Stability of Discrete-Time Switched Homogeneous Systems on Cones and Conewise Homogeneous Inclusions

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Abstract

This paper presents a stability analysis of switched homogenous systems on cones under arbitrary and optimal switching rules with extensions to conewise homogeneous or linear inclusions. Several interrelated approaches, such as the joint spectral radius approach and the generating function approach, are exploited to derive necessary and sufficient stability conditions and to develop efficient algorithms for stability tests. Specifically, the generalized joint spectral radius and the generalized joint lower spectral radius are introduced to characterize the radii of domains of strong and weak attraction. Furthermore, strong and weak generating functions and their radii of convergence are employed to derive stability conditions; their analytic properties, numerically effective approximations and convergence analysis are established. Extensions to conewise homogeneous or linear inclusions are made to address state dependent switching dynamics. Relations between different stability notions in the strong or weak sense are studied; Lyapunov techniques are used for stability analysis of the conewise linear inclusions.

1 Introduction

Stability analysis of hybrid and switched dynamical systems is a fundamental problem in systems and control and has received growing interest in the last few years, driven by a number of important applications in complex systems with hierarchical and multi-modal structure [18, 29, 35]. Switched systems and their extensions, i.e., differential or discrete inclusions [11], are subject to possible abrupt changes in dynamics and possess inherent nonsmoothness that complicates the study of their stability. Numerous techniques have been proposed for the stability analysis of switched systems, particularly switched linear systems, e.g., the Lie-algebraic approach [17], the Lyapunov framework [8, 13, 20, 26], the geometric approach [2, 18], the joint spectral radius method [15, 23, 32], the variational approach [19, 22], and the recent generating function approach [14, 27, 28].

Most literature on switched systems concentrates on those on the Euclidean space. However, many applied systems have their states confined within certain regions. A prominent example is positive systems [10] that model a wide range of engineering, biological, and economic systems. In a positive system, the system trajectories are restricted to the nonnegative orthant of $\mathbb{R}^n$. Thus, when analyzing its stability, it is more meaningful and less conservative to concentrate on this restricted set. Certain tools, such as the common Lyapunov function approach [6, 9, 12, 21], have been developed for switched positive systems and their extension, i.e., switched systems over cones.

This paper performs stability analysis for discrete-time switched systems on closed cones and their extensions (namely, conewise inclusions) from several different, albeit interconnected, perspectives, such as the joint spectral radius approach and the generating function approach. The

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focus is on switched homogeneous systems (SHSs) on cones. This class of switched systems includes not only the well studied switched linear systems (SLSs) on $\mathbb{R}^n$ as a special case but also a wide spectrum of switched nonlinear systems on constrained regions. A key feature of the SHSs on cones is the scaling property. By using this property and other techniques, we show that various results for the SLSs can be generalized to switched nonlinear systems in a unified framework, which also leads to important new results. Particularly, we consider stability notions under two switching policies: arbitrary switching which is the most studied switching law, and the optimal switching which is much less explored and more difficult to characterize. We refer to these stability notions as strong and weak stabilities, respectively. Unlike the SLSs whose local and global stabilities are equivalent, the SHSs of homogeneous degree strictly greater than one demonstrate strong nonlinearity such that their strong (resp. weak) stability is only valid on a certain domain called the domain of strong (resp. weak) attraction. One of the goals of this paper is to develop necessary and sufficient conditions and efficient algorithms to characterize the strong and weak stabilities of the SHSs, e.g., their domains of attraction. Specifically, the paper addresses the following critical issues that constitute the contributions of the paper as compared to the existing results:

• For the SLS, it is well known that the joint spectral radius of subsystem matrices determines the exponential growth rate and the exponential stability of the SLS under arbitrary switching. This paper generalizes this notion to nonlinear SHSs on cones. Particularly, the generalized joint spectral radius (GJSR) and the generalized joint lower spectral radius (GJLSR) are introduced via such techniques as the extended Fekete's Lemma. These radii are exploited to determine the domains of strong and weak attraction of the SHS on cones, respectively.

• A generating function approach is proposed to develop a unified, numerically efficient framework for the stability characterization of the SHSs on cones. Informally speaking, a generating function is a suitably defined power series whose coefficients are determined from the systems trajectories under certain switching rules [14]; its radius of convergence characterizes the exponential growth rate of the system trajectories. A generating function is closely related to the value function of a properly defined optimal control problem and admits efficient numerical approximations. Originally introduced for the SLS on the Euclidean space $\mathbb{R}^n$ in [14], the generating function notion is extended to both the SLSs on cones and the (more general) nonlinear SHSs on cones. For the former, we establish a connection with the generating function on an invariant subspace containing the cone and extend algorithms for the SLS on $\mathbb{R}^n$ to that on a convex closed cone. For the latter, due to the nonlinearity and switching induced nonsmoothness, we develop various new techniques, e.g., semicontinuity and Dini's Theorem, to establish analytic properties, stability implications, effective numerical approximations and convergence analysis of the generating functions.

• The third part of the paper extends the stability analysis of the SHSs to that of discrete-time conewise homogeneous or linear inclusions (CHIs or CLIs), even more general switched systems with state-dependent switchings. The state domain of a CHI is covered by finitely many cones, with different homogeneous dynamics defined on each cone. The state dynamics takes multiple values on the intersections of cones, thus making the system an inclusion. The strong and weak stability notions are investigated; their subtle connections are illustrated via examples. A Lyapunov framework is developed for the stability of the CLIs, and its relation with the SLSs is established.

The paper is organized as follows. The SHSs on cones and their stability concepts are introduced in Section 2; the notions of the GJSR and GJLSR are defined and used to characterize the domains of attraction. Sections 3 and 4 focus on the strong and weak generating functions for the SLSs and the nonlinear SHSs on cones respectively. Numerical approximations and convergence analysis are developed for the strong and weak generating functions. In Sections 5 and 6, stabilities of the CHIs and CLIs are studied. Conclusions are drawn in Section 7.

