigma$. For a fixed $z$, $G(\cdot,z)$ is nondecreasing in $\lambda\geq 0$. Indeed, due to the supremum in (3), $G(\cdot,z)$ can be viewed intuitively as the power series in $\lambda$ corresponding to the "most divergent" trajectories of the SLS starting from $z$. Thus, by Lemma 1, its radius of convergence defined by

$$\lambda^*(z):=\sup\{\lambda\geq 0\,|\,G(\lambda,z)<\infty\}\quad (4)$$

is expected to characterize the fastest exponential growth rate of the SLS trajectories starting from $z$. We call $\lambda^*(z)$ the *radius of strong convergence* of the SLS at $z$,

For each fixed $\lambda\geq 0$, $G(\lambda,z)$ is a function of $z$ only:

$$G_\lambda(z):=G(\lambda,z),\quad \forall z\in\mathbb{R}^n.\quad (5)$$

By definition (3), $G_\lambda(\cdot)$ is nonnegative and homogeneous of degree two, with $G_0(z)=\|z\|^2$. Since $G_\lambda(\cdot)$ is nondecreasing in $\lambda$, we have $G_\lambda(z)\geq\|z\|^2$, $\forall z$, for $\lambda\geq 0$. We will also refer to $G_\lambda(z)$ as the strong generating function of the SLS (1).

### *A. Properties of General $G_\lambda(z)$*

We first prove some useful properties of the function $G_\lambda(z)$.

*Proposition 1:* $G_\lambda(z)$ has the following properties:

1. (Bellman Equation): Let $\lambda\geq 0$ be arbitrary. Then $G_\lambda(z)=\|z\|^2+\lambda\cdot\max_{i\in\mathcal{M}}G_\lambda(A_iz)$, $\forall z\in\mathbb{R}^n$.
2. (Sub-Additivity): Let $\lambda\geq 0$ be arbitrary. Then $\sqrt{G_\lambda(z_1+z_2)}\leq\sqrt{G_\lambda(z_1)}+\sqrt{G_\lambda(z_2)}$, $\forall z_1,z_2\in\mathbb{R}^n$.
3. (Convexity): For each $\lambda\geq 0$, both $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ are convex functions of $z$ on $\mathbb{R}^n$.
4. (Invariant Subspace): For each $\lambda\geq 0$, the set $\mathcal{G}_\lambda:=\{z\in\mathbb{R}^n\,|\,G_\lambda(z)<\infty\}$ is a subspace of $\mathbb{R}^n$ invariant under $\{A_i\}_{i\in\mathcal{M}}$, i.e., $A_i\mathcal{G}_\lambda\subseteq\mathcal{G}_\lambda$ for all $i\in\mathcal{M}$.
5. For each $\lambda\geq 0$, $G_\lambda(z)<\infty$ for all $z\in\mathbb{R}^n$ implies that $G_\lambda(z)\leq c\|z\|^2$ for all $z\in\mathbb{R}^n$ for some finite constant $c$.
6. For $0\leq\lambda<(\max_{i\in\mathcal{M}}\|A_i\|^2)^{-1}$, $G_\lambda(z)<\infty$, $\forall z$.

*Proof:* 1. Note that $G_\lambda(\cdot)$ is the value function of an infinite horizon optimal control problem maximizing the functional $\sum_{t=0}^\infty\lambda^t\|x(t;z,\sigma)\|^2$. Property 1 is a direct consequence of the dynamic programming principle.

2. For a fixed $\lambda\geq 0$, since $x(t;z,\sigma)$ is linear in $z$, we have

$$\begin{aligned}G_\lambda(z_1+z_2)&=\sup_\sigma\sum_{t=0}^\infty\lambda^t\|x(t;z_1,\sigma)+x(t;z_2,\sigma)\|^2\\ &\leq G_\lambda(z_1)+2\sup_\sigma\sum_{t=0}^\infty\lambda^t\|x(t;z_1,\sigma)\|\,\|x(t;z_2,\sigma)\|+G_\lambda(z_2)\\ &\leq G_\lambda(z_1)+2\sqrt{G_\lambda(z_1)}\sqrt{G_\lambda(z_2)}+G_\lambda(z_2),\ \forall z_1,z_2\in\mathbb{R}^n.\end{aligned}$$

The Cauchy-Schwartz inequality is used in the last step. Taking the square root yields the desired conclusion.

3. For a fixed $\lambda\geq 0$, $G_\lambda(z)$ is convex in $z$ as by (3) it is the pointwise supremum of a family of convex (indeed, quadratic) functions of $z$ indexed by $\sigma$. The convexity of $\sqrt{G_\lambda(z)}$ follows from sub-additivity as $\sqrt{G_\lambda(\alpha_1z_1+\alpha_2z_2)}\leq\sqrt{G_\lambda(\alpha_1z_1)}+\sqrt{G_\lambda(\alpha_2z_2)}=\alpha_1\sqrt{G_\lambda(z_1)}+\alpha_2\sqrt{G_\lambda(z_2)}$, for any $z_1,z_2\in\mathbb{R}^n$ and $\alpha_1,\alpha_2\geq 0$ with $\alpha_1+\alpha_2=1$.

4. This follows directly from Properties 1 and 2.

5. Assume $\lambda$ is such that $G_\lambda(z)<\infty$, $\forall z\in\mathbb{R}^n$. Write an arbitrary $z\in\mathbb{S}^{n-1}$ in a standard basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^n$ as $z=\sum_{i=1}^n\alpha_i\,\mathbf{e}_i$, where $\sum_{i=1}^n\alpha_i^2=1$. In light of sub-additivity, we have $G_\lambda(z)\leq[\sum_{i=1}^n\sqrt{G_\lambda(\alpha_i\mathbf{e}_i)}]^2\leq n\sum_{i=1}^n\alpha_i^2G_\lambda(\mathbf{e}_i)\leq c$, where $c:=n\cdot\max_{1\leq i\leq n}G_\lambda(\mathbf{e}_i)<\infty$ by our assumption on $\lambda$. By homogeneity, we have $G_\lambda(z)\leq c\|z\|^2$, $\forall z\in\mathbb{R}^n$.

6. We simply note that any trajectory $x(t;z,\sigma)$ of the SLS satisfies $\|x(t;z,\sigma)\|^2\leq(\max_{i\in\mathcal{M}}\|A_i\|^2)^t\|z\|^2$, $\forall t$. ■

From Proposition 1, $\mathcal{G}_\lambda$ is a subspace of $\mathbb{R}^n$ that decreases monotonically from $\mathcal{G}_0=\mathbb{R}^n$ at $\lambda=0$ to $\mathcal{G}_\infty:=\cap_{\lambda\geq 0}\mathcal{G}_\lambda$ as $\lambda\to\infty$. Let $\lambda_1<\lambda_2<\cdots<\lambda_d$ for some integer $d<n$ be the exact values of $\lambda$ at which $\mathcal{G}_\lambda$ shrinks. Then the set of all distinct $\mathcal{G}_\lambda$ forms a filtration of subspaces of $\mathbb{R}^n$ as: $\mathcal{G}_0=\mathcal{G}_{\lambda_1^-}\supsetneq\mathcal{G}_{\lambda_1^+}=\mathcal{G}_{\lambda_2^-}\supsetneq\mathcal{G}_{\lambda_2^+}=\cdots=\mathcal{G}_{\lambda_d^-}\supsetneq\mathcal{G}_{\lambda_d^+}=\mathcal{G}_\infty$, where $\mathcal{G}_{\lambda_j^-}:=\lim_{\lambda\uparrow\lambda_j}\mathcal{G}_{\lambda_j}$ and $\mathcal{G}_{\lambda_j^+}:=\lim_{\lambda\downarrow\lambda_j}\mathcal{G}_{\lambda_j}$ for each $j$. Since each subspace $\mathcal{G}_{\lambda_j^-}$ (or $\mathcal{G}_{\lambda_j^+}$) is invariant under $\{A_i\}_{i\in\mathcal{M}}$, the SLS (1) restricted on it defines a sub-SLS. Intuitively, the restricted SLS on $\mathcal{G}_\infty$ is "the most exponentially stable" as its trajectories have the slowest exponential growth rate. On bigger subspaces, the restricted SLSs will be "less exponentially stable" as they contain faster growing trajectories. Equivalently, a suitable change of coordinates can simultaneously transform $\{A_i\}_{i\in\mathcal{M}}$ into the same row block upper echelon form, with their last row blocks corresponding to the restricted SLS on $\mathcal{G}_\infty$; their last two row blocks corresponding to the restricted SLS on $\mathcal{G}_{\lambda_d-}$, and so on.

From the above discussion, the radius of strong convergence $\lambda^*(z)$ at $z\in\mathbb{R}^n$ can have at most $d+1\leq n$ distinct values: $\{\lambda_1,\ldots,\lambda_d,\infty\}$. In particular, if the SLS is *irreducible*, namely, it has no nontrivial invariant subspaces other than $\mathbb{R}^n$ and $\{0\}$ (which occurs with probability one for randomly generated SLSs), then $d=1$, and $G_\lambda(z)$ is either finite everywhere or infinite everywhere for any $\lambda\geq 0$.

*Remark 2:* In the Multiplicative Ergodic Theorem for non-switched dynamical systems, $\{-\log\sqrt{\lambda_1},\ldots,-\log\sqrt{\lambda_d},\infty\}$ are called the Lyapunov exponents of the systems [22].

*Example 1:* Consider a SLS on $\mathbb{R}^2$ with two subsystems:

$$A_1=\begin{bmatrix}\frac{7}{6} & -\frac{5}{6}\\ -\frac{5}{6} & \frac{7}{6}\end{bmatrix},\quad A_2=\begin{bmatrix}\frac{5}{3} & \frac{4}{3}\\ \frac{4}{3} & \frac{5}{3}\end{bmatrix}.\quad (6)$$

Starting from any initial $z=(z_1,z_2)^T$, let $x(t;z,\sigma_1)$ and $x(t;z,\sigma_2)$ be the state trajectories under the switching sequences $\sigma_1=(1,1,\ldots)$ and $\sigma_2=(2,2,\ldots)$, respectively. Then it can be proved (though by no mean trivially) that $G_\lambda(z)$ is $\max_{i=1,2}\sum_{t=0}^\infty\lambda^t\|x(t;z,\sigma_i)\|^2$, or more explicitly,

$$\begin{cases}\max\Big\{\frac{9(z_1+z_2)^2}{2(9-\lambda)}+\frac{(z_1-z_2)^2}{2(1-4\lambda)},\\ \qquad\qquad\frac{(z_1+z_2)^2}{2(1-9\lambda)}+\frac{9(z_1-z_2)^2}{2(9-\lambda)}\Big\}, & \text{if } 0\leq\lambda<\frac{1}{9}\\ \frac{(z_1-z_2)^2}{2(1-4\lambda)}\cdot 1_{\{z_1+z_2=0\}}+\infty\cdot 1_{\{z_1+z_2\neq 0\}}, & \text{if } \frac{1}{9}\leq\lambda<\frac{1}{4}\\ \infty, & \text{if } \lambda\geq\frac{1}{4}.\end{cases}$$

Here, $1_{\{z_1+z_2=0\}}$ denotes the indicator function for the set $\{(z_1,z_2)\in\mathbb{R}^2\,|\,z_1+z_2=0\}$. Similarly for $1_{\{z_1+z_2\neq 0\}}$.

Thus, $\mathcal{G}_\lambda$ is $\mathbb{R}^2$ for $0 \le \lambda < \frac{1}{9}$; the 1D subspace $\mathcal{V} := \{(\alpha, -\alpha)^T \mid \alpha \in \mathbb{R}\}$ for $\frac{1}{9} \le \lambda < \frac{1}{4}$; and $\{0\}$ for $\lambda \ge \frac{1}{4}$. Each of these is an invariant subspace of $\mathbb{R}^2$ for $\{A_1, A_2\}$. For example, $\mathcal{V}$ is a common eigenspace of $A_1$ and $A_2$. In fact, $A_1$ and $A_2$ commute and can be simultaneously diagonalized as $Q^{-1}A_1Q$ =diag$(\frac{1}{3}, 2)$ and $Q^{-1}A_2Q$ =diag$(3, \frac{1}{3})$ by $Q = \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$. Under the transformation by $Q$, $\mathcal{V}$ becomes the vertical axis. See also [38] for results on the nice reachability of such SLSs.

### *B. Radius of Strong Convergence*

We next define a quantity that characterizes the stability of the SLS under arbitrary switching.

*Definition 3:* The radius of strong convergence of the SLS (1), denoted by $\lambda^* \in (0, \infty]$, is defined as

$$\lambda^* := \sup\left\{\lambda \ge 0 \mid G_\lambda(z) < \infty,\ \forall z \in \mathbb{R}^n\right\}.$$

By Property 5 of Proposition 1, $\lambda^*$ can also be defined as $\lambda^* = \sup\{\lambda \ge 0 \mid G_\lambda(z) < c\|z\|^2, \forall z \in \mathbb{R}^n,$ for some finite $c\}$. By Property 6, $\lambda^* \ge (\max_{i\in\mathcal{M}} \|A_i\|^2)^{-1} > 0$. It is possible that $\lambda^* = \infty$. This is the case, for example, if all solutions $x(t; z, \sigma)$ of the SLS converge to the origin within a finite time uniformly in $z$ and $\sigma$. For the SLS in Example 1, $\lambda^* = \frac{1}{9}$.

The following theorem shows that the radius of strong convergence is sufficient for determining the exponential stability of the SLS (1) under arbitrary switching.

*Theorem 2:* The following statements are equivalent:

1) The SLS (1) is exponentially stable under arbitrary switching.
2) Its radius of strong convergence $\lambda^* > 1$.
3) The strong generating function at $\lambda = 1$, $G_1(z)$, is finite for all $z \in \mathbb{R}^n$.