2 Switched Homogeneous Systems on Cones

A discrete-time *switched homogeneous system* (SHS) on a closed (but not necessarily convex) cone $\mathcal{C} \subseteq \mathbb{R}^n$ is defined by

$$x(t+1) = F_{\sigma(t)}(x(t)), \quad t \in \mathbb{Z}_+, \tag{1}$$

where $\sigma(t) \in \mathcal{M} := \{1, \ldots, m\}$ for each $t \in \mathbb{Z}_+$ is the switching sequence; and for each $i \in \mathcal{M}$, F_i is a continuous $\mathbb{R}^n$-valued function on an open set containing $\mathcal{C}$ and is positively homogeneous on $\mathcal{C}$ of degree $\nu \geq 1$ with $\nu \in \mathbb{R}$, i.e., $F_i(0) = 0$, and $F_i(\alpha x) = \alpha^\nu F_i(x)$, $\forall\, \alpha \geq 0$, $x \in \mathcal{C}$. We assume that each vector field F_i is positively invariant on $\mathcal{C}$, i.e., $F_i(\mathcal{C}) \subseteq \mathcal{C}$. This assumption ensures that a trajectory starting from $\mathcal{C}$ remains in $\mathcal{C}$ for all positive times under any switching sequence. As an example, on the positive cone $\mathcal{C} = \mathbb{R}^n_+$, all F_i may be chosen to be vector-valued maps with entries being homogeneous multivariate polynomials with positive coefficients. In the simplest case when the homogeneous degree $\nu = 1$, the SHS (1) becomes the well studied *switched linear system* (SLS) on $\mathcal{C}$: $x(t+1) = A_{\sigma(t)}x(t)$, $t \in \mathbb{Z}_+$, where $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{M}$, are matrices satisfying $A_i\mathcal{C} \subseteq \mathcal{C}$. As an important class of SHSs on cones, the SLSs will be the focus of Section 3 later on.

Denote by $x(t; z, \sigma)$ the solution of the SHS (1) under the switching sequence σ starting from $z \in \mathcal{C}$. It is easy to see that for any $\alpha \geq 0$, $x(t; \alpha z, \sigma) = \alpha^{\nu^t} x(t; z, \sigma)$, $\forall\, t \in \mathbb{Z}_+$. Thus, for $z \neq 0$,

$$x(t; z, \sigma) = \|z\|^{\nu^t} x\left(t; \frac{z}{\|z\|}, \sigma\right), \ \forall\ t \in \mathbb{Z}_+. \tag{2}$$

In what follows, we derive some preliminary bounds on the growth of $\|x(t; z, \sigma)\|$. Since each F_i is continuous and the intersection $\mathcal{C} \cap \mathbb{S}^{n-1}$ is compact, the quantity defined below

$$\mu_1 := \max_{i\in\mathcal{M}} \max_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \|F_i(z)\| \tag{3}$$

must be finite. It follows immediately from the above definition and the ν-homogeneity of F_i that

$$\|F_i(z)\| \leq \mu_1 \|z\|^\nu, \quad \forall\, z \in \mathcal{C},\, i \in \mathcal{M}. \tag{4}$$

Thus $\|x(1; z, \sigma)\| = \|F_{\sigma(0)}(z)\| \leq \mu_1\|z\|^\nu$. Further, $\|x(2; z, \sigma)\| = \|F_{\sigma(1)}(x(1; z, \sigma))\| \leq \mu_1\|x(1; z, \sigma)\|^\nu$ $\leq \mu_1(\mu_1\|z\|^\nu)^\nu \leq (\mu_1)^{\nu+1}\|z\|^{\nu^2}$. By induction, we have the following growth estimate under any σ:

$$\|x(t; z, \sigma)\| \leq (\mu_1)^{\sum_{j=0}^{t-1}\nu^j} \|z\|^{\nu^t}, \ \ \forall\ z \in \mathcal{C},\ t \in \mathbb{Z}_+, \tag{5}$$

where $(\mu_1)^{\sum_{j=0}^{t-1}\nu^j}$ characterizes the super-exponential growth of system trajectories when $\nu > 1$. In Section 2.3, more accurate growth estimates will be established.

It is worth mentioning that the SHS (1) is a special case of the discrete-time conewise homogeneous inclusion (CHI) to be discussed in Section 5. Hence the SHSs share favorable properties of the CHIs, e.g., the equivalence of strong stability notions (cf. Theorem 5.1).

2.1 Stability Notions of the SHSs on Cones

It is clear that when $\nu = 1$ (i.e., the SLS case), the local and global stability notions are equivalent. However, this is not the case for $\nu > 1$, because of the super-exponential growth in (5). In the following, we introduce local stability notions first.

Definition 2.1 (Local Strong Stability of SHS). At $x_e = 0$, the SHS (1) on $\mathcal{C}$ is called

- *locally strongly stable* if for any $\varepsilon > 0$, a neighborhood $\mathcal{N}$ of x_e in $\mathcal{C}$ exists such that for any $z \in \mathcal{N}$, $\|x(t; z, \sigma)\| < \varepsilon$, $\forall\ t \in \mathbb{Z}_+$, under any switching sequence σ;

- *locally strongly convergent* if for each z in a neighborhood $\mathcal{N}$ of x_e in $\mathcal{C}$, $x(t; z, \sigma) \to 0$ as $t \to \infty$ under any switching sequence σ;
- *locally strongly asymptotically stable* if it is locally strongly stable and strongly convergent;
- *locally strongly exponentially stable* (with the parameters $\kappa \geq 0$ and $r \in [0, 1)$) if for any z in a neighborhood $\mathcal{N}$ of x_e in $\mathcal{C}$, $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, $\forall\, t \in \mathbb{Z}_+$, under any switching sequence σ.

In the above definitions of strong stabilities, by replacing "any switching sequence" with "at least one switching sequence," we can define the corresponding local weak stabilities, namely, *the local weak stability*, *local weak convergence*, *local weak asymptotic stability*, and *local weak exponential stability*. In this paper, these strong stabilities are also called stabilities under arbitrary switching; while the weak ones are often referred to as stabilities under optimal switching. In the following, we show that for $\nu > 1$, the SHS is always locally strongly exponentially stable on $\mathcal{C}$.

Lemma 2.1. Let $\nu > 1$. The SHS (1) is strongly exponentially stable with the parameters $\kappa = 1$ and any $r \in (0, 1)$ on the set $\mathcal{N} := \left\{ z \,\middle|\, \|z\| \leq (r/\mu_1)^{\frac{1}{\nu-1}} \right\} \cap \mathcal{C}$.

Proof. It follows from (5) that for any $z \in \mathcal{N}$ and any switching sequence σ,

$$\|x(t; z, \sigma)\| \leq (\mu_1)^{\frac{\nu^t-1}{\nu-1}} \|z\|^{\nu^t-1} \|z\| \leq (\mu_1)^{\frac{\nu^t-1}{\nu-1}} \left(\frac{r}{\mu_1}\right)^{\frac{\nu^t-1}{\nu-1}} \|z\| = r^{\frac{\nu^t-1}{\nu-1}} \|z\| \leq r^t \|z\|, \quad \forall\, t \in \mathbb{Z}_+,$$

where $\frac{\nu^t-1}{\nu-1} = \sum_{j=0}^{t-1} \nu^j \geq t$ and $r \in (0, 1)$ is used in the last inequality. □

A justification of the choice of $\mathcal{N}$ in the above lemma is that if $\|z\| \leq (r/\mu_1)^{1/(\nu-1)}$, then by (4), $\|F_i(z)\| \leq \mu_1 \|z\|^\nu = \mu_1 \|z\|^{\nu-1} \|z\| \leq r\|z\|$, i.e., each F_i is a contraction on $\mathcal{N}$.

As Lemma 2.1 suggests, the neighborhood $\mathcal{N}$ in Definition 2.1 where the stability holds may not be small. To highlight the domain of stability, we introduce the following concepts.

Definition 2.2 (Non-local Strong Stability of SHS). For a given (possibly large) set $\mathcal{S} \subseteq \mathcal{C}$, the SHS (1) on $\mathcal{C}$ is called

- *strongly asymptotically stable on* $\mathcal{S}$ if for any $z \in \mathcal{S}$ and any switching sequence σ, $\lim_{t\to\infty} x(t; z, \sigma) = 0$. In this case, the set $\mathcal{S}$ is called a *domain of strong attraction* [3];
- *strongly uniformly asymptotically stable on* $\mathcal{S}$ if for any $\varepsilon > 0$, there exists $T_\varepsilon \in \mathbb{Z}_+$ (independent of z and σ) such that for any $z \in \mathcal{S}$ and any switching sequence σ, $\|x(t; z, \sigma)\| < \varepsilon$ for all $t \geq T_\varepsilon$;
- *strongly exponentially stable on* $\mathcal{S}$ if there exist $\kappa \geq 1$ and $r \in (0, 1)$ (independent of z and σ) such that for any $z \in \mathcal{S}$ and any switching sequence σ, $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, $\forall\, t \in \mathbb{Z}_+$.

Similarly, weak asymptotic (resp. uniform asymptotic, exponential) stability on the set $\mathcal{S}$ can be defined for the SHS (1) by replacing "any switching sequence" with "at least one switching sequence (possibly dependent on z)" in the above definitions. Especially, if the SHS is weakly asymptotically stable on $\mathcal{S}$, then $\mathcal{S}$ is called a *domain of weak attraction*.

We next show that the above three strong (resp. weak) stability notions are equivalent on the intersection of $\mathcal{C}$ and any closed ball; see a related result for $\nu = 1$ in [1]. Denote by $\mathcal{B}_\rho := \{x \in \mathbb{R}^n \mid \|x\| < \rho\}$ the open ball of radius $\rho > 0$, and let $\overline{\mathcal{B}}_\rho := \operatorname{cl} \mathcal{B}_\rho = \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$ be its closure.

Proposition 2.1. Suppose $\nu > 1$. If the SHS (1) is weakly asymptotically stable on $\overline{\mathcal{B}}_\rho \cap \mathcal{C}$ for some $\rho > 0$, then it is also weakly exponentially stable on $\overline{\mathcal{B}}_\rho \cap \mathcal{C}$.

Proof. Since the SHS is weakly asymptotically stable on $\overline{\mathcal{B}}_\rho \cap \mathcal{C}$, for each $z \in \rho\,\mathbb{S}^{n-1} \cap \mathcal{C}$, a switching sequence σ_z exists such that $x(t; z, \sigma_z) \to 0$ as $t \to \infty$. For fixed σ and t, since $x(t; z, \sigma)$ is a continuous function of z, we have $\|x(T_z; z', \sigma_z)\| \le \rho/2$ for some finite time $T_z \in \mathbb{Z}_+$ and all z' in an open neighborhood U_z of z. As all such U_z constitute an open cover of the compact set $\rho\,\mathbb{S}^{n-1} \cap \mathcal{C}$, there is a finite subcover, i.e., there exist finitely many points $\{z_1^*, \dots, z_p^*\}$ in $\rho\,\mathbb{S}^{n-1} \cap \mathcal{C}$ and their open neighborhoods $\{U_{z_i^*}\}_{i=1}^p$ covering $\rho\,\mathbb{S}^{n-1} \cap \mathcal{C}$ such that on each $U_{z_i^*}$, a switching sequence σ_i^* and a finite time T_i^* can be found with $\|x(T_i^*; z, \sigma_i^*)\| \le \rho/2$ for all $z \in U_{z_i^*}$. Define

$$\alpha := \max_{i \in \mathcal{M}} \max_{z \in \mathrm{cl}\, U_{z_i^*}} \max_{0 \le t \le T_i^*} \|x(t; z, \sigma_i^*)\| < \infty. \tag{6}$$

For any initial state $z \in \rho\,\mathbb{S}^{n-1} \cap \mathcal{C}$, $z \in U_{z_i^*}$ for some $i \in \{1, \dots, p\}$. By the definition of σ_i^* and T_i^*, $x(T_1) := x(T_i^*; z, \sigma_i^*)$ satisfies $\|x(T_1)\| \le \rho/2$. Assume $x(T_1) \ne 0$ without loss of generality. Then, $\rho\, x(T_1)/\|x(T_1)\| \in U_{z_j^*}$ for some $j \in \{1, \dots, p\}$. By (2), $x(T_2) := x(T_j^*; x(T_1), \sigma_j^*)$ satisfies

$$\|x(T_2)\| = \left(\frac{\|x(T_1)\|}{\rho}\right)^{\nu^{T_j^*}} \left\| x\left(T_j^*; \frac{\rho\, x(T_1)}{\|x(T_1)\|}, \sigma_j^*\right) \right\| \le \left(\frac{1}{2}\right)^{\nu^{T_j^*}} \frac{\rho}{2}.$$

Similarly, $\rho x(T_2)/\|x(T_2)\| \in U_{z_k^*}$ for some $k \in \{1, \dots, p\}$, and $x(T_3) := x(T_k^*; x(T_2), \sigma_k^*)$ satisfies

$$\|x(T_3)\| = \left(\frac{\|x(T_2)\|}{\rho}\right)^{\nu^{T_k^*}} \left\| x\left(T_k^*; \frac{\rho\, x(T_2)}{\|x(T_2)\|}, \sigma_k^*\right) \right\| \le \left(\frac{1}{2}\right)^{\nu^{T_j^* + T_k^*}} \frac{\rho}{2}.$$