*Proof:* To show $1) \Rightarrow 2)$, suppose there exist constants $\kappa \ge 1$ and $r \in [0, 1)$ such that $\|x(t; z, \sigma)\| \le \kappa r^t \|z\|$, $\forall t$, for all trajectory $x(t; z, \sigma)$ of the SLS. Then for any $\lambda < r^{-2}$,

$$\begin{aligned} G_\lambda(z) &= \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^2 \\ &\le \sum_{t=0}^{\infty} \lambda^t \kappa^2 r^{2t} \|z\|^2 = \frac{\kappa^2}{1 - \lambda r^2} \|z\|^2 < \infty,\ \forall z \in \mathbb{R}^n. \end{aligned}$$

It follows that $\lambda^* \ge r^{-2} > 1$. The implication $2) \Rightarrow 3)$ follows directly from the definition of $\lambda^*$. Finally, to show $3) \Rightarrow 1)$, suppose $G_1(z) < \infty, \forall z$. By Proposition 1, $G_1(z) \le c\|z\|^2$, $\forall z$, for some finite constant $c$. Thus, for any trajectory $x(t; z, \sigma)$ of the SLS, $\sum_{t=0}^{\infty} \|x(t; z, \sigma)\|^2 \le c\|z\|^2$. This implies that $\|x(t; z, \sigma)\| \le \sqrt{c}\|z\|$, $\forall t$; and that $x(t; z, \sigma) \to 0$ as $t \to \infty$. Consequently, the SLS is asymptotically stable, hence exponentially stable [6], under arbitrary switching. ∎

Theorem 2 implies the following stronger conclusions.

*Corollary 1:* Given a SLS with a radius of strong convergence $\lambda^*$, for any $r > (\lambda^*)^{-1/2}$, there exists a constant $\kappa_r$ such that $\|x(t; z, \sigma)\| \le \kappa_r r^t \|z\|$, $\forall t$, for all trajectories $x(t; z, \sigma)$ of the SLS. Furthermore, $(\lambda^*)^{-1/2}$ is also the smallest value for the previous statement to hold.

*Proof:* Suppose $r > (\lambda^*)^{-1/2}$. The scaled SLS with subsystem dynamics matrices $\{A_i/r\}_{i\in\mathcal{M}}$ is easily seen to have its strong generating function to be $G(\lambda/r^2, z)$; hence, its radius of strong convergence is $r^2\lambda^* > 1$. By Theorem 2, the scaled SLS is exponentially stable under arbitrary switching. In particular, all its trajectories $\tilde{x}(t; z, \sigma)$ satisfy $\|\tilde{x}(t; z, \sigma)\| \le \kappa_r \|z\|$, $\forall t$, for some $\kappa_r > 0$. For all trajectories $x(t; z, \sigma)$ of the original SLS, since $\tilde{x}(t; z, \sigma) = r^{-t} x(t; z, \sigma)$, we must have $\|x(t; z, \sigma)\| = r^t \|\tilde{x}(t; z, \sigma)\| \le \kappa_r r^t \|z\|$, $\forall t$. The second conclusion is a direct consequence of Theorem 2. ∎

In other words, the maximum exponential growth rate of all the trajectories of the SLS is $(\lambda^*)^{-1/2}$. Later on in Theorem 3, we will study how to infer the constant $\kappa_r$ from $G_\lambda(z)$.

*Remark 3:* For the SLS (1), the *joint spectral radius* [17], [18] of the subsystem dynamics matrices $\{A_i\}_{i\in\mathcal{M}}$ is defined by $\rho^* := \lim_{k\to\infty} \sup\left\{\|A_{i_1} \cdots A_{i_k}\|^{1/k},\ i_1, \ldots, i_k \in \mathcal{M}\right\}$; and the *Lyapunov exponent* of the corresponding linear difference inclusion is $\gamma^* := \sup_{\sigma, z\neq 0} \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t; z, \sigma)\|$ (see [15]). These two quantities also characterize the maximal exponential growth rate of the SLS trajectories, and are related to $\lambda^*$ by: $(\lambda^*)^{-1/2} = \rho^* = e^{\gamma^*}$. In this sense, Corollary 1 is equivalent to [17, Prop. 1.4] and to [18, Prop. 4.17].

### *C. Smoothness of Finite $G_\lambda(z)$*

When $\lambda$ is in the range of $[0, \lambda^*)$, the function $G_\lambda(z)$ is finite everywhere. We shall focus on such finite $G_\lambda(z)$, and establish some smoothness properties of them in this section.

We first introduce a few notions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *directionally differentiable* at $z_0 \in \mathbb{R}^n$ if its (one-sided) directional derivative at $z_0$ along any direction $v \in \mathbb{R}^n$ defined as $f'(z_0; v) := \lim_{\tau\downarrow 0} \frac{f(z_0+\tau v) - f(z_0)}{\tau}$ exists. If $f$ is both directionally differentiable at $z_0$ and locally Lipschitz continuous in a neighborhood of $z_0$, it is called *B(ouligand)-differentiable* at $z_0$. Finally, $f$ is *semismooth* at $z_0$ if it is B-differentiable in a neighborhood of $z_0$ and the following limit holds: $\lim_{\substack{z\to z_0 \\ z\neq z_0}} \frac{|f'(z; z-z_0) - f'(z_0; z-z_0)|}{\|z - z_0\|} = 0$.

*Proposition 2:* For $\lambda \in [0, \lambda^*)$, both $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ are convex, locally Lipschitz continuous, and semismooth on $\mathbb{R}^n$. Moreover, $\sqrt{G_\lambda(z)}$ is globally Lipschitz continuous.

*Proof:* The convexity of $G_\lambda(z)$ and $\sqrt{G_\lambda(z)}$ has been proved in Proposition 1. Being convex, they must also be semismooth according to [39, Prop. 7.4.5]. Finally, using the sub-additivity in Proposition 1, we obtain, $\forall z, \Delta z \in \mathbb{R}^n$,

$$-\sqrt{G_\lambda(-\Delta z)} \le \sqrt{G_\lambda(z + \Delta z)} - \sqrt{G_\lambda(z)} \le \sqrt{G_\lambda(\Delta z)}.$$

Thus, $\left|\sqrt{G_\lambda(z+\Delta z)} - \sqrt{G_\lambda(z)}\right| \le \sqrt{G_\lambda(\pm\Delta z)} \le \sqrt{c}\|\Delta z\|$ for some finite constant $c$ as $\lambda < \lambda^*$, i.e., $\sqrt{G_\lambda(z)}$ is globally Lipschitz continuous on $\mathbb{R}^n$ with the Lipschitz constant $\sqrt{c}$. As a result, $\sqrt{G_\lambda(z)}$, hence $G_\lambda(z)$, is also locally Lipschitz continuous on $\mathbb{R}^n$. ∎

Note that for $\lambda > \lambda^*$, $G_\lambda(z)$ can not be continuous on $\mathbb{R}^n$. Indeed, in this case, $G_\lambda(z_0) = \infty$ at some $z_0 \in \mathbb{R}^n$. Thus as $k \to \infty$, the sequence $\frac{z_0}{k} \to 0$, but $G_\lambda(\frac{z_0}{k}) \to \infty \neq 0$.

### *D. Quadratic Bounds of Finite $G_\lambda(z)$*

For $\lambda \in [0, \lambda^*)$, $G_\lambda(z)$ is finite everywhere, hence quadratically bounded by Proposition 1. Define

$$g_\lambda := \sup\{G_\lambda(z) \mid \|z\| = 1\}, \quad \lambda \in [0, \lambda^*). \tag{7}$$

By homogeneity, $g_\lambda$ can be equivalently defined as the smallest constant $c$ such that $G_\lambda(z) \le c\|z\|^2$, $\forall z \in \mathbb{R}^n$.

It is easy to see that $g_\lambda$ is finite and strictly increasing on $[0,\lambda^*)$ (we exclude the trivial case where all $A_i$ are zero), with $g_0 = 1$. We next prove an affine lower bound of $1/g_\lambda$.

*Lemma 2:* $\frac{1}{g_\lambda} \ge 1-\lambda\cdot\max_{i\in\mathcal{M}}\|A_i\|^2$, $\forall \lambda\in[0,\lambda^*)$. Thus,

$$\frac{\lambda}{1-1/g_\lambda} \ge \frac{1}{\max_{i\in\mathcal{M}}\|A_i\|^2}, \quad \forall \lambda \in (0,\lambda^*). \tag{8}$$

*Proof:* Let $\lambda \in [0,\lambda^*)$. For an arbitrary trajectory $x(t;z,\sigma)$ of the SLS, $\|x(t;z,\sigma)\|^2 \le (\max_{i\in\mathcal{M}}\|A_i\|^2)^t\|z\|^2$ for all $t$. Therefore, for $0\le\lambda<(\max_{i\in\mathcal{M}}\|A_i\|^2)^{-1}$,

$$\sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma)\|^2 \le \frac{1}{1-\lambda\cdot\max_{i\in\mathcal{M}}\|A_i\|^2}\|z\|^2,\ \forall\sigma.$$

By definition, we have $g_\lambda \le (1-\lambda\cdot\max_{i\in\mathcal{M}}\|A_i\|^2)^{-1}$, which is the desired conclusion for $0\le\lambda<(\max_{i\in\mathcal{M}}\|A_i\|^2)^{-1}$. When $(\max_{i\in\mathcal{M}}\|A_i\|^2)^{-1}\le\lambda<\lambda^*$, the desired conclusion is trivial: $\frac{1}{g_\lambda}\ge 0\ge 1-\lambda\cdot\max_{i\in\mathcal{M}}\|A_i\|^2$. ■

The following auxiliary result on general power series is proved in Appendix A.

*Lemma 3:* Let $\{w_t\}_{t=0,1,\ldots}$ be a sequence of nonnegative scalars such that for some $\lambda_0>0$ and $\beta\ge1$,

$$\sum_{t=0}^{\infty} w_{t+s}\lambda_0^t \le \beta w_s, \quad s=0,1,\ldots. \tag{9}$$

Then the power series $\sum_{t=0}^\infty w_t\lambda^t$ has its radius of convergence at least $\lambda_1 := \lambda_0/(1-1/\beta)$. Moreover,

$$\sum_{t=0}^{\infty} w_t\lambda^t \le \frac{\beta w_0}{1-(\beta-1)(\lambda/\lambda_0-1)} < \infty,\ \forall\lambda\in[\lambda_0,\lambda_1).$$

Using Lemma 3, we obtain the following estimate of $g_\lambda$.

*Proposition 3:* The function $\lambda/(1-1/g_\lambda)$ is nondecreasing for $\lambda\in(0,\lambda^*)$, and is upper bounded by

$$\frac{\lambda}{1-1/g_\lambda} \le \lambda^*, \quad \forall\lambda\in(0,\lambda^*). \tag{10}$$

*Proof:* Consider a fixed $\lambda_0\in(0,\lambda^*)$. Let $x(t;z,\sigma)$ be an arbitrary trajectory of the SLS. For each $s=0,1,\ldots$, by the definition of $g_{\lambda_0}$, the trajectory $x(t+s;z,\sigma)$, $t=0,1,\ldots$, of the SLS starting from the initial state $x(s;z,\sigma)$ satisfies

$$\sum_{t=0}^{\infty}\lambda_0^t\|x(t+s;z,\sigma)\|^2 \le G_{\lambda_0}(x(s;z,\sigma)) \le g_{\lambda_0}\|x(s;z,\sigma)\|^2.$$

Hence, the sequence $\{w_t := \|x(t;z,\sigma)\|^2\}_{t=0,1,\ldots}$ satisfies the condition (9) with $\beta=g_{\lambda_0}$. By Lemma 3, we have

$$\sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma)\|^2 \le \frac{g_{\lambda_0}}{1-(g_{\lambda_0}-1)(\lambda/\lambda_0-1)}\|z\|^2,$$

for $\lambda\in[\lambda_0,\lambda_1)$, where $\lambda_1 := \lambda_0/(1-1/g_{\lambda_0})$. As the trajectory $x(t;z,\sigma)$ is arbitrary, we conclude that

$$G_\lambda(z) \le \frac{g_{\lambda_0}}{1-(g_{\lambda_0}-1)(\lambda/\lambda_0-1)}\|z\|^2 < \infty,\ \forall\lambda\in[\lambda_0,\lambda_1).$$

This implies $\lambda^*\ge\lambda_1$, i.e., the desired conclusion (10); and

$$g_\lambda \le \frac{g_{\lambda_0}}{1-(g_{\lambda_0}-1)(\lambda/\lambda_0-1)} \ \Rightarrow\ \frac{\lambda}{1-1/g_\lambda} \ge \frac{\lambda_0}{1-1/g_{\lambda_0}},$$

for $\lambda\in[\lambda_0,\lambda_1)$. Since $\lambda_0\in(0,\lambda^*)$ is arbitrary and $\lambda_1>\lambda_0$, this proves the monotonicity of $\lambda/(1-1/g_\lambda)$ on $(0,\lambda^*)$. ■

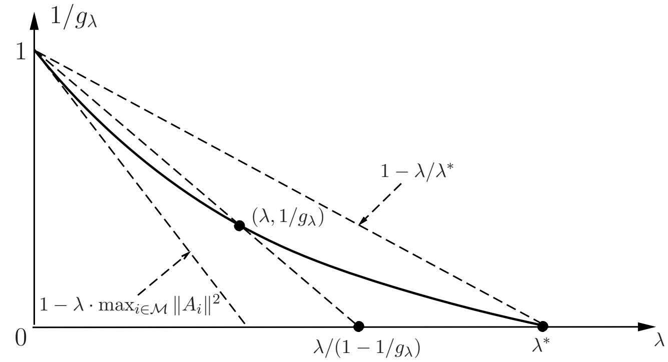


Fig. 1. Plot of the function $1/g_\lambda$ (in solid line).

In Appendix B, the following generic properties of the functions $g_\lambda$ and $1/g_\lambda$ are proved.