Continuing this process via induction, we obtain a switching sequence σ_z that is the concatenation of $\sigma_i^*, \sigma_j^*, \sigma_k^*, \dots$, under which the trajectory $x(t; z, \sigma_z)$ at times $0 < T_1 < T_2 < T_3 < \cdots$ satisfies

$$\|x(T_i; z, \sigma_z)\| \le \left(\frac{1}{2}\right)^{\nu^{T_i - T_1}} \frac{\rho}{2} = \frac{\rho}{2}\left(\frac{1}{2^{\nu^{-T_1}}}\right)^{\nu^{T_i}} \le \frac{\rho}{2}\left(\frac{1}{2^{\nu^{-T_1}}}\right)^{T_i}, \quad i = 1, 2, \dots,$$

which is exponentially decaying. Using (2) and the definition of α in (6), we obtain $\|x(t; z, \sigma_z)\| \le \kappa r^t$, $\forall\, t \in \mathbb{Z}_+$, for some properly defined constants $\kappa > 0$ dependent on α and $r = 1/2^{\nu^{-T_1}} \in (0, 1)$.

Finally we consider a general nonzero initial state $z \in \overline{\mathcal{B}}_\rho \cap \mathcal{C}$. Noting that $z' := \rho z/\|z\| \in \rho\,\mathbb{S}^{n-1} \cap \mathcal{C}$, by the above result, there exists a switching sequence $\sigma_{z'}$ such that

$$\|x(t; z, \sigma_{z'})\| = \left(\frac{\|z\|}{\rho}\right)^{\nu^t} \left\| x\left(t; \frac{\rho\, z}{\|z\|}, \sigma_{z'}\right) \right\| \le \frac{\kappa}{\rho}\, r^t \|z\|, \quad \forall\, t \in \mathbb{Z}_+.$$

This proves that the SHS is weakly exponentially stable on $\overline{\mathcal{B}}_\rho \cap \mathcal{C}$. □

Corollary 2.1. Suppose $\nu > 1$. The following hold for the SHS on $\mathcal{S} = \overline{\mathcal{B}}_\rho \cap \mathcal{C}$ for $\rho > 0$:

weak asymptotic stability $\Leftrightarrow$ weak uniform asymptotic stability $\Leftrightarrow$ weak exponential stability

Proposition 2.2. Let $\nu > 1$. The following hold for the SHS on $\mathcal{S} = \overline{\mathcal{B}}_\rho \cap \mathcal{C}$ for $\rho > 0$:

strong asymptotic stability $\Leftrightarrow$ strong uniform asymptotic stability $\Leftrightarrow$ strong exponential stability

Proof. It suffices to prove that strong asymptotic stability implies strong uniform asymptotic stability, which further implies strong exponential stability. To show the first implication, suppose that the SHS (1) is strongly, but not uniformly, asymptotically stable on $\mathcal{S}$. Then there exist a constant $\varepsilon_0 > 0$, a sequence of initial states $\{z_k\} \subseteq \mathcal{S}$, a sequence of switching sequences $\{\sigma^k\}$, and a strictly increasing time sequence $\{t_k\}$ with $t_k \uparrow \infty$ such that $\|x(t_k; z_k, \sigma^k)\| \geq \varepsilon_0$ for all k. Due to the local strong exponential stability given by Lemma 2.1, there exists a constant $\widetilde{\rho} > 0$ such that $z \in \mathcal{B}_{\widetilde{\rho}} \cap \mathcal{C} \Rightarrow \|x(t; z, \sigma)\| < \varepsilon_0$ for all $t \in \mathbb{Z}_+$ and any σ. This implies that for each k, $\|x(t; z_k, \sigma^k)\| \geq \widetilde{\rho}$ for all $t \in \{0, 1, \ldots, t_k\}$. In view of the compactness of $\mathcal{S}$, the closedness of $\mathcal{C}$, the continuity of F_i, and the growth rate estimate (5), it follows from a similar argument as in the proof of [27, Proposition 1] with an extension to the homogeneous vector fields (also cf. Theorem 5.1) that there exist $z_* \in \mathcal{S}$ and a switching sequence σ^* such that $\|x(t; z_*, \sigma^*)\| \geq \widetilde{\rho}$ for all $t \in \mathbb{Z}_+$. But this contradicts the strong asymptotic stability of the SHS on $\mathcal{S}$.

To show that strong uniform asymptotic stability implies strong exponential stability, let $\rho' = \rho + \varepsilon$ for some $\varepsilon > 0$. By the strong uniform asymptotic stability, we can find $T_* \in \mathbb{Z}_+$, dependent on ρ' only, such that for any $z \in \mathcal{S}$ and under any switching sequence σ, $\|x(T_*; z, \sigma)\| \leq \rho'/2$. By extending the argument in the proof of Proposition 2.1 (though there is no need to use the open covering argument), we have, for any $z \in \mathcal{C}$ and under any switching sequence σ,

$$\|x(t; z, \sigma)\| \leq \alpha \left(\frac{\|z\|}{\rho'}\right)^{\nu^t} \leq \frac{\alpha}{\rho'} \left(\frac{\|z\|}{\rho'}\right)^{\nu^t - 1} \|z\| \leq \frac{\alpha}{\rho'} \left(\frac{\|z\|}{\rho'}\right)^{t} \|z\|, \quad \forall\, t \in \mathbb{Z}_+,$$

where $\alpha := \max_{z \in \rho' \mathbb{S}^{n-1} \cap \mathcal{C}} \max_{t \in [0, T_*], \sigma} \|x(t; z, \sigma)\|$. This implies the strong exponential stability on $\mathcal{S}$. $\square$

Proposition 2.3. If the SHS (1) is strongly asymptotically stable on $\mathcal{B}_\rho \cap \mathcal{C}$ with $\rho > 0$, then it is strongly exponentially stable on $\mathcal{B}_{\rho - \varepsilon} \cap \mathcal{C}$ for any $\varepsilon \in (0, \rho)$.

Proof. For the given ρ and $\varepsilon \in (0, \rho)$, let $\rho' = \rho - \frac{\varepsilon}{2}$. Since the SHS is strongly asymptotically stable on $\mathcal{B}_\rho \cap \mathcal{C}$, so is it on the compact set $\overline{\mathcal{B}}_{\rho'} \cap \mathcal{C}$. The rest follows directly from Proposition 2.2. $\square$

2.2 Domains of Attraction of the SHSs on Cones with $\nu > 1$

It is well known that the SHS of homogeneous degree $\nu > 1$ is *not* globally asymptotically stable in general. A question arises naturally: *what is the largest possible domain of attraction in the strong or weak sense*? Specifically, we define two radii of the domains of attraction:

(1) The radius of the domain of strong attraction:

$$\rho^* := \sup\{\rho > 0 \,|\, \text{ the SHS is strongly asymptotically stable on } \mathcal{B}_\rho \cap \mathcal{C}\,\}.$$

(2) The radius of the domain of weak attraction:

$$\rho_* := \sup\{\rho > 0 \,|\, \text{ the SHS is weakly asymptotically stable on } \mathcal{B}_\rho \cap \mathcal{C}\,\}.$$

By Propositions 2.1 and 2.3, the asymptotic stability in the above definitions can be equivalently replaced by exponential stability. Clearly, $0 < \rho^* \leq \rho_* \leq \infty$. The last inequality is strict if

$$\zeta := \min_{i \in \mathcal{M}} \min_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} \|F_i(z)\| \tag{7}$$

is strictly positive, since $\|F_i(z)\| \geq \zeta \|z\|^\nu$, $\forall\, z \in \mathcal{C}$, $i \in \mathcal{M}$. It then follows from a similar argument for (5) that $\|x(t; z, \sigma)\| \geq \zeta^{\frac{\nu^t - 1}{\nu - 1}} \|z\|^{\nu^t}, \forall\, z \in \mathcal{C}, t \in \mathbb{Z}_+$ under any switching sequence σ. We also have, by the proof of Lemma 2.1, that if $\|z\| \geq (1/\zeta)^{1/(\nu-1)}$, then $\|x(t; z, \sigma)\| \geq \|z\|, \forall\, \sigma$. This implies that $\rho_* \leq (1/\zeta)^{1/(\nu-1)} < \infty$.

The next proposition shows that the boundary $\rho^* \mathbb{S}^{n-1} \cap \mathcal{C}$ (resp. $\rho_* \mathbb{S}^{n-1} \cap \mathcal{C}$) contains an initial state whose trajectories are not strongly (resp. weakly) convergent. Thus ρ^* (resp. ρ_*) is exactly the first radius where the SHS starts to lose the strong (resp. weak) asymptotic stability. This result will be used in Section 4 for stability analysis via generating functions.

Proposition 2.4. Let ρ^* and ρ_* be finite. Then the following hold:

(1) There exist $z_0 \in \rho^*\mathbb{S}^{n-1} \cap \mathcal{C}$ and a switching sequence σ_{z_0} such that $x(t; z_0, \sigma_{z_0})$ does not converge to 0 as $t \to \infty$. Thus the SHS is *not* strongly asymptotically stable on $\overline{\mathcal{B}}_{\rho^*} \cap \mathcal{C}$.

(2) There exists $z_0 \in \rho_*\mathbb{S}^{n-1} \cap \mathcal{C}$ such that $x(t; z_0, \sigma)$ does not converge to the origin as $t \to \infty$ under any σ. Thus the SHS is *not* weakly asymptotically stable on $\overline{\mathcal{B}}_{\rho_*} \cap \mathcal{C}$.

Proof. (1) We prove this result via contradiction. Suppose that the SHS is strongly asymptotically stable on $\overline{\mathcal{B}}_{\rho^*} \cap \mathcal{C}$. By Proposition 2.2, the SHS is strongly exponentially stable on $\overline{\mathcal{B}}_{\rho^*} \cap \mathcal{C}$. Hence, there exist $\kappa > 0$ and $r \in [0, 1)$ such that for any $z \in \overline{\mathcal{B}}_{\rho^*} \cap \mathcal{C}$ and under any σ, $\|x(t; z, \sigma)\| \le \kappa r^t \|z\|, \forall\, t$. Therefore, there exists $T_* > 0$ such that for any $z \in \rho^*\mathbb{S}^{n-1} \cap \mathcal{C}$, $\|x(T_*; z, \sigma)\| \le \rho^*/3$ under any σ. Note that there are finitely many switching segments on the time window $\{0, \ldots, T_*\}$ and that $x(T_*; z, \sigma)$ is continuous in z for each fixed σ; the latter is due to the continuity of F_i on an open set containing $\mathcal{C}$. Following a similar open covering argument in Proposition 2.1, we obtain finitely many points $z_1^*, \ldots, z_p^*$ in $\rho^*\mathbb{S}^{n-1} \cap \mathcal{C}$ and the open balls $\mathcal{B}(z_i^*, \varepsilon_i)$ (with respect to the topology of $\mathbb{R}^n$) centered at z_i^* with the radius $\varepsilon_i > 0$ covering $\rho^*\mathbb{S}^{n-1} \cap \mathcal{C}$ such that $\|x(T_*; z, \sigma)\| \le \rho^*/2.5$ under any σ, for all z in each open ball.

For a given $\chi > 0$, let the set $\mathcal{U}_\chi := \{z \in \mathcal{C} \,:\, \rho^* \le \|z\| \le \rho^* + \chi\}$. Define $\overline{\chi} := \sup\{\chi : \mathcal{U}_\chi \subseteq \cup_{i=1}^p \mathcal{B}(z_i^*, \varepsilon_i)\}$. We claim that $\overline{\chi} > 0$. Geometrically speaking, the claim implies that the open covering of the boundary surface $\rho^*\mathbb{S}^{n-1} \cap \mathcal{C}$ can be extended to a thin layer growing above the boundary surface. We prove the claim by contradiction. Suppose $\overline{\chi} = 0$. Then there exists a sequence (z_k) in $\mathcal{C}$ such that $\rho^* \le \|z_k\| \le \rho^* + 1/k$ and $z_k \notin \cup_{i=1}^p \mathcal{B}(z_i^*, \varepsilon_i)$ for all $k \in \mathbb{N}$. Since the sequence (z_k) is bounded, it has a convergent subsequence which can be assumed to be (z_k) itself. Thus the limit of (z_k) satisfies $\|z^*\| = \rho^*$. Since $\mathcal{C}$ is closed, we have $z^* \in \rho^*\mathbb{S}^{n-1} \cap \mathcal{C}$. Therefore, z^* must be in the interior of some open ball $\mathcal{B}(z_i^*, \varepsilon_i)$. As $(z_k) \to z^*$, $z_k \in \mathcal{B}(z_i^*, \varepsilon_i)$ for all k sufficiently large. This yields a contradiction.