*Proposition 4:* The function $1/g_\lambda$ is strictly decreasing, semismooth, and Lipschitz continuous with Lipschitz constant $\max_{i\in\mathcal{M}}\|A_i\|^2$ on $[0,\lambda^*)$. Moreover, $1/g_\lambda\to0$ as $\lambda\uparrow\lambda^*$. Correspondingly, $g_\lambda$ is strictly increasing, convex, semismooth, and locally Lipschitz continuous on $[0,\lambda^*)$, with $g_\lambda\to\infty$ as $\lambda\uparrow\lambda^*$.

*Remark 4:* Since $g_\lambda\to\infty$ as $\lambda\uparrow\lambda^*$, the generating function $G_{\lambda^*}(z)$ must have infinite value at some $z\in\mathbb{R}^n$, according to Property 5 of Proposition 1. This implies that, as $\lambda$ increases, $\lambda^*$ is precisely the first value at which $G_\lambda(\cdot)$ starts to have infinite values.

Fig. 1 plots the graph of a generic $1/g_\lambda$ as a function of $\lambda\in[0,\lambda^*)$. According to (8) and (10), the graph of $1/g_\lambda$ is sandwiched between those of two affine functions: $1-\lambda\ \max_{i\in\mathcal{M}}\|A_i\|^2$ from the left and $1-\lambda/\lambda^*$ from the right. In addition, by Proposition 3, the ray (middle dashed line) emitting from the point $(0,1)$ and passing through $(\lambda,1/g_\lambda)$ intersects the $\lambda$-axis at the point $(\frac{\lambda}{1-1/g_\lambda},0)$ that moves monotonically to the right towards $(\lambda^*,0)$ as $\lambda$ increases in $[0,\lambda^*)$, i.e., the ray rotates around its starting point $(1,0)$ counterclockwise monotonically. It is conjectured that the function $1/g_\lambda$ is indeed convex on $[0,\lambda^*)$ for any SLS.

It is easy to show that the directional derivative of $g_\lambda$ at $\lambda=0$ is $g'_\lambda(0_+)=\max_{i\in\mathcal{M}}\|A_i\|^2$. Hence the directional derivative of $1/g_\lambda$ at $\lambda=0$ is: $(1/g_\lambda)'(0_+)=-\max_{i\in\mathcal{M}}\|A_i\|^2$. In Fig. 1, this means that the graph of $1/g_\lambda$ is tangential to the leftmost dashed ray emitting from $(0,1)$.

### *E. Norms Induced by Finite $G_\lambda(z)$*

As an immediate result of Proposition 1, for $\lambda\in[0,\lambda^*)$, $\sqrt{G_\lambda(z)}$ is finite, sub-additive, and homogeneous of degree one; thus it defines a norm on the vector space $\mathbb{R}^n$:

$$\|z\|_{G_\lambda} := \sqrt{G_\lambda(z)}, \quad \forall z\in\mathbb{R}^n. \tag{11}$$

As $\lambda$ increases, the norm $\|\cdot\|_{G_\lambda}$ increases, hence its unit ball shrinks. See Fig. 2 for the plots of such unit balls for the SLS (6) in Example 1. The vector norm $\|\cdot\|_{G_\lambda}$ induces a matrix norm for $A\in\mathbb{R}^{n\times n}$ by: $\|A\|_{G_\lambda} := \sup_{z\neq0}\{\|Az\|_{G_\lambda}/\|z\|_{G_\lambda}\}$.

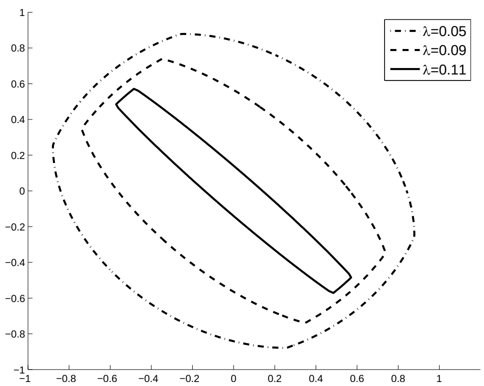


Fig. 2. Unit balls of $\|\cdot\|_{G_\lambda}$ for the SLS (6).

For $\lambda \in [0, \lambda^*)$, define the constant:

$$d_\lambda := \sup_{\|z\|=1} \max_{i\in\mathcal{M}} \|A_i z\|^2_{G_\lambda} = \sup_{\|z\|=1} \max_{i\in\mathcal{M}} G_\lambda(A_i z). \quad (12)$$

*Lemma 4:* For each $\lambda \in [0, \lambda^*)$, the norm $\|\cdot\|_{G_\lambda}$ satisfies:

$$\max_{i\in\mathcal{M}} \|A_i\|_{G_\lambda} = \sqrt{d_\lambda/(1+\lambda d_\lambda)}.$$

*Proof:* Using the Bellman equation, we write

$$\begin{aligned}\max_{i\in\mathcal{M}} \|A_i\|^2_{G_\lambda} &= \max_{i\in\mathcal{M}} \sup_{\|z\|=1} \frac{G_\lambda(A_i z)}{G_\lambda(z)} = \sup_{\|z\|=1} \frac{\max_{i\in\mathcal{M}} G_\lambda(A_i z)}{G_\lambda(z)} \\ &= \sup_{\|z\|=1} \frac{\max_{i\in\mathcal{M}} G_\lambda(A_i z)}{1+\lambda\cdot \max_{i\in\mathcal{M}} G_\lambda(A_i z)} \\ &= \frac{\sup_{\|z\|=1}\max_{i\in\mathcal{M}} G_\lambda(A_i z)}{1+\lambda\cdot\sup_{\|z\|=1}\max_{i\in\mathcal{M}} G_\lambda(A_i z)} = \frac{d_\lambda}{1+\lambda d_\lambda}.\end{aligned}$$

Note that the second last step follows as $x/(1+\lambda x)$ is continuous and increasing in $x \in \mathbb{R}_+$ for any $\lambda \geq 0$. ■

The above results yields bounds on both the exponential growth rate of the SLS trajectories and $\lambda^*$ as follows.

*Corollary 2:* Suppose $\lambda \in [0, \lambda^*)$. Then

$$\|x(t; z, \sigma)\| \leq \sqrt{g_\lambda}\left(\frac{d_\lambda}{1+\lambda d_\lambda}\right)^{t/2} \|z\|, \quad t = 0, 1, \ldots,$$

holds for all trajectories of the SLS (1). As a result, we have

$$\lambda^* > \lambda + \frac{1}{d_\lambda}, \quad \forall \lambda \in [0, \lambda^*). \quad (13)$$

*Proof:* We note that $\|z\| \leq \|z\|_{G_\lambda} \leq \sqrt{g_\lambda}\|z\|$, $\forall z \in \mathbb{R}^n$, i.e., the norm $\|\cdot\|_{G_\lambda}$ is equivalent to the Euclidean norm $\|\cdot\|$. For any trajectory $x(t; z, \sigma)$ of the SLS, we then have

$$\begin{aligned}\|x(t; z, \sigma)\| &\leq \|x(t; z, \sigma)\|_{G_\lambda} = \|A_{\sigma(t-1)}\cdots A_{\sigma(0)} z\|_{G_\lambda} \\ &\leq \|A_{\sigma(t-1)}\|_{G_\lambda}\cdots\|A_{\sigma(0)}\|_{G_\lambda}\cdot\|z\|_{G_\lambda}, \ \forall t.\end{aligned}$$

Applying Lemma 4 and noting that $\|z\|_{G_\lambda} \leq \sqrt{g_\lambda}\|z\|$, we obtain the first conclusion. This in turn implies that $\lambda^* > (1+\lambda d_\lambda)/d_\lambda$, which is the desired conclusion (13). ■

As $\lambda \uparrow \lambda^*$, by (13), we must have $d_\lambda \to \infty$; hence $d_\lambda/(1+\lambda d_\lambda) \to 1/\lambda^*$. The following then holds.

*Theorem 3:* For any $\varepsilon > 0$, there exists a $\lambda \in (0, \lambda^*)$ sufficiently close to $\lambda^*$ such that

$$\|x(t; z, \sigma)\| \leq \sqrt{g_\lambda}(r_\lambda)^t\|z\|, \quad t = 0, 1, \ldots,$$

for all trajectories $x(t; z, \sigma)$ of the SLS (1), where

$$r_\lambda = \left(\frac{d_\lambda}{1+\lambda d_\lambda}\right)^{1/2} \leq (\lambda^*)^{-1/2} + \varepsilon.$$

*Remark 5:* Once $G_\lambda(z)$ is computed at any $\lambda \in [0, \lambda^*)$, (13) gives a lower bound of $\lambda^*$. An upper bound of $\lambda^*$ can be derived by using a result in [27, Lemma 3.3] as

$$\lambda^* \leq \left[\inf_{z\neq 0} \max_{i\in\mathcal{M}} \frac{G_\lambda(A_i z)}{G_\lambda(z)}\right]^{-1}. \quad (14)$$

This upper bound, together with the lower bound in (13), gives an interval for the possible values of $\lambda^*$. As $\lambda \uparrow \lambda^*$, it can be shown that the two bounds converge towards each other; and the corresponding norm $\|\cdot\|_{G_\lambda}$ approaches asymptotically a *Barabanov norm* [23], [27] of $\{A_i\}_{i\in\mathcal{M}}$.

### F. Algorithms for Computing $G_\lambda(z)$

We next present some algorithms for computing the finite strong generating functions. The idea is that $G_\lambda(z)$ as the value function of an infinite horizon optimal control problem can be approximated by those of a sequence of finite horizon problems. Specifically, for each $k = 0, 1, \ldots$, define

$$G^k_\lambda(z) := \max_\sigma \sum_{t=0}^k \lambda^t \|x(t; z, \sigma)\|^2, \ \forall z \in \mathbb{R}^n. \quad (15)$$

Maximum is used here instead of supremum as only the first $k$ steps of $\sigma$ affect the summation.

*Proposition 5:* For any $\lambda \geq 0$ and $k = 0, 1, \ldots$, $G^k_\lambda(z)$ is a convex function of $z$ on $\mathbb{R}^n$ satisfying

$$G^0_\lambda(z) \leq G^1_\lambda(z) \leq G^2_\lambda(z) \leq \cdots \leq G_\lambda(z), \ \forall z \in \mathbb{R}^n.$$

Moreover, for $\lambda \in [0, \lambda^*)$, $G^k_\lambda(z)$ as $k \to \infty$ converges exponentially fast to $G_\lambda(z)$: $\forall k = 0, 1, \ldots$,

$$|G^k_\lambda(z) - G_\lambda(z)| \leq g_\lambda(1 - 1/g_\lambda)^{k+1}\|z\|^2, \ \forall z \in \mathbb{R}^n.$$

*Proof:* The convexity proof is identical to that of Proposition 1, hence omitted. Fix $\lambda \geq 0$ and $z \in \mathbb{R}^n$, and let $\sigma_k$ be a switching sequence achieving the maximum in (15). Then,

$$G^k_\lambda(z) = \sum_{t=0}^k \lambda^t \|x(t; z, \sigma_k)\|^2 \leq \sum_{t=0}^{k+1} \lambda^t \|x(t; z, \sigma_k)\|^2 \leq G^{k+1}_\lambda(z).$$

Similarly, we can show $G^k_\lambda(z) \leq G_\lambda(z)$, hence the monotonicity. When $\lambda \in [0, \lambda^*)$, $G_\lambda(z)$ is finite for each $z \in \mathbb{R}^n$. Let $\hat{x}(t) := x(t; z, \sigma)$ be a trajectory under an optimal switching sequence $\sigma$ that achieves the supremum in (3), i.e., $G_\lambda(z) = \sum_{t=0}^\infty \lambda^t \|\hat{x}(t)\|^2$. By the Bellman equation,

$$G_\lambda(\hat{x}(k-1)) - \lambda\cdot G_\lambda(\hat{x}(k)) = \|\hat{x}(k-1)\|^2 \geq \frac{G_\lambda(\hat{x}(k-1))}{g_\lambda},$$

for $k = 1, 2, \ldots$, where the last step follows from the definition of $g_\lambda$. Rearranging and by induction, we obtain

$$\begin{aligned}G_\lambda(\hat{x}(k)) &\leq \lambda^{-1}(1 - 1/g_\lambda)\, G_\lambda(\hat{x}(k-1)) \leq \cdots \\ &\leq \lambda^{-k}(1 - 1/g_\lambda)^k G_\lambda(z) \leq \lambda^{-k} g_\lambda (1 - 1/g_\lambda)^k \|z\|^2,\end{aligned}$$

**Algorithm 1** Computing $G_\lambda(z)$ on Grid Points of $\mathbb{S}^{n-1}$

Let $\mathcal{S} = \{z_j\}_{j=1}^N$ be a set of grid points of $\mathbb{S}^{n-1}$;
Initialize $k = 0$, and $\widehat{G}_\lambda^0(z_j) = 1$ for all $z_j \in \mathcal{S}$;
**repeat**
  $k \leftarrow k + 1$;
  **for** each $z_j \in \mathcal{S}$ **do**
    **for** each $i \in \mathcal{M}$ **do**
      Find a minimal subset $\mathcal{S}_{ij}$ of $\mathcal{S}$ whose elements span a convex cone containing $A_i z_j$;
      Express $A_i z_j$ as $\sum_{z_\ell \in \mathcal{S}_{ij}} \alpha_{ij}^\ell z_\ell$ with $\alpha_{ij}^\ell > 0$;
      Set $g_{ij} = \sum_{z_\ell \in \mathcal{S}_{ij}} \alpha_{ij}^\ell \sqrt{\widehat{G}_\lambda^{k-1}(z_\ell)}$;
    **end for**
    Set $\widehat{G}_\lambda^k(z_j) = 1 + \lambda \cdot \max_{i \in \mathcal{M}} g_{ij}^2$;
  **end for**
**until** $k$ is large enough
**return** $\widehat{G}_\lambda^k(z_j)$ for all $z_j \in \mathcal{S}$

for $k = 0, 1, \ldots$. The optimality of $\hat{x}(t)$ then implies that

$$\sum_{t=k}^{\infty} \lambda^t \|\hat{x}(t)\|^2 = \lambda^k \sum_{t=0}^{\infty} \lambda^t \|\hat{x}(t+k)\|^2 = \lambda^k G_\lambda(\hat{x}(k)) \\ \leq g_\lambda (1 - 1/g_\lambda)^k \|z\|^2, \quad \forall k = 0, 1, \ldots.$$

Consequently, $G_\lambda(z) \geq G_\lambda^k(z) \geq \sum_{t=0}^k \lambda^t \|\hat{x}(t)\|^2 = G_\lambda(z) - \sum_{t=k+1}^\infty \lambda^t \|\hat{x}(t)\|^2 \geq G_\lambda(z) - g_\lambda(1 - 1/g_\lambda)^{k+1} \|z\|^2$. ■

Therefore, $G_\lambda^k(z)$ for $k$ large enough provide approximations of $G_\lambda(z)$ within arbitrary precision. A recursive procedure to compute the functions $G_\lambda^k(z)$ is as follows:

$$G_\lambda^0(z) = \|z\|^2 \\ G_\lambda^k(z) = \|z\|^2 + \lambda \cdot \max_{i \in \mathcal{M}} G_\lambda^{k-1}(A_i z), \quad k = 1, 2, \ldots. \quad (16)$$

To implement the recursion numerically, one first represents each $G_\lambda^{k-1}(z)$ by its values on some fine grid points of the unit sphere, and then carries out the recursion (16) by estimating conservatively the values of $G_\lambda^{k-1}(A_i z)$ at those $A_i z$ not aligned with the direction of any grid point, using convexity and homogeneity of the function $\sqrt{G_\lambda^{k-1}(z)}$. The above idea is summarized in Algorithm 1. Similar ray gridding techniques have also been used in the previous studies [40], [41].