Due to the above claim, there exists $\widetilde{\rho} > \rho^*$ such that $(\widetilde{\rho}\,\mathbb{S}^{n-1} \cap \mathcal{C}) \subseteq \cup_{i=1}^p \mathcal{B}(z_i^*, \varepsilon_i)$ and that for any $z \in \widetilde{\rho}\,\mathbb{S}^{n-1} \cap \mathcal{C}$, $\|x(T_*; z, \sigma)\| \le \widetilde{\rho}/2$ under any σ. The rest of the proof is similar to the argument below (6) in Proposition 2.1 with "at least one switching" replaced by "arbitrary switching". Thus the SHS is strongly asymptotically stable on $\overline{\mathcal{B}}_{\widetilde{\rho}} \cap \mathcal{C}$, contradicting the definition of ρ^*.

(2) The proof is is similar to the above argument and that of Proposition 2.1, and is omitted. □

2.3 Generalized Joint Spectral Radii of the SHSs on Cones

The notion of the *joint spectral radius* (JSR) has been introduced to characterize the maximum exponential growth rate of the trajectories of SLSs on $\mathbb{R}^n$ with a set of matrices $\{A_i\}_{i \in \mathcal{M}}$; see, e.g., [15, 32, 34]. Specifically, the JSR is defined as $\mu^* := \lim_{k \to \infty} \big(\sup\{\|A_{i_1} \cdots A_{i_k}\|^{1/k} \,:\, i_1, \ldots, i_k \in \mathcal{M}\}\big)$. It is known that the SLS is strongly exponentially stable if and only if $\mu^* < 1$. Other related stability measures of the SLS under different switching policies include the *joint spectral sub-radius* [15, 32] and the *joint lower spectral radius* [31]. In this section, we extend these notions to the SHSs on cones, which treat the above mentioned quantities as special cases.

2.3.1 Generalized Joint Spectral Radius of the SHSs on Cones

For notational simplicity, denote $h_k := \sum_{i=0}^{k-1} \nu^i$ for $k \in \mathbb{N}$. Define, for each $k \in \mathbb{N}$,

$$\mu_k := \sup\Big\{ \big\|F_{i_1} \circ \cdots \circ F_{i_k}(z)\big\|^{1/h_k} \,:\, i_1, \ldots, i_k \in \mathcal{M},\ z \in \mathbb{S}^{n-1} \cap \mathcal{C} \Big\}. \tag{8}$$

Note that μ_1 defined in (3) becomes a special case of the above definition for $k = 1$. The following result, which relies on the generalized Fekete's Lemma [25, Section 2.6], shows that the sequence (μ_k) converges to the infimum of the set $\{\mu_k\}$ as $k \to \infty$.

Theorem 2.1. The limit of the sequence (μ_k) exists with $\lim_{k\to\infty} \mu_k = \inf\{\mu_k\}$.

Proof. Without loss of generality, we assume that each $\mu_k > 0$. Note that for any $p, q \in \mathbb{N}$, $i_1, \ldots, i_p$, $i_{p+1}, \ldots, i_{p+q} \in \mathcal{M}$ and any $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, we have

$$\Big\| \underbrace{F_{i_1} \circ \cdots \circ F_{i_p}}_{p \text{ terms}} \circ \underbrace{F_{i_{p+1}} \circ \cdots \circ F_{i_{p+q}}}_{q \text{ terms}}(z) \Big\| \;\leq\; (\mu_p)^{h_p} \Big\| F_{i_{p+1}} \circ \cdots \circ F_{i_{p+q}}(z) \Big\|^{\nu^p} \;\leq\; (\mu_p)^{h_p} (\mu_q)^{h_q \nu^p}.$$

This implies $(\mu_{p+q})^{h_{p+q}} \leq (\mu_p)^{h_p}(\mu_q)^{h_q\nu^p}$. Hence, by defining $c_k := h_k \log \mu_k$, $k \in \mathbb{N}$, we have $c_{p+q} \leq c_p + \nu^p \cdot c_q$, $\forall\ p, q \in \mathbb{N}$. Fix $\ell \in \mathbb{N}$. Using the above inequality repeatedly, we obtain $c_{2\ell} \leq c_\ell + \nu^\ell c_\ell \leq (1 + \nu^\ell)c^\ell$, $c_{3\ell} \leq c_\ell + \nu^\ell c_{2\ell} \leq (1 + \nu^\ell + \nu^{2\ell})c_\ell$, ... In general, we have

$$c_{p\ell} \leq \Big(1 + \nu^\ell + \cdots + \nu^{(p-1)\ell}\Big)\, c_\ell = \frac{h_{p\ell}}{h_\ell} c_\ell \quad \Rightarrow \quad \frac{c_{p\ell}}{h_{p\ell}} \leq \frac{c_\ell}{h_\ell}, \quad \forall\ p \in \mathbb{N}.$$

Furthermore, for any $p \in \mathbb{N}$ and $q = 0, \ldots, \ell - 1$,

$$\frac{c_{p\ell+q}}{h_{p\ell+q}} \leq \frac{c_q + \nu^q c_{p\ell}}{h_{p\ell+q}} \leq \frac{c_q}{h_{p\ell+q}} + \frac{\nu^q \cdot h_{p\ell}}{h_{p\ell+q}} \frac{c_\ell}{h_\ell} \leq \frac{c_q}{h_{p\ell+q}} + \frac{c_\ell}{h_\ell},$$

where the last inequality is due to $\nu^q \cdot h_{p\ell} \leq h_{p\ell+q}$.

Let $p' \in \mathbb{N}$ be arbitrary. Then any $k \in \mathbb{N}$ with $k \geq p'\ell$ can be written as $k = p\ell + q$ for some $p \geq p'$ and $q \in \{0, \ldots, \ell - 1\}$. Using the above inequality, we obtain

$$\sup\left\{\frac{c_k}{h_k} : k \geq p'\ell\right\} \leq \sup_{p\in\mathbb{N},\, p\geq p'} \ \sup_{q\in\{0,\ldots,\ell-1\}} \left(\frac{c_q}{h_{p\ell+q}} + \frac{c_\ell}{h_\ell}\right) \leq \frac{c_\ell}{h_\ell} + \sup_{q\in\{0,\ldots,\ell-1\}} \frac{c_q}{h_{p'\ell+q}}.$$

Letting $p' \to \infty$ and noting that $h_{p'\ell+q} \to \infty$ while c_q is finite for each q, we have

$$\limsup_{k\to\infty} \frac{c_k}{h_k} = \lim_{p'\to\infty} \sup\left\{\frac{c_k}{h_k} : k \geq p'\ell\right\} \leq \frac{c_\ell}{h_\ell}.$$

Since $\ell \in \mathbb{N}$ is arbitrary, $\limsup c_k/h_k \leq \inf\{c_\ell/h_\ell \,|\, \ell \in \mathbb{N}\}$. On the other hand, it is obvious that $\liminf c_k/h_k \geq \inf\{c_\ell/h_\ell \,|\, \ell \in \mathbb{N}\}$. Therefore, the following limit exists: $\lim_{k\to\infty} \log \mu_k = \lim_{k\to\infty} \frac{c_k}{h_k} = \inf_{\ell\in\mathbb{N}} \frac{c_\ell}{h_\ell} = \inf_{k\in\mathbb{N}} \log \mu_k$. The theorem thus follows by the continuity of the logarithmic function. □

We remark that in the above proof, it is possible that $\inf\{c_k/h_k\} = -\infty$. In this case, (μ_k) converges to 0 as $k \to \infty$. In all other cases, (μ_k) converges to a finite positive number. We define the *generalized joint spectral radius* (GJSR) of the set of ν-homogeneous continuous maps $\{F_i\}_{i\in\mathcal{M}}$, or alternatively, the GJSR of the SHS (1) on the closed cone $\mathcal{C}$, as $\mu^* := \lim_{k\to\infty} \mu_k$. By Theorem 2.1, $\mu^* = \inf\{\mu_k\}$. When $\mathcal{C} = \mathbb{R}^n$ and $F_i(x) = A_i x$, μ^* is exactly the standard JSR of $\{A_i\}_{i\in\mathcal{M}}$.

2.3.2 Generalized Joint Lower Spectral Radius of the SHSs on Cones

Define, for each $k \in \mathbb{N}$,

$$a_k \;:=\; \sup_{z\in\,\mathbb{S}^{n-1}\cap\mathcal{C}} \left(\inf\left\{ \big\|F_{i_1} \circ \cdots \circ F_{i_k}(z)\big\|^{1/h_k} \;:\; i_1, \ldots, i_k \in \mathcal{M}\right\}\right). \tag{9}$$

The following result shows that the sequence (a_k) converges to the infimum of the set $\{a_k\}$.

Theorem 2.2. The limit of the sequence (a_k) exists with $\lim_{k\to\infty} a_k = \inf\{a_k\}$.

Proof. Let $a_* = \inf\{a_k\}$. For any $\varepsilon > 0$, there exists a_ℓ such that $a_* \le a_\ell < a_* + \varepsilon$. Hence,

$$\sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \Big(\inf\big\{ \big\|F_{i_1}\circ\cdots\circ F_{i_\ell}(z)\big\| \,:\, i_1,\ldots,i_\ell\in\mathcal{M}\big\}\Big) \;\le\; (a_*+\varepsilon)^{h_\ell}.$$

Therefore, for any $z \in \mathbb{S}^{n-1}\cap\mathcal{C}$, there exists a switching sequence segment $\sigma_{z,\ell}$ of length ℓ such that $\|x(\ell; z, \sigma_{z,\ell})\| \le (a_*+\varepsilon)^{h_\ell}$. By the continuity of F_i and the compactness of $\mathbb{S}^{n-1}\cap\mathcal{C}$, there exists a constant $\kappa \ge 1$ (independent of z and $\sigma_{z,\ell}$) such that for all $z \in \mathbb{S}^{n-1}\cap\mathcal{C}$, $\|x(t; z, \sigma_{z,\ell})\| \le \kappa(a_*+\varepsilon)^{h_t}, \forall\, t = 0,\ldots,\ell$. For a given $z \in \mathbb{S}^{n-1}\cap\mathcal{C}$, define $\widetilde{z} := x(\ell; z, \sigma_{z,\ell})/\|x(\ell; z, \sigma_{z,\ell})\| \in \mathbb{S}^{n-1}\cap\mathcal{C}$, where $x(\ell; z, \sigma_{z,\ell})$ is assumed to be nonzero without loss of generality. Then, there exists another switching sequence segment $\sigma_{\widetilde{z},\ell}$ of length ℓ such that $\|x(\ell; \widetilde{z}, \sigma_{\widetilde{z},\ell})\| \le (a_*+\varepsilon)^{h_\ell}$, and

$$\big\|x(t; x(\ell; z, \sigma_{z,\ell}), \sigma_{\widetilde{z},\ell})\big\| = \big\|x(t; \widetilde{z}, \sigma_{\widetilde{z},\ell})\big\| \cdot \big\|x(\ell; z, \sigma_{z,\ell})\big\|^{\nu^t} \le \kappa(a_*+\varepsilon)^{h_t}\cdot(a_*+\varepsilon)^{\nu^t\cdot h_\ell} = \kappa(a_*+\varepsilon)^{h_{t+\ell}},$$

for all $t = 0,\ldots,\ell$, where $h_t + \nu^t h_\ell = h_{t+\ell}$ is used. Let $\sigma_{z,2\ell}$ be the concatenation of $\sigma_{z,\ell}$ and $\sigma_{\widetilde{z},\ell}$. Thus, under the switching sequence segment $\sigma_{z,2\ell}$ of length 2ℓ, $\|x(2\ell; z, \sigma_{z,2\ell})\| \le (a_*+\varepsilon)^{h_{2\ell}}$ and $\|x(t; z, \sigma_{z,2\ell})\| \le \kappa(a_*+\varepsilon)^{h_t}$, $\forall\, t = 0,\ldots,2\ell$. Repeating this argument and using induction, it can be shown that for any $z \in \mathbb{S}^{n-1}\cap\mathcal{C}$, there exists a switching sequence $\widetilde{\sigma}_z$ such that $\|x(t; z, \widetilde{\sigma}_z)\| \le \kappa(a_*+\varepsilon)^{h_t}$, $\forall\, t\in\mathbb{Z}_+$. This implies

$$a_k \;\le\; \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \big\|x(k; z, \widetilde{\sigma}_z)\big\|^{1/h_k} \;\le\; \kappa^{1/h_k}\cdot(a_*+\varepsilon), \quad \forall\, k\in\mathbb{N}.$$

Therefore, in light of $\kappa^{1/h_k} \to 1$ as $k\to\infty$, we obtain $\limsup_{k\to\infty}(a_k) \le (a_*+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, $\limsup_{k\to\infty}(a_k) \le a_* = \inf\{a_k\}$. This proves the desired conclusion. □

We define the *generalized joint lower spectral radius* (GJLSR) of the set of ν-homogeneous continuous maps $\{F_i\}_{i\in\mathcal{M}}$, or alternatively, the GJLSR of the SHS (1) on $\mathcal{C}$, as $a_* := \lim_{k\to\infty} a_k$. It is worth pointing out that μ^* and a_* are independent of the vector norm $\|\cdot\|$ used in their definitions.