Algorithm 1 returns a sequence of mappings $\widehat{G}_\lambda^k : \mathcal{S} \to \mathbb{R}_+$, $k = 0, 1, \ldots$, where $\mathcal{S}$ is a set of grid points of the unit sphere. They provide upperbounds of $G_\lambda^k(z)$ on $\mathcal{S}$ as follows.

*Proposition 6:* $G_\lambda^k(z_j) \leq \widehat{G}_\lambda^k(z_j)$, $\forall z_j \in \mathcal{S}$, $\forall k = 0, 1, \ldots$

*Proof:* We prove by induction. At $k = 0$, we have $G_\lambda^0(z_j) = \widehat{G}_\lambda^0(z_j) = 1$ for all $z_j \in \mathcal{S}$. Suppose the conclusion is true for $0, 1, \ldots, k-1$. For any $z_j \in \mathcal{S}$, let $\mathcal{S}_{ij}$ and $\alpha_{ij}^\ell$ be as given in Algorithm 1. Then $\widehat{G}_\lambda^k(z_j) = 1 + \lambda \cdot \max_{i \in \mathcal{M}} g_{ij}^2$, where by the induction hypothesis and sub-additivity,

$$g_{ij} \geq \sum_{z_\ell \in \mathcal{S}_{ij}} \alpha_{ij}^\ell \sqrt{G_\lambda^{k-1}(z_\ell)} = \sum_{z_\ell \in \mathcal{S}_{ij}} \sqrt{G_\lambda^{k-1}(\alpha_{ij}^\ell z_\ell)} \\ \geq \sqrt{G_\lambda^{k-1}\Big(\sum_{z_\ell \in \mathcal{S}_{ij}} \alpha_{ij}^\ell z_\ell\Big)} = \sqrt{G_\lambda^{k-1}(A_i z_j)}.$$

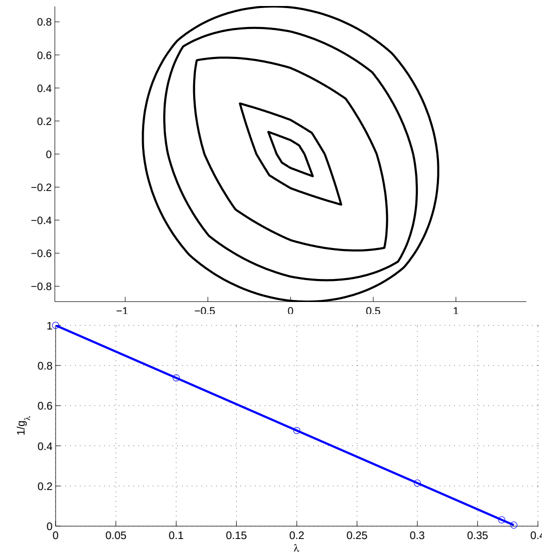

Fig. 3. Top: Unit balls of $\|\cdot\|_{G_\lambda}$ for the SLS in Example 2 at $\lambda = 0.1, 0.2, 0.3, 0.37, 0.38$ (inward). Bottom: Plot of $1/g_\lambda$.

Using the Bellman equation in Proposition 1, we then have $\widehat{G}_\lambda^k(z_j) \geq 1 + \lambda \cdot \max_{i \in \mathcal{M}} G_\lambda^{k-1}(A_i z_j) = G_\lambda^k(z_j)$. Thus, the conclusion also holds for $k$. This completes the proof. ■

Combining Proposition 6 and Theorem 2, we have the following stability test.

*Corollary 3:* A sufficient condition for the SLS (1) to be exponentially stable under arbitrary switching is that the mappings $\widehat{G}_1^k : \mathcal{S} \to \mathbb{R}_+$, $k = 0, 1, \ldots$, obtained by Algorithm 1 are uniformly bounded.

By repeatedly applying Algorithm 1 to a sequence of $\lambda$ whose values increase according to $\lambda_{\text{next}} = \lambda/(1 - 1/g_\lambda)$ or by (13), increasingly accurate underestimates of $\lambda^*$ can be obtained. In view of Section III-E, this procedure is somewhat similar to the norm iteration proposed in [27], although the iterations in [27] are performed through a max-relaxation scheme and in this paper through the Bellman equation.

As $\lambda$ approaches $\lambda^*$, however, the computation time of Algorithm 1 will get exponentially longer for two reasons. First, the convergence of $G_\lambda^k(z)$ to $G_\lambda(z)$ is much slower by Proposition 5. Second, errors of $\hat{G}_\lambda^k(z)$ over-approximating $G_\lambda^k(z)$ will accumulate quickly in time. Thus, a denser grid and more iterations are generally needed to ensure a given accuracy. This is not surprising given that the problem of finding $\lambda^*$ (or the JSR) is known to be NP-hard [24]. Hence the complexity of Algorithm 1, like other approximation algorithms, will grow exponentially with respect to the state dimension and the required accuracy. Some recently developed algorithms for estimating the JSR (e.g., [29]) have the desirable feature of providing prescribed performance guarantee of the computed estimates. Besides (13) and (14), we are currently working on developing further performance assurance for Algorithm 1.

## G. Examples

*Example 2:* The following example is taken from [25]:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

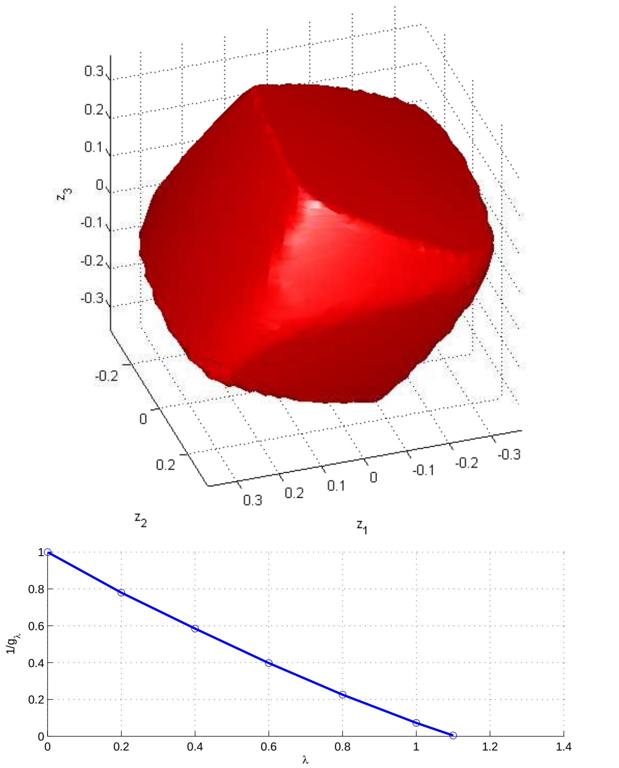


Fig. 4. Top: Unit ball of $\|\cdot\|_{G_1}$ for the SLS in Example 3. Bottom: Plot of $1/g_\lambda$.

Algorithm 1 is used to compute the functions $G_\lambda(z)$ of this SLS for different values of $\lambda$: $\lambda =$0.1, 0.2, 0.3, 0.37, and 0.38. The results are shown in Fig. 3, where the top and bottom figures plot respectively the unit balls of the norm $\|\cdot\|_{G_\lambda}$ and the graph of the function $1/g_\lambda$. From the plot, the graph of $1/g_\lambda$ is very close to a straight line. Thus, by computing $1/g_\lambda$ at two different $\lambda$ and extrapolating, one can obtain an accurate estimate of $\lambda^*$. This gives some justification to the fact that the joint spectral radius in this case is available analytically [18]: $\rho^* = \frac{1+\sqrt{5}}{2}$. Thus $\lambda^* = 1/(\rho^*)^2 \simeq 0.3820$.

*Example 3:* Consider the following SLS in $\mathbb{R}^3$:

$$A_1 = \begin{bmatrix} 0.5 & 0 & -0.7 \\ 0 & 0.3 & 0 \\ 0 & -0.4 & -0.6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.4 & 0.2 & 0.3 \\ 0 & 0 & 0.3 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0.9 & 0.2 & 0.3 \\ -0.2 & 0.3 & -0.5 \end{bmatrix}.$$

Overestimates of the function $G_\lambda(z)$ are computed by applying Algorithm 1 on $75^2$ grid points of the unit sphere. The unit ball corresponding to the estimated norm $\sqrt{G_1(z)}$ is shown at the top of Fig. 4; the bottom figure depicts the computed $1/g_\lambda$ for $\lambda = 0.2$, 0.4, 0.6, 0.8, 1, and 1.1. Since $g_\lambda$ at $\lambda = 1.1$ is finite, $\lambda^* > 1.1$, hence the given SLS is exponentially stable under arbitrary switching. An extrapolation of the function $1/g_\lambda$ provides an estimate of $\lambda^*$ at around 1.1064.

### *H. Generalized Strong Generating Functions*

The definition of strong generating functions can be generalized. Let $q$ be a positive integer, and let $\|\cdot\|$ now denote an arbitrary norm on $\mathbb{R}^n$. Define a generalized strong generating function as $G_{\lambda,q}(z) := \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t;z,\sigma)\|^q$. When $q = 2$ and $\|\cdot\|$ is the Euclidean norm, $G_{\lambda,q}(z)$ reduces to $G_\lambda(z)$ defined in (3). The function $G_{\lambda,q}(z)$ retains most of the properties of $G_\lambda(z)$ in Proposition 1. For instance, for any $\lambda \geq 0$, $[G_{\lambda,q}(z)]^{1/q}$ is subadditive, positively homogeneous, and convex in $z$; it is further finite, globally Lipschitz continuous, and semismooth whenever $\lambda$ is smaller than

$$\lambda^{q,*} := \sup\{\lambda \geq 0 \,|\, G_{\lambda,q}(z) < \infty, \ \forall z \in \mathbb{R}^n\}.$$

We call $\lambda^{q,*}$ the radius of strong convergence corresponding to $G_{\lambda,q}(z)$, and note that its definition does not depend on the choice of the norm $\|\cdot\|$. Similar to Theorem 2, we can show that the SLS is exponentially stable under arbitrary switching if and only if $\lambda^{q,*} > 1$. Indeed, $\lambda^{q,*}$ is related to $\lambda^*$ and the JSR $\rho^*$ as: $(\lambda^{q,*})^{1/q} = (\lambda^*)^{1/2} = (\rho^*)^{-1}$, for all $q$.

Although different choices of $q$ and $\|\cdot\|$ lead to equivalent stability tests, the numerical robustness of such tests may vary. Noting that any nonnegative sequence $\{w_t\}_{t=0,1,\dots}$ satisfies $\sum_{t=0}^{\infty}(w_t)^{qr} \leq \left(\sum_{t=0}^{\infty}(w_t)^q\right)^r$, we have $\lambda^{qr,*} \geq (\lambda^{q,*})^r$, $\forall q, r = 1, 2, \dots$. For a barely exponentially stable SLS with a convergence radius $\lambda^{2,*}$ only slightly above 1, by choosing $r > 1$, the new radius $\lambda^{2r,*} \geq (\lambda^{2,*})^r$ has a larger gap with 1; hence it may lead to a more robust stability test.

## IV. Weak Generating Functions

### *A. Definition and Properties*

The *weak generating function* $H : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ of the SLS (1) is defined as

$$H(\lambda, z) := \inf_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t;z,\sigma)\|^2, \quad \forall \lambda \geq 0, \ z \in \mathbb{R}^n, \quad (17)$$

where the infimum is over all switching sequences $\sigma$ of the SLS. Then $H(\lambda, z)$ is monotonically increasing in $\lambda$, with $H(0, z) = \|z\|^2$ when $\lambda = 0$. The threshold

$$\lambda_*(z) := \sup\{\lambda \geq 0 \,|\, H(\lambda, z) < \infty\}$$

is called the *radius of weak convergence* of the SLS at $z$. For each fixed $\lambda \geq 0$, write $H_\lambda(z) := H(\lambda, z)$, $\forall z \in \mathbb{R}^n$. Then $H_\lambda(\cdot)$ is homogeneous of degree two, with $H_0(\cdot) = \|\cdot\|^2$.

Some properties of the function $H_\lambda(z)$ are listed below. It is noted that many properties (e.g. convexity) of the strong generating function $G_\lambda(z)$ are not valid for $H_\lambda(z)$.

*Proposition 7:* $H_\lambda(z)$ has the following properties.

1. (Bellman Equation): Let $\lambda \geq 0$ be arbitrary. Then $H_\lambda(z) = \|z\|^2 + \lambda \cdot \min_{i\in\mathcal{M}} H_\lambda(A_i z)$, $\forall z \in \mathbb{R}^n$.
2. (Invariant Subset): For each $\lambda \geq 0$, the set $\mathcal{H}_\lambda := \{z \in \mathbb{R}^n \,|\, H_\lambda(z) = \infty\}$ is a subset of $\mathbb{R}^n$ invariant under $\{A_i\}_{i\in\mathcal{M}}$, i.e., $A_i\mathcal{H}_\lambda \subseteq \mathcal{H}_\lambda$ for all $i \in \mathcal{M}$.
3. For $0 \leq \lambda < (\min_{i\in\mathcal{M}} \|A_i\|^2)^{-1}$, $H_\lambda(z) \leq c\|z\|^2$ for some finite constant $c > 1$.

*Proof:* Property 1 is proved by applying the dynamic programming principle to the optimal control problem of minimizing $\sum_{t=0}^{\infty} \lambda^t \|x(t;z,\sigma)\|^2$. Property 2 follows directly from Property 1. For Property 3, by choosing $\sigma_0 := (i_0, i_0, i_0, \dots)$

with no switching where $i_0 = \text{argmin}_{i\in\mathcal{M}}\|A_i\|$, we have $\|x(t;z,\sigma_0)\|^2 \le (\min_{i\in\mathcal{M}}\|A_i\|^2)^t\|z\|^2$, $\forall t$. Therefore, for $0 \le \lambda < (\min_{i\in\mathcal{M}}\|A_i\|^2)^{-1}$,

$$H_\lambda(z) \le \sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma_0)\|^2 \le \frac{1}{1-\lambda\cdot\min_{i\in\mathcal{M}}\|A_i\|^2}\|z\|^2,$$

which is finite and bounded by a quadratic function. ■

Note that $\mathcal{H}_\lambda$, unlike $\mathcal{G}_\lambda$, cannot be a subspace of $\mathbb{R}^n$ as it does not contain the origin.

### *B. Radius of Weak Convergence*

*Definition 4:* The radius of weak convergence of the SLS (1), denoted by $\lambda_* \in (0,\infty]$, is defined by

$$\lambda_* := \sup\{\lambda \ge 0 \,|\, H_\lambda(z) < \infty,\ \forall z \in \mathbb{R}^n\}.$$

By Proposition 7, we must have $\lambda_* \ge (\min_{i\in\mathcal{M}}\|A_i\|^2)^{-1}$. The value of $\lambda_*$ could reach $\infty$ if starting from any $z$, a switching sequence $\sigma_z$ exists so that $x(t;z,\sigma_z)$ reaches the origin within a uniform time $T<\infty$.

The next result shows that a function $H_\lambda(z)$ finite everywhere on $\mathbb{R}^n$ must be bounded by a quadratic function. Thus, $\lambda_*$ can also be defined to be $\sup\{\lambda \ge 0 \,|\, H_\lambda(z) \le c\|z\|^2,\ \forall z\in\mathbb{R}^n$, for some constant $c\}$.

*Proposition 8:* For each $\lambda \ge 0$, the following statements are equivalent:

1) $H_\lambda(z) < \infty$, $\forall z \in \mathbb{R}^n$;
2) $H_\lambda(z) \le c\|z\|^2$, $\forall z \in \mathbb{R}^n$, for some positive constant $c$;
3) The scaled SLS with subsystem matrices $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$ is exponentially stable under optimal switching.

*Proof:* It is obvious that 2) $\Rightarrow$ 1). We show in the following that 1) $\Rightarrow$ 3), and 3) $\Rightarrow$ 2).

To prove 1) $\Rightarrow$ 3), assume $\lambda \ge 0$ is such that 1) holds. Then for any $z\in\mathbb{R}^n$, there exists a switching sequence $\sigma_z$ such that $\sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma_z)\|^2 < \infty$. Consider the scaled SLS with subsystem dynamics matrices $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$, and denote by $\widetilde{x}(t;z,\sigma_z) = \lambda^{t/2}x(t;z,\sigma_z)$ its solution starting from $z$ under $\sigma_z$. Since $\sum_{t=0}^{\infty}\|\widetilde{x}(t;z,\sigma_z)\|^2 < \infty$, $\widetilde{x}(t;z,\sigma_z) \to 0$ as $t\to\infty$ for each $z$. By Remark 1, the scaled SLS is exponentially stable under optimal switching.

To show 3) $\Rightarrow$ 2), assume the scaled SLS is exponentially stable under optimal switching. Then there exist constants $\kappa > 0$ and $r\in(0,1)$ such that for each $z$, the trajectory $\widetilde{x}(t;z,\sigma_z)$ of the scaled SLS under at least one switching sequence $\sigma_z$ satisfies $\|\widetilde{x}(t;z,\sigma_z)\|^2 \le \kappa r^t\|z\|^2$, $\forall t$. As a result, $\forall z$,

$$H_\lambda(z) \le \sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma_z)\|^2 = \sum_{t=0}^{\infty}\|\widetilde{x}(t;z,\sigma_z)\|^2 \le \frac{\kappa\|z\|^2}{1-r},$$

which is exactly the conclusion of 2) with $c = \kappa/(1-r)$. ■

We next show that the radius of weak convergence $\lambda_*$ characterizes the exponential stability of the SLS under optimal switching just as $\lambda^*$ does for the exponential stability under arbitrary switching.

*Theorem 4:* The SLS (1) is exponentially stable under optimal switching if and only if $\lambda_* > 1$.

*Proof:* To prove necessity, we observe that for a SLS exponentially stable under optimal switching with the parameters $\kappa > 0$ and $r\in[0,1)$, by a similar argument as in the proof of Theorem 2, we must have $H_\lambda(z) \le \frac{\kappa^2}{1-\lambda r^2}\|z\|^2$, $\forall z$, for $\lambda < r^{-2}$; hence $\lambda_* \ge r^{-2} > 1$. To show sufficiency, suppose $\lambda_* > 1$. Then at $\lambda = 1$, $H_1(z)$ is finite for all $z$. By Proposition 8, this implies that the scaled SLS with subsystem matrices $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$, which is also the original SLS (1), is exponentially stable under optimal switching. ■

Following similar steps as in the proof of Corollary 1, we can prove the subsequent result.

*Corollary 4:* Consider the SLS (1). For any $r > (\lambda_*)^{-1/2}$, there is a constant $\kappa_r$ such that starting from each $z\in\mathbb{R}^n$, $\|x(t;z,\sigma_z)\| \le \kappa_r r^t\|z\|$, $\forall t$, for at least one switching sequence $\sigma_z$. Furthermore, $(\lambda_*)^{-1/2}$ is the smallest possible value for the previous statement to be true.

To sum up, the maximum exponential growth rate over all initial states of the trajectories of the SLS (1) under optimal switching is given by precisely $(\lambda_*)^{-1/2}$. Another related (but generally larger) quantity characterizing such a rate is the *joint spectral subradius* of $\{A_i\}_{i\in\mathcal{M}}$ defined by [17]:

$$\check{\rho} := \lim_{k\to\infty}\inf\left\{\|A_{i_1}\cdots A_{i_k}\|^{1/k},\ i_1,\ldots,i_k\in\mathcal{M}\right\}. \quad (18)$$

Difference between these two rates is shown via the following SLS (inspired from [42, pp. 1135]) with subsystem matrices:

$$A_1 = \begin{bmatrix}\frac{2}{3} & 0\\ 0 & \frac{3}{2}\end{bmatrix},\quad A_2 = QA_1Q^T,\quad A_3 = QA_2Q^T,$$

where $Q\in\mathbb{R}^{2\times 2}$ is the rotation matrix of $60^\circ$ counterclockwise. Let $\mathcal{C}$ be the symmetric cone consisting of all those $x\in\mathbb{R}^2\simeq\mathbb{C}$ with phase angle within the range of either $[-30^\circ,30^\circ]$ or $[150^\circ,210^\circ]$. It is easy to check that $\|A_1x\| \le \sqrt{\frac{43}{48}}\|x\|$ for $x\in\mathcal{C}$; $\|A_2x\| \le \sqrt{\frac{43}{48}}\|x\|$ for $x\in Q\mathcal{C}$; and $\|A_3x\| \le \sqrt{\frac{43}{48}}\|x\|$ for $x\in Q^2\mathcal{C}$. Since $\mathbb{R}^2 = \mathcal{C}\cup Q\mathcal{C}\cup Q^2\mathcal{C}$, a state feedback switching policy can be designed as follows: $\sigma(x) = 1$, 2, and 3, for $x$ in $\mathcal{C}$, $Q\mathcal{C}$, and $Q^2\mathcal{C}$, respectively (if $x$ is in more than one set, any tie breaking rule can be applied). Then starting from any $z$, the resulting closed-loop trajectory satisfies $\|x(t+1;z,\sigma)\| \le \sqrt{\frac{43}{48}}\cdot\|x(t;z,\sigma)\|$ for all $t$. Thus, the SLS is exponentially stable under $\sigma(x)$, and we have $(\lambda_*)^{-1/2} \le \sqrt{\frac{43}{48}}$, i.e., $\lambda_* > \frac{48}{43}$. In comparison, since each of the subsystem matrices has determinant one, any finite product $A_{i_1}\cdots A_{i_k}$ also has determinant one, hence norm at least one. By (18), we must have $\check{\rho}\ge 1$, hence $\check{\rho} > (\lambda_*)^{-1/2}$. This gap can be explained as $\check{\rho}$ defined by (18) satisfies

$$\begin{aligned}\check{\rho} &= \lim_{k\to\infty}\inf_{i_1,\ldots,i_k\in\mathcal{M}}\sup_{\|x\|=1}\|A_{i_1}\cdots A_{i_k}x\|^{1/k}\\ &\ge \lim_{k\to\infty}\sup_{\|x\|=1}\inf_{i_1,\ldots,i_k\in\mathcal{M}}\|A_{i_1}\cdots A_{i_k}x\|^{1/k},\end{aligned}$$

where the right hand side is exactly $(\lambda_*)^{-1/2}$. In other words, the JSS $\check{\rho}$ and the rate $(\lambda_*)^{-1/2}$ studied in this paper characterize the smallest possible worst-case exponential growth rate over all initial states of the SLS trajectories achievable by *open-loop* switching policies of the form $(i_1,\ldots,i_k,i_1,\ldots,i_k,\ldots)$ and by *closed-loop* switching policies with state feedback switching laws, respectively.

### C. Quadratic Bounds of Finite $H_\lambda(z)$

For each $\lambda \in [0, \lambda_*)$, define the constant $h_\lambda$ as the smallest constant $c$ such that $H_\lambda(z) \le c\|z\|^2$ for all $z$:

$$h_\lambda := \sup\{H_\lambda(z) \mid \|z\| = 1\}. \quad (19)$$

Obviously, $h_\lambda$ is strictly increasing in $\lambda$, with $h_0 = 1$.

Similar to Proposition 3 for $g_\lambda$, we have the following estimates of $h_\lambda$.

*Proposition 9:* The function $\lambda/(1-1/h_\lambda)$ is nondecreasing for $\lambda \in (0, \lambda_*)$, and is upper bounded by

$$\frac{\lambda}{1-1/h_\lambda} \le \lambda_*, \quad \forall \lambda \in (0, \lambda_*). \quad (20)$$

*Proof:* Let $\lambda_0 \in (0, \lambda_*)$. For any $z \in \mathbb{R}^n$, let $\sigma_z$ be a switching sequence so that the resulting trajectory $x(t; z, \sigma_z)$ achieves the infimum in (17): $H_{\lambda_0}(z) = \sum_{t=0}^{\infty} \lambda_0^t \|x(t; z, \sigma_z)\|^2 < \infty$. By the optimality of $\sigma_z$, for each $s = 0, 1, \ldots$, the time shifted trajectory of the SLS, $x(t+s; z, \sigma_z)$, $\forall t$, starting from the initial state $x(s; z, \sigma_z)$ is also optimal: $\sum_{t=0}^{\infty} \lambda_0^t \|x(t+s; z, \sigma_z)\|^2 = H_{\lambda_0}(x(s; z, \sigma_z)) \le h_{\lambda_0}\|x(s; z, \sigma_z)\|^2$. In other words, the sequence defined by $\{w_t := \|x(t; z, \sigma_z)\|^2\}_{t=0,1,\ldots}$ satisfies the condition (9) with $\beta = h_{\lambda_0}$. By Lemma 3, we then have

$$\sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma_z)\|^2 \le \frac{h_{\lambda_0}}{1-(h_{\lambda_0}-1)(\lambda-\lambda_0)/\lambda_0}\|z\|^2,$$

for $\lambda \in [\lambda_0, \lambda_1)$, where $\lambda_1 := \lambda_0/(1-1/h_{\lambda_0})$. Therefore,

$$H_\lambda(z) \le \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma_z)\|^2 \le \frac{h_{\lambda_0}\|z\|^2}{1-(h_{\lambda_0}-1)(\lambda-\lambda_0)/\lambda_0}$$

is finite for all $z \in \mathbb{R}^n$ and $\lambda \in [\lambda_0, \lambda_1)$. This implies that $\lambda_* \ge \lambda_1$, which is the desired (20); and that for $\lambda \in [\lambda_0, \lambda_1)$,

$$h_\lambda \le \frac{h_{\lambda_0}}{1-(h_{\lambda_0}-1)(\lambda-\lambda_0)/\lambda_0} \;\Rightarrow\; \frac{\lambda}{1-1/h_\lambda} \ge \frac{\lambda_0}{1-1/h_{\lambda_0}}.$$

This proves the monotonicity of $\lambda/(1-1/h_\lambda)$ on $(0, \lambda_*)$. ■

The following result follows directly from Proposition 9.

*Corollary 5:* For each $\lambda \in [0, \lambda_*)$, $1/h_\lambda \le 1 - \lambda/\lambda_*$. As a result, $1/h_\lambda \to 0$ and $h_\lambda \to \infty$ as $\lambda \uparrow \lambda_*$.

The directional derivative of $h_\lambda$ at $\lambda = 0$ is computed in Appendix C as follows.

*Lemma 5:* The directional derivative of $h_\lambda$ at 0 exists and is given by $h'_\lambda(0_+) := \lim_{\lambda \downarrow 0} \frac{h_\lambda - h_0}{\lambda} = \sup_{\|z\|=1} \min_{i \in \mathcal{M}} \|A_i z\|^2$.

As a result of Lemma 5,

$$\lim_{\lambda \downarrow 0} \frac{\lambda}{1-1/h_\lambda} = \frac{1}{h'_\lambda(0_+)} = \frac{1}{\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2}.$$

Since by Proposition 9, $\lambda/(1-1/h_\lambda)$ is nondecreasing in $\lambda$ on $(0, \lambda_*)$, the above limit implies that, for $\lambda \in (0, \lambda_*)$,

$$\frac{\lambda}{1-1/h_\lambda} \ge \lim_{\lambda \downarrow 0} \frac{\lambda}{1-1/h_\lambda} = \frac{1}{\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2}.$$

This leads to the following estimate of $h_\lambda$, which is the counterpart of Lemma 2 for $g_\lambda$.

*Corollary 6:* $\frac{1}{h_\lambda} \ge 1 - \lambda \cdot \sup_{\|z\|=1} \min_{i \in \mathcal{M}} \|A_i z\|^2$, $\forall \lambda \in [0, \lambda_*)$.

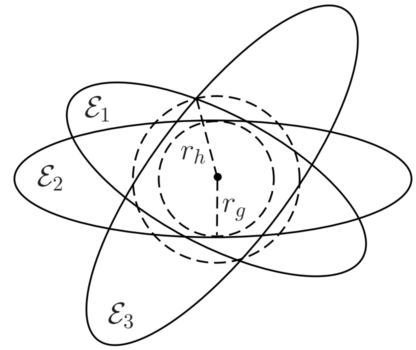


Fig. 5. Geometric interpretation of the Lipschitz constants in Propositions 4 and 10: $\max_{i\in\mathcal{M}} \|A_i\|^2 = 1/r_g^2$ and $\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2 = 1/r_h^2$.

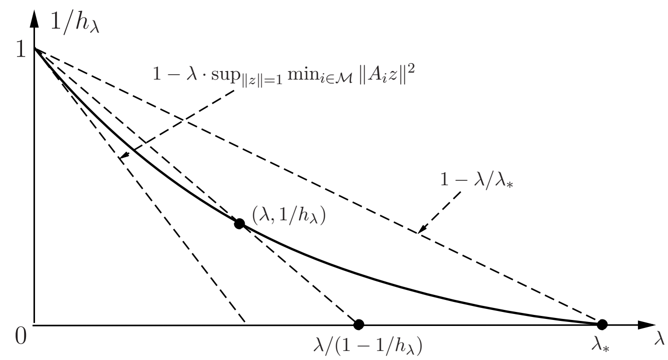


Fig. 6. Plot of the function $1/h_\lambda$ (in solid line).

Similar to the proof of Proposition 4 in Appendix B, we can use Proposition 9 and Corollary 6 to prove the following.

*Proposition 10:* The function $1/h_\lambda$ defined on $[0, \lambda_*)$ is strictly decreasing and Lipschitz continuous with Lipschitz constant $\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2$. The function $h_\lambda$ is strictly increasing and locally Lipschitz continuous on $[0, \lambda_*)$.

*Remark 6:* The Lipschitz constants of $1/g_\lambda$ in Proposition 4 and of $1/h_\lambda$ in Proposition 10 are given as $\sup_{\|z\|=1} \max_{i\in\mathcal{M}} \|A_i z\|^2$ and $\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2$, respectively. An overestimate of the latter is given by $\min_{i\in\mathcal{M}} \sup_{\|z\|=1} \|A_i z\|^2 = \min_{i\in\mathcal{M}} \|A_i\|^2$. For a geometric interpretation, associate each matrix $A_i$ with an ellipsoid $\mathcal{E}_i := \{z \in \mathbb{R}^n \mid \|A_i z\|^2 \le 1\}$ in $\mathbb{R}^n$. The intersection of all such ellipsoids, $\cap_{i\in\mathcal{M}} \mathcal{E}_i$, is a convex set that can embed a maximal ball centered at 0 with the radius $r_g$; whereas their union $\cup_{i\in\mathcal{M}} \mathcal{E}_i$ can embed a maximal ball centered at 0 with the radius $r_h$ (see Fig. 5). Further, let $r_\ell := \max_{i\in\mathcal{M}}\{$the length of the shortest principal axis of $\mathcal{E}_i\}$. The two Lipschitz constants and the overestimate can then be expressed as $\max_{i\in\mathcal{M}} \|A_i\|^2 = 1/r_g^2$, $\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2 = 1/r_h^2$, and $\min_{i\in\mathcal{M}} \|A_i\|^2 = 1/r_\ell^2$.

A generic plot of the function $1/h_\lambda$ for $\lambda \in [0, \lambda_*)$ is shown in Fig. 6. The function decreases strictly from 1 at $\lambda = 0$ to 0 as $\lambda \uparrow \lambda_*$. Its graph is sandwiched by those of two affine functions: $1 - \lambda/\lambda_*$ from the right, and $1 - \lambda \cdot \sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2$ from the left. Moreover, as $\lambda$ increases from 0 towards $\lambda_*$, the ray emitting from the point $(1, 0)$ and passing through the point $(\lambda, 1/h_\lambda)$

rotates counterclockwise monotonically, and intersects the $\lambda$-axis at a point whose $\lambda$-coordinate, $\lambda/(1-1/h_\lambda)$, provides an asymptotically tight underestimate of $\lambda_*$.

### D. Approximating Finite $H_\lambda(z)$

For each $\lambda \in [0, \lambda_*)$, the function $H_\lambda(z)$ is finite everywhere on $\mathbb{R}^n$. We next show that it is the limit of a sequence of functions $H_\lambda^k(z)$, $k = 0, 1, \ldots$, defined by

$$H_\lambda^k(z) := \min_\sigma \sum_{t=0}^{k} \lambda^t \|x(t; z, \sigma)\|^2, \quad \forall z \in \mathbb{R}^n. \tag{21}$$

As the value functions of an optimal control problem with variable finite horizons, $H_\lambda^k(z)$ can be computed recursively as follows: $H_\lambda^0(z) = \|z\|^2$, $\forall z \in \mathbb{R}^n$; and for $k = 1, 2, \ldots$,

$$H_\lambda^k(z) = \|z\|^2 + \lambda \cdot \min_{i \in \mathcal{M}} H_\lambda^{k-1}(A_i z), \quad \forall z \in \mathbb{R}^n. \tag{22}$$

Equivalently, we can write

$$H_\lambda^k(z) = \min\{z^T P z : \ P \in \mathcal{P}_k\}, \quad \forall z \in \mathbb{R}^n, \tag{23}$$

where $\mathcal{P}_k$, $k = 0, 1, \ldots$, is a sequence of sets of positive definite matrices defined by: $\mathcal{P}_0 = \{I\}$; and for $k = 1, 2, \ldots$,

$$\mathcal{P}_k = \{I + \lambda A_i^T P A_i \,|\, P \in \mathcal{P}_{k-1}, \, i \in \mathcal{M}\}. \tag{24}$$

*Proposition 11:* The sequence of functions $H_\lambda^k(z)$ is monotonic: $H_\lambda^0 \le H_\lambda^1 \le H_\lambda^2 \le \cdots \le H_\lambda$; and for $\lambda \in [0, \lambda_*)$, it converges exponentially fast to $H_\lambda(z)$: for $k = 0, 1, \ldots$,

$$|H_\lambda^k(z) - H_\lambda(z)| \le h_\lambda^2 (1 - 1/h_\lambda)^{k+1} \|z\|^2, \ \forall z \in \mathbb{R}^n. \tag{25}$$

*Proof:* Fix $k \ge 1$. For each $z \in \mathbb{R}^n$, let $\sigma_k$ be a switching sequence achieving the minimum in (21). Then, $H_\lambda^k(z) = \sum_{t=0}^{k} \lambda^t \|x(t; z, \sigma_k)\|^2 \ge \sum_{t=0}^{k-1} \lambda^t \|x(t; z, \sigma_k)\|^2 \ge H_\lambda^{k-1}(z)$. Similarly, we have $H_\lambda^k(z) \le H_\lambda(z)$, proving the monotonicity.

Next assume $\lambda \in [0, \lambda_*)$. For any $z \in \mathbb{R}^n$ and $k = 0, 1, \ldots$, we note that $\|z\|^2 \le H_\lambda^k(z) \le H_\lambda(z) \le h_\lambda \|z\|^2$. Let $\sigma_k$ be such that $\hat{x}(t) := x(t; z, \sigma_k)$ achieves the minimum in (21). For each $s = 0, 1, \ldots, k-1$, since $\hat{x}(t)$ is also optimal over the time horizon $t = s, s+1, \ldots, k$, we have

$$\begin{aligned} H_\lambda^{k-s}(\hat{x}(s)) &= \|\hat{x}(s)\|^2 + \lambda \sum_{t=0}^{k-s-1} \lambda^t \|\hat{x}(t+s+1)\|^2 \\ &= \|\hat{x}(s)\|^2 + \lambda H_\lambda^{k-s-1}(\hat{x}(s+1)). \end{aligned}$$

Since $\|\hat{x}(s)\|^2 \ge H_\lambda^{k-s}(\hat{x}(s))/h_\lambda$, the above implies that

$$H_\lambda^{k-s-1}(\hat{x}(s+1)) \le \lambda^{-1}(1 - 1/h_\lambda) H_\lambda^{k-s}(\hat{x}(s)).$$

Applying this inequality for $s = k-1, \ldots, 0$, we have

$$\begin{aligned} &\|\hat{x}(k)\|^2 = H_\lambda^0(\hat{x}(k)) \le \lambda^{-1}(1 - 1/h_\lambda) H_\lambda^1(\hat{x}(k-1)) \\ &\le \cdots \le \lambda^{-k}(1 - 1/h_\lambda)^k H_\lambda^k(z) \le \lambda^{-k} h_\lambda (1 - 1/h_\lambda)^k \|z\|^2. \end{aligned}$$

**Algorithm 2** Computing Over Approximations of $H_\lambda^k(z)$.

Initialize $k = 0$, $\tilde{\mathcal{P}}_0^\varepsilon = \{I_n\}$, and $\mathcal{P}_0^\varepsilon = \tilde{\mathcal{P}}_0^\varepsilon$;
**repeat**
  $k \leftarrow k + 1$;
  $\tilde{\mathcal{P}}_k^\varepsilon = \{I + \lambda A_i^T P A_i \,|\, i \in \mathcal{M}, \, P \in \mathcal{P}_{k-1}^\varepsilon\}$;
  Find an $\varepsilon$-equivalent subset $\mathcal{P}_k^\varepsilon \subset \tilde{\mathcal{P}}_k^\varepsilon$;
**until** $k$ is large enough
**return** $H_\lambda^{k,\varepsilon}(z) = \min\{z^T P z \,|\, P \in \mathcal{P}_k^\varepsilon\}$.

Using this inequality, and adopting the switching sequence that first follows $\sigma_k$ for $k$ steps and thereafter follows an infinite-horizon optimal $\sigma_*$ starting from $\hat{x}(k)$, we obtain

$$\begin{aligned} H_\lambda(z) &\le \sum_{t=0}^{k} \lambda^t \|\hat{x}(t)\|^2 + \sum_{t=k+1}^{\infty} \lambda^t \|x(t-k; \hat{x}(k), \sigma_*)\|^2 \\ &= H_\lambda^k(z) + \lambda^k \left[H_\lambda(\hat{x}(k)) - \|\hat{x}(k)\|^2\right] \\ &\le H_\lambda^k(z) + (h_\lambda - 1)\lambda^k \|\hat{x}(k)\|^2 \\ &\le H_\lambda^k(z) + h_\lambda^2 (1 - 1/h_\lambda)^{k+1} \|z\|^2. \end{aligned}$$

This, together with $H_\lambda^k(z) \le H_\lambda(z)$, proves (25). ■

By (25), for any $\lambda_0 \in [0, \lambda_*)$, $H_\lambda^k(z)$, $k = 0, 1, \ldots$, is a sequence of functions of $(\lambda, z)$ converging uniformly to $H_\lambda(z)$ on $[0, \lambda_0] \times \mathbb{S}^{n-1}$. As each $H_\lambda^k(z)$ is continuous in $(\lambda, z)$ on $[0, \lambda_0] \times \mathbb{S}^{n-1}$, so is $H_\lambda(z)$. Using homogeneity and the arbitrariness of $\lambda_0 \in [0, \lambda_*)$, we obtain the following result.

*Corollary 7:* The function $H_\lambda(z) = H(\lambda, z)$ is continuous in $(\lambda, z)$ on $[0, \lambda_*) \times \mathbb{R}^n$.

### E. Relaxation Algorithm for Computing $H_\lambda(z)$

According to (25), $H_\lambda^k(z)$ for large $k$ provide increasingly accurate estimates of $H_\lambda(z)$. By (23), to characterize $H_\lambda^k(z)$, it suffices to compute the set $\mathcal{P}_k$. To deal with the rapidly increasing size of $\mathcal{P}_k$ as $k$ increases, we introduce the following complexity reduction technique, inspired by [43], [44]. A subset $\mathcal{P}_k^\varepsilon \subseteq \mathcal{P}_k$ is called *ε-equivalent* to $\mathcal{P}_k$ for some $\varepsilon > 0$ if

$$H_\lambda^{k,\varepsilon}(z) := \min_{P \in \mathcal{P}_k^\varepsilon} z^T P z \le \min_{P \in \mathcal{P}_k} z^T P z + \varepsilon \|z\|^2, \ \forall z \in \mathbb{R}^n.$$

A sufficient (though not necessary) condition for this to hold is that, for each $P \in \mathcal{P}_k$, $P + \varepsilon I_n$ is bounded from below by a convex combination of matrices in $\mathcal{P}_k^\varepsilon$, i.e., there exist constants $\alpha_Q \ge 0$, $\forall Q \in \mathcal{P}_k^\varepsilon$, adding up to 1 such that $P + \varepsilon I_n \succeq \sum_{Q \in \mathcal{P}_k^\varepsilon} \alpha_Q \cdot Q$. This leads to a natural procedure of removing matrices from $\mathcal{P}_k$ iteratively until a minimal $\varepsilon$-equivalent subset $\mathcal{P}_k^\varepsilon$ is achieved. By applying this procedure at each step of the iteration (24), we obtain Algorithm 2, which returns approximations of $H_\lambda^k(z)$ for all $k$ with uniformly bounded approximation errors as follows.

*Proposition 12:* Assume $\lambda \in [0, \lambda_*)$. Then for $k = 0, 1, \ldots$,

$$H_\lambda^k(z) \le H_\lambda^{k,\varepsilon}(z) \le (1 + \varepsilon) H_\lambda^k(z), \quad \forall z \in \mathbb{R}^n. \tag{26}$$

*Proof:* Obviously, (26) holds for $k = 0$. Assume it holds for some $k - 1 \ge 0$. Define, for $z \in \mathbb{R}^n$,

$$\tilde{H}_\lambda^{k,\varepsilon}(z) := \min_{P \in \tilde{\mathcal{P}}_k^\varepsilon} z^T P z = \|z\|^2 + \lambda \cdot \min_{i \in \mathcal{M}} H_\lambda^{k-1,\varepsilon}(A_i z).$$

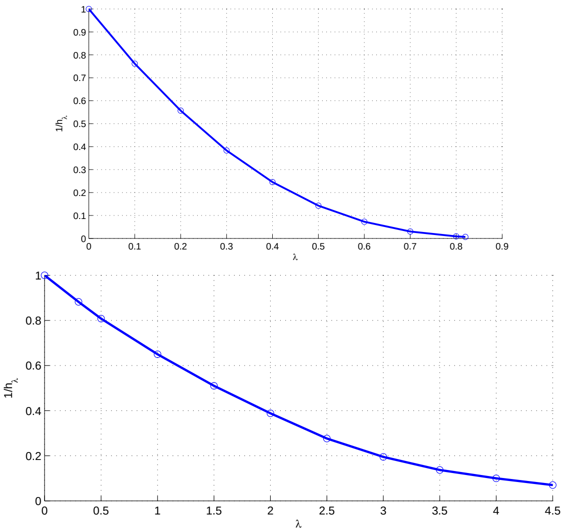


Fig. 7. Plots of $1/h_\lambda$ for SLSs in Example 2 (top) and Example 3 (bottom).

By our hypothesis, $H_\lambda^{k-1,\varepsilon}(A_i z) \le (1+\varepsilon) H_\lambda^{k-1}(A_i z)$. Thus,

$$\tilde{H}_\lambda^{k,\varepsilon}(z) \le \|z\|^2 + \lambda(1+\varepsilon) H_\lambda^{k-1}(A_i z), \ \forall z \in \mathbb{R}^n, \ i \in \mathcal{M}.$$

Since $\mathcal{P}_k^\varepsilon$ is $\varepsilon$-equivalent to $\tilde{\mathcal{P}}_k^\varepsilon$, we then have,

$$H_\lambda^{k,\varepsilon}(z) \le \tilde{H}_\lambda^{k,\varepsilon}(z) + \varepsilon\|z\|^2 \le (1+\varepsilon)\big[\|z\|^2 + \lambda H_\lambda^{k-1}(A_i z)\big],$$

for each $i \in \mathcal{M}$. By (22), we have $H_\lambda^{k,\varepsilon}(z) \le (1+\varepsilon) H_\lambda^k(z)$, $\forall z$. That $H_\lambda^{k,\varepsilon}(z) \ge H_\lambda^k(z)$ can be trivially proved. ■

The estimated $1/h_\lambda$ obtained by Algorithm 2 are plotted in Fig. 7 for the SLSs in Example 2 (top) and Example 3 (bottom). In both cases, $1/h_\lambda$ decreases from 1 at $\lambda = 0$ to 0 at $\lambda = \lambda_*$. (For Example 3, high computational complexity of Algorithm 2 prevents us from getting accurate estimates for $\lambda$ close to $\lambda_*$). We observe that in each case, $1/h_\lambda$ is convex and "more curved" than the plots of $1/g_\lambda$. It is conjectured that the function $1/h_\lambda$ is convex on $[0, \lambda_*)$ for all SLSs.

According to Propositions 11 and 12, by choosing $k$ large enough and $\varepsilon$ small enough, Algorithm 2 can return estimates of $H_\lambda(z)$, hence $\lambda_*$, with an error as small as possible. See [44] for the expressions of the approximation error in terms of $k$, $\varepsilon$, and matrices $A_i$'s. On the other hand, to attain a higher accuracy, the computation time of Algorithm 2 still grows exponentially, as is reflected by the rapidly increasing size of the set $P_k^\varepsilon$, despite our relaxation effort. Our numerical experiments suggest that computing $\lambda_*$ is no less challenging than computing the joint spectral subradius, which in itself is an NP-hard problem [20], [24]. A future direction of our research is to prove this formally and to see if the algorithms for computing the joint spectral subradius (e.g. [29]) can be adapted to compute the radius of weak convergence $\lambda_*$.

## V. Mean Generating Functions

The notion of generating functions can also be extended to the SLS (2) under the random switching probability $p$. Define its *mean generating function* $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ as

$$F(\lambda, z) := \mathbf{E}\left[\sum_{t=0}^\infty \lambda^t \|\mathbf{x}(t; z, p)\|^2\right] = \sum_{t=0}^\infty \lambda^t\, \mathbf{E}\left[\|\mathbf{x}(t; z, p)\|^2\right], \tag{27}$$

for $z \in \mathbb{R}^n$, $\lambda \ge 0$. For each $\lambda \ge 0$, $F_\lambda(z) := F(\lambda, z)$ is the averaged sum of the power series along the random system trajectory; thus, its value lies between two extremes:

$$H_\lambda(z) \le F_\lambda(z) \le G_\lambda(z), \quad \forall z \in \mathbb{R}^n. \tag{28}$$

The absence of maximum or minimum in the definition of $F_\lambda(z)$ makes its characterization much easier compared to $G_\lambda(z)$ and $H_\lambda(z)$. For example, $F_\lambda(z)$ is quadratic in $z \in \mathbb{R}^n$:

$$F_\lambda(z) = z^T \mathbf{E}\Big[I + \sum_{t=1}^\infty \lambda^t \prod_{s=0}^{t-1} \mathbf{A}(s)^T \prod_{s=t-1}^{0} \mathbf{A}(s)\Big] z. \tag{29}$$

We write $F_\lambda(z) = z^T Q_\lambda z$, where $Q_\lambda \in \mathbb{R}^{n\times n}$ is positive definite (with possibly infinite entries) and increasing in $\lambda$.

The function $F_\lambda(z)$ has a similar set of properties as $G_\lambda(z)$.

*Proposition 13:* $F_\lambda(z)$ has the following properties.

1. (Bellman Equation): Let $\lambda \ge 0$ be arbitrary. Then $F_\lambda(z) = \|z\|^2 + \lambda \sum_{i\in\mathcal{M}} p_i\, F_\lambda(A_i z)$, $\forall z \in \mathbb{R}^n$.
2. (Sub-Additivity and Convexity): For any $\lambda \ge 0$, $\sqrt{F_\lambda(z_1+z_2)} \le \sqrt{F_\lambda(z_1)} + \sqrt{F_\lambda(z_1)}$, $\forall z_1, z_2 \in \mathbb{R}^n$. As a result, $\sqrt{F_\lambda(z)}$ is a convex function of $z$ on $\mathbb{R}^n$.
3. (Invariant Subspace): For each $\lambda \ge 0$, the set $\mathcal{F}_\lambda := \{z \in \mathbb{R}^n \,|\, F_\lambda(z) < \infty\}$ is a subspace of $\mathbb{R}^n$ invariant under $\{A_i\}_{i\in\mathcal{M}'}$, where $\mathcal{M}' := \{i \in \mathcal{M} \,|\, p_i > 0\}$.
4. For any $\lambda \ge 0$, if $F_\lambda(z)$ is finite for all $z$, then $F_\lambda(z) \le c\|z\|^2$ for some finite constant $c$.

The proof of Proposition 13 is straightforward, hence omitted.

The radius of convergence of $F_\lambda(z)$ is defined as

$$\lambda_p^* := \sup\{\lambda \ge 0 \,|\, F_\lambda(z) < \infty, \ \forall z \in \mathbb{R}^n\},$$

which depends on the probability distribution $p$. By (27), $\lambda_p^*$ is the minimal radius of convergence of the deterministic series $\big\{\mathbf{E}\big[\|\mathbf{x}(t; z, p)\|^2\big]\big\}_{t=0,1,\ldots}$ over all $z \in \mathbb{R}^n$.

*Theorem 5:* The following statements are equivalent:

1) The SLS (2) under the random switching probability $p$ is mean square exponentially stable;
2) Its mean generating function $F_\lambda(z)$ has a radius of convergence $\lambda_p^* > 1$;
3) The mean generating function at $\lambda = 1$, $F_1(z)$, is finite everywhere on $\mathbb{R}^n$.

*Proof:* To show 1) $\Rightarrow$ 2), suppose the SLS (2) is mean square exponentially stable, i.e., there exist $\kappa > 0$ and $r \in [0, 1)$ such that $\mathbf{E}\left[\|\mathbf{x}(t; z, p)\|^2\right] \le \kappa r^t \|z\|^2$, $\forall t$, $\forall z$. Then,

$$F_\lambda(z) \le \sum_{t=0}^\infty \lambda^t \kappa r^t \|z\|^2 \le \frac{\kappa}{1-\lambda r}\|z\|^2 < \infty, \ \forall z \in \mathbb{R}^n,$$

whenever $0 \le \lambda < r^{-1}$. Consequently, $\lambda_p^* \ge r^{-1} > 1$. That 2) $\Rightarrow$ 3) is obvious. Finally, to show 3) $\Rightarrow$ 1), we note that $F_1(z) = \sum_{t=0}^\infty \mathbf{E}\left[\|\mathbf{x}(t; z, p)\|^2\right] < \infty$ implies that $\mathbf{E}\left[\|\mathbf{x}(t; z, p)\|^2\right] \to 0$ as $t \to \infty$, for all $z$. It follows that the SLS (2) is mean square asymptotically stable, hence mean square exponentially stable by [36, Theorem 4.1.1]. ■

For $\lambda \in [0, \lambda_p^*)$, $F_\lambda(\cdot)$ is finite everywhere. Define

$$f_\lambda := \sup\{F_\lambda(z) \mid \|z\| = 1\} < \infty, \quad \lambda \in [0, \lambda_p^*). \tag{30}$$

Then, $f_\lambda = \sigma_{\max}(Q_\lambda)$, the largest eigenvalue of $Q_\lambda$ in (29).

*Proposition 14:* For $\lambda \in (0, \lambda_p^*)$, the function $\lambda/(1-1/f_\lambda)$ is nondecreasing and upper bounded by $\lambda/(1-1/f_\lambda) \le \lambda_p^*$.

*Proof:* Let $\lambda_0 \in (0, \lambda_p^*)$. For each $z \in \mathbb{R}^n$, define the sequence $w_t := \mathbf{E}\left[\|\mathbf{x}(t; z, p)\|^2\right]$, $t = 0, 1, \ldots$, which satisfies

$$\sum_{t=0}^\infty w_{t+s}\lambda_0^t = \mathbf{E}\left[\sum_{t=0}^\infty \lambda_0^t \|\mathbf{x}(s+t; z, p)\|^2\right] = \mathbf{E}[F_{\lambda_0}(\mathbf{x}(s; z, p))]$$
$$\le \mathbf{E}\left[f_{\lambda_0}\|\mathbf{x}(s; z, p)\|^2\right] = f_{\lambda_0} w_s, \ \forall s = 0, 1, \ldots.$$

Applying Lemma 3 with $\beta = f_{\lambda_0}$, we conclude that $F_\lambda(z) = \sum_{t=0}^\infty w_t \lambda^t$ is finite for $\lambda \in [\lambda_0, \lambda_1)$, where $\lambda_1 := \lambda_0/(1 - 1/f_{\lambda_0})$. Therefore, $\lambda_p^* \ge \lambda_1$, which is the second conclusion. That $\lambda/(1-1/f_\lambda)$ is nondecreasing is proved in exactly the same way as in Proposition 9 (with $h_\lambda$ replaced by $f_\lambda$). ■

As a result, we have the following estimate.

*Corollary 8:* For each $\lambda \in [0, \lambda_p^*)$, $1/f_\lambda \le 1 - \lambda/\lambda_p^*$. Hence, $1/f_\lambda \to 0$ and $f_\lambda \to \infty$ as $\lambda \uparrow \lambda_p^*$.

We next compute the directional derivative of $f_\lambda$ at $\lambda = 0$.

*Lemma 6:* The directional derivative of $f_\lambda$ at 0 exists and is given by $f'_\lambda(0_+) := \lim_{\lambda \downarrow 0} \dfrac{f_\lambda - f_0}{\lambda} = \sigma_{\max}\left(\sum_{i \in \mathcal{M}} p_i A_i^T A_i\right)$.

*Proof:* Let $z \in \mathbb{S}^{n-1}$ be arbitrary. For small $\lambda > 0$, we can use (27) to write $\frac{F_\lambda(z)-1}{\lambda} = z^T\left(\sum_{i\in\mathcal{M}} p_i A_i^T A_i\right) z + O(\lambda)$, where $O(\lambda) \ge 0$ is uniform in $z$. Therefore, we can exchange the order of the limit and supremum below to obtain

$$f'_\lambda(0_+) = \lim_{\lambda \downarrow 0} \sup_{\|z\|=1} \frac{F_\lambda(z) - 1}{\lambda} = \sup_{\|z\|=1} \lim_{\lambda \downarrow 0} \frac{F_\lambda(z) - 1}{\lambda},$$

which is exactly $\sup_{\|z\|=1} z^T\left(\sum_{i\in\mathcal{M}} p_i A_i^T A_i\right) z$. ■

Similar to Corollary 6 and Proposition 10, we can show the following two results.

*Corollary 9:* $\dfrac{1}{f_\lambda} \ge 1 - \lambda\, \sigma_{\max}\left(\sum_{i\in\mathcal{M}} p_i A_i^T A_i\right), \forall \lambda \in [0, \lambda_p^*)$.

*Proposition 15:* The function $1/f_\lambda$ defined on $[0, \lambda_p^*)$ is strictly decreasing and Lipschitz continuous with Lipschitz constant $\sigma_{\max}\left(\sum_{i\in\mathcal{M}} p_i A_i^T A_i\right)$. Hence, the function $f_\lambda$ is strictly increasing and locally Lipschitz continuous on $[0, \lambda_p^*)$.

From (28), we have $h_\lambda \le f_\lambda \le g_\lambda$, thus $1/g_\lambda \le 1/f_\lambda \le 1/h_\lambda$. The Lipschitz constants of the functions $1/h_\lambda$, $1/f_\lambda$, and $1/g_\lambda$ in Propositions 10, 15, and 4, respectively, satisfy

$$\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2 \ \le\ \sigma_{\max}\left(\sum_{i\in\mathcal{M}} p_i A_i^T A_i\right) \ \le\ \max_{i\in\mathcal{M}} \|A_i\|^2.$$

By setting the probability distribution $p$ to be $p_{i_*} = 1$ for $i_* = \arg\max_{i\in\mathcal{M}} \|A_i\|^2$ and $p_i = 0$ for $i \ne i_*$, the second inequality becomes equality. On the other hand, the first inequality is in general strict regardless of the choice of $p$, due to the generally lossy nature of the S-procedure [45]. Indeed, when the cardinality $|\mathcal{M}| \ge 3$ and the state dimension $n \ge 2$, there exist matrices $\{A_i\}_{i\in\mathcal{M}}$ and a constant $\gamma > 0$ such that: (i) $\min_{i\in\mathcal{M}} z^T A_i^T A_i z \le \gamma\|z\|^2$, $\forall z \in \mathbb{R}^n$; (ii) there exists no $p$ such that $\sum_{i\in\mathcal{M}} p_i A_i^T A_i \preceq \gamma I$. Then for all $p$, $\sup_{\|z\|=1} \min_{i\in\mathcal{M}} \|A_i z\|^2 \le \gamma < \sigma_{\max}\left(\sum_{i\in\mathcal{M}} p_i A_i^T A_i\right)$.

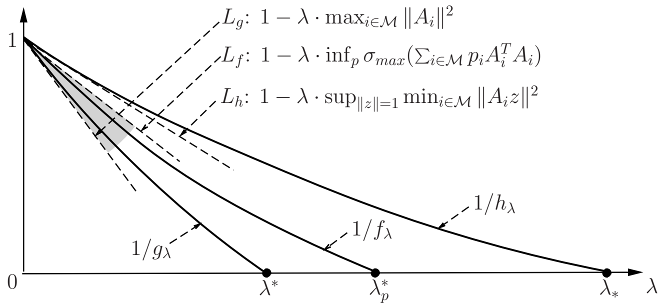


Fig. 8. Plot of the function $1/f_\lambda$.

A general plot of the function $1/f_\lambda$ is shown in Fig. 8. The graph of $1/f_\lambda$ is sandwiched between those of $1/g_\lambda$ and $1/h_\lambda$. By Proposition 14, the ray emitting from $(0,1)$ and passing through $(\lambda, 1/f_\lambda)$ rotates counterclockwise monotonically as $\lambda$ increases. The discussions in the preceding paragraph further imply that the graph of $1/f_\lambda$ leaves $(0,1)$ along a direction within the shaded conic region bounded by the lines $L_g$ and $L_f$; whereas generally a gap exists between $L_f$ and the asymptotic direction $L_h$ in which $1/h_\lambda$ leaves $(0,1)$. Algorithms based on the Bellman equation can also be devised to compute $F_\lambda(z)$, hence $\lambda_p^*$. The details are omitted here.

## VI. Conclusion

Generating functions (more precisely their radii of convergence) of switched linear systems provide effective characterizations of the growth rates of the system trajectories, and in particular their exponential stability, under various switching rules. Numerical algorithms for their computation have also been developed based on their many properties derived here.

## Appendix A
## Proof of Lemma 3

*Proof:* Write $\sum_{t=0}^\infty w_{t+s}\lambda_0^t = w_s + \lambda_0 \sum_{t=0}^\infty w_{t+s+1}\lambda_0^t$. Since by assumption the left hand side is at most $\beta w_s$, we have

$$\sum_{t=0}^\infty w_{t+s+1}\lambda_0^t \le \frac{\beta - 1}{\lambda_0} w_s, \quad s = 0, 1, \ldots. \tag{31}$$

Define the power series: $W(\lambda) := \sum_{t=0}^\infty w_t \lambda^t$, $\lambda \in \mathbb{R}$. Denote by $R_W$ its radius of convergence. Since $W(\lambda_0) = \sum_{t=0}^\infty w_t \lambda_0^t \le \beta w_0 < \infty$, we have $R_W > \lambda_0$. We cannot have $R_W = \lambda_0$ since the power series $W(\lambda)$ with nonnegative coefficients must have its radius of convergence as a singular point, at which $W(\lambda)$ diverges (see [47, Theorem 5.7.1]).

As a power series defines an analytic function within its radius of convergence [47], $W(\lambda)$ is analytic hence infinite time differentiable at $\lambda_0$. Its first order derivative at $\lambda_0$ is

$$W'(\lambda_0) = \sum_{t=1}^\infty t w_t \lambda_0^{t-1} = \sum_{t=0}^\infty (t+1) w_{t+1}\lambda_0^t$$
$$= \sum_{t=0}^\infty \sum_{s=0}^t w_{t+1}\lambda_0^t = \sum_{s=0}^\infty \lambda_0^s \sum_{t=0}^\infty w_{t+s+1}\lambda_0^t.$$

Applying (31), we obtain the estimate

$$W'(\lambda_0) \le \frac{\beta-1}{\lambda_0}\sum_{s=0}^{\infty} w_s\lambda_0^s \le \beta\frac{\beta-1}{\lambda_0}w_0. \qquad (32)$$

Similarly, the second order derivative $W''(\lambda_0)$ is

$$W''(\lambda_0) = \sum_{t=0}^{\infty}(t+2)(t+1)w_{t+2}\lambda_0^t$$
$$= 2\sum_{t=0}^{\infty}\sum_{s=0}^{t}(s+1)w_{t+2}\lambda_0^t = 2\sum_{s=0}^{\infty}(s+1)\lambda_0^s\sum_{t=0}^{\infty}w_{t+s+2}\lambda_0^t.$$

Using (31) and (32), we can derive the estimate $W''(\lambda_0) \le 2\frac{\beta-1}{\lambda_0}\sum_{s=0}^{\infty}(s+1)w_{s+1}\lambda_0^s = 2\frac{\beta-1}{\lambda_0}W'(\lambda_0) \le 2\beta\left(\frac{\beta-1}{\lambda_0}\right)^2 w_0$. By induction, we can show, for $k = 0, 1, 2, \ldots$,

$$W^{(k)}(\lambda_0) \le k!\,\beta\left(\frac{\beta-1}{\lambda_0}\right)^k w_0. \qquad (33)$$

Let $\bar{W}(\lambda) := \sum_{k=0}^{\infty}\frac{1}{k!}W^{(k)}(\lambda_0)(\lambda-\lambda_0)^k$ be the Taylor series expansion of $W(\lambda)$ at $\lambda_0$. For $\lambda \ge \lambda_0$ and close to $\lambda_0$, $\bar{W}(\lambda) = W(\lambda)$ as $W(\lambda)$ is analytic at $\lambda_0$; and by (33),

$$\bar{W}(\lambda) \le \sum_{k=0}^{\infty}\beta\left(\frac{\beta-1}{\lambda_0}\right)^k w_0(\lambda-\lambda_0)^k$$
$$\le \frac{\beta w_0}{1-(\beta-1)(\lambda/\lambda_0-1)},$$

where the last inequality holds if $(\beta-1)(\lambda/\lambda_0-1) < 1$, or equivalently, if $\lambda \in [\lambda_0, \lambda_1)$ with $\lambda_1$ defined in the lemma statement. Therefore, the power series $\bar{W}(\lambda)$ centered at $\lambda_0$ is convergent, and thus defines an analytic function, on $[\lambda_0, \lambda_1)$. The concatenation of $W(\lambda)$ and $\bar{W}(\lambda)$ yields an analytic continuation of $W(\lambda)$ defined at least on $[0, \lambda_1)$. Being a power series of nonnegative coefficients, $W(\lambda)$ does not admit any analytic continuation beyond its radius of convergence $R_W$ [47, Definition 5.7.1]. Hence, $R_W \ge \lambda_1$. This implies in particular that any $\lambda \in [\lambda_0, \lambda_1)$ is within the radius of convergence of both $W(\lambda)$ and $\bar{W}(\lambda)$ and as such we must have $W(\lambda) = \bar{W}(\lambda)$, $\forall\lambda \in [\lambda_0, \lambda_1)$. This together with the last inequality above proves the desired conclusion. ∎

## Appendix B
## Proof of Proposition 4

*Proof:* The monotonicity of $g_\lambda$, hence of $1/g_\lambda$, is obvious as $G_\lambda(z)$ is nondecreasing in $\lambda$. For convexity, let $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2$ for some $\lambda_1, \lambda_2 \in [0, \lambda^*)$, $\alpha \in [0, 1]$; and let $z \in \mathbb{S}^{n-1}$ be such that $G_\lambda(z) = g_\lambda$. Since $G_\lambda(z)$ is a convex function of $\lambda \in [0, \lambda^*)$ for any fixed $z$ (it is the maximum of a family of convex functions of $\lambda \ge 0$), we have $g_\lambda = G_\lambda(z) \le \alpha G_{\lambda_1}(z) + (1-\alpha)G_{\lambda_2}(z) \le \alpha g_{\lambda_1} + (1-\alpha)g_{\lambda_2}$. This proves the convexity of $g_\lambda$ and hence its semismoothness [39, Prop. 7.4.5]. Being the composition of two semismooth functions $g_\lambda$ and $x \mapsto 1/x$, $1/g_\lambda$ is also semismooth [39, Prop. 7.4.4].

Pick any $\lambda_0, \lambda \in (0, \lambda^*)$ with $\lambda_0 < \lambda$. Proposition 3 implies that $\lambda/(1-1/g_\lambda) \ge \lambda_0/(1-1/g_{\lambda_0})$. Thus by Lemma 2,

$$\frac{1}{g_\lambda} - \frac{1}{g_{\lambda_0}} \ge -(\lambda-\lambda_0)\frac{1-1/g_{\lambda_0}}{\lambda_0} \ge -(\lambda-\lambda_0)\max_{i\in\mathcal{M}}\|A_i\|^2,$$

By Lemma 2, the above inequality also holds for $\lambda_0 = 0$. This proves the Lipschitz continuity of $1/g_\lambda$, hence the local Lipschitz continuity of $g_\lambda$, on $[0, \lambda^*)$.

Finally, by (10), $0 \le 1/g_\lambda \le 1 - \lambda/\lambda^*$ for $\lambda \in (0, \lambda^*)$. Thus, $\lim_{\lambda\uparrow\lambda^*} 1/g_\lambda = 0$. Consequently, $\lim_{\lambda\uparrow\lambda^*} g_\lambda = \infty$. ∎

## Appendix C
## Proof of Lemma 5

*Proof:* Fix an arbitrary $z \in \mathbb{S}^{n-1}$ and let $\lambda > 0$ be small. Recalling the definition of $H_\lambda(z)$, we write

$$\frac{H_\lambda(z)-1}{\lambda} = \frac{\inf_\sigma\sum_{t=0}^{\infty}\lambda^t\|x(t;z,\sigma)\|^2 - 1}{\lambda}$$
$$= \inf_\sigma\sum_{t=1}^{\infty}\lambda^{t-1}\|x(t;z,\sigma)\|^2 = \inf_\sigma\sum_{t=0}^{\infty}\lambda^t\|x(t+1;z,\sigma)\|^2.$$

Since for all $\sigma$, $\|x(t;z,\sigma)\|^2 \le (\max_{i\in\mathcal{M}}\|A_i\|^2)^t$, $\forall t$; and $\inf_\sigma\|x(1;z,\sigma)\|^2 = \min_{i\in\mathcal{M}}\|A_iz\|^2$, the above implies

$$\min_{i\in\mathcal{M}}\|A_iz\|^2 \le \frac{H_\lambda(z)-1}{\lambda} \le \min_{i\in\mathcal{M}}\|A_iz\|^2 + O(\lambda),$$

for some term $O(\lambda)$ uniform in $\sigma$ and $z$. Thus, we can exchange the order of limit and supremum below to obtain:

$$h'_\lambda(0_+) = \lim_{\lambda\downarrow 0}\frac{h_\lambda-1}{\lambda} = \lim_{\lambda\downarrow 0}\sup_{\|z\|=1}\frac{H_\lambda(z)-1}{\lambda}$$
$$= \sup_{\|z\|=1}\lim_{\lambda\downarrow 0}\frac{H_\lambda(z)-1}{\lambda} = \sup_{\|z\|=1}\min_{i\in\mathcal{M}}\|A_iz\|^2.$$

This completes the proof of Lemma 5. ∎

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