2.3.3 Application of the GJSR and GJLSR to the SHSs with $\nu > 1$ on Cones

The GJSR and GJLSR characterize the growth rates of the trajectories of the SHSs on cones. Indeed, it follows from the definitions of μ^* and a_* that for any $\varepsilon > 0$, (i) there exists $\kappa^*_\varepsilon \ge 1$ such that for all $z \in \mathcal{C}$, $\|x(t; z, \sigma)\| \le \kappa^*_\varepsilon\,(\mu^*+\varepsilon)^{h_t}\|z\|^{\nu^t}, \forall\, t$ under any σ; and (ii) there exists $\kappa_{*,\varepsilon} \ge 1$ such that for any $z \in \mathcal{C}$, $\|x(t; z, \sigma_z)\| \le \kappa_{*,\varepsilon}\,(a_*+\varepsilon)^{h_t}\|z\|^{\nu^t}, \forall\, t$ under some σ_z. Noting from (i) that $\|x(t; z, \sigma)\| \le \kappa^*_\varepsilon\left[(\mu^*+\varepsilon)^{h_t}\|z\|^{\nu^t-1}\right]\|z\|$ and $h_t = \frac{\nu^t-1}{\nu-1}$, we deduce that if $\|z\| < (\mu^*+\varepsilon)^{-\frac{1}{\nu-1}}$, then $x(t; z, \sigma)$ converges to the origin under any σ. This hints that the radius of the domain of strong attraction $\rho^* = (\mu^*)^{-\frac{1}{\nu-1}}$; the similar observation can be made for ρ_*. The following theorems rigorously justify these observations; a generating function approach will be discussed in Section 4.

Theorem 2.3. Let $\nu > 1$. The radius of the domain of strong attraction of the SHS (1) on $\mathcal{C}$ is

$$\rho^* = (\mu^*)^{-\frac{1}{\nu-1}},$$

where μ^* is the GJSR of the SHS (1). Thus, if $\mu^* = 0$, then the domain of strong attraction is $\mathcal{C}$.

Proof. Denote $\widehat{\rho} := (\mu^*)^{-1/(\nu-1)}$, which is infinity if $\mu^* = 0$. We claim that the SHS (1) is strongly asymptotically stable on $\mathcal{B}_\rho\cap\mathcal{C}$ for any $\rho\in(0,\widehat{\rho})$. Indeed, for any given $\rho\in(0,\widehat{\rho})$, we can always find $\eta > 0$ such that $(\mu^*+\eta)^{-1/(\nu-1)} = \rho$. Since $\lim_{k\to\infty}\mu_k = \mu^*$, $\mu_k \le \mu^* + \eta/2$ for all $k \ge T$ for some sufficiently large $T\in\mathbb{N}$. Thus for any $z\in\mathcal{B}_\rho\cap\mathcal{C}$ and any σ, we have

$$\begin{aligned}\|x(t; z, \sigma)\| \;&\le\; \mu_t^{h_t}\|z\|^{\nu^t} \;\le\; \Big(\mu^*+\frac{\eta}{2}\Big)^{h_t}\rho^{\nu^t-1}\|z\| \;=\; \Big(\mu^*+\frac{\eta}{2}\Big)^{h_t}(u^*+\eta)^{-\frac{\nu^t-1}{\nu-1}}\,\|z\| \\ &\le\; \left(\frac{\mu^*+\eta/2}{\mu^*+\eta}\right)^{h_t}\|z\| \;\le\; \left(\frac{\mu^*+\eta/2}{\mu^*+\eta}\right)^{t}\|z\|, \quad \forall\, t\ge T,\end{aligned}$$

where $h_t = \frac{\nu^t-1}{\nu-1}$ is used. This yields the strong exponential (hence asymptotic) stability on $\mathcal{B}_\rho \cap \mathcal{C}$. As $\rho \in (0, \widehat{\rho})$ is arbitrary, it follows that $\rho^* \geq \widehat{\rho}$. As a result, when $\mu^* = 0$, we must have $\rho^* = \infty$.

Next we show that $\rho^* \leq \widehat{\rho}$, or equivalently, the SHS (1) is not strongly asymptotically stable on $\rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$ for any $\rho > \widehat{\rho}$. Without loss of generality we assume $\mu^* > 0$, hence $\widehat{\rho} < \infty$. For any given $\rho > \widehat{\rho}$, there exists $\eta \in (0, \mu^*)$ such that $\rho = (\mu^* - \eta)^{-\frac{1}{\nu-1}}$. Then for any $z \in \rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$ and under any switching sequence σ,

$$\|x(t; z, \sigma)\| \;=\; \|z\|^{\nu^t} \|x(t; z/\|z\|, \sigma)\| \;=\; (\mu^* - \eta)^{-\frac{\nu^t}{(\nu-1)}} \, \|x\,(t; z/\|z\|, \sigma)\| \,. \tag{10}$$

Since $\mathbb{S}^{n-1} \cap \mathcal{C}$ is a compact set and F_i's are continuous maps, we deduce from (8) that for any fixed time $t \in \mathbb{N}$, there exist some $z^*_t \in \rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$ and some switching sequence σ^t such that $\big\|x(t; z^*_t/\|z^*_t\|, \sigma^t)\big\| = (\mu_t)^{h_t} \geq (\mu^*)^{h_t}$. For this trajectory, equation (10) implies

$$\|x(t; z^*_t, \sigma^t)\| \geq (\mu^* - \eta)^{-\frac{\nu^t}{(\nu-1)}} \, (\mu^*)^{h_t} = (\mu^* - \eta)^{-\frac{1}{\nu-1}} \left(\frac{\mu^*}{\mu^* - \eta}\right)^{h_t} \geq \rho \left(\frac{\mu^*}{\mu^* - \eta}\right)^{t}.$$

This shows that the SHS is not strongly uniformly asymptotically stable on $\rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$, and thus not strongly asymptotically stable in view of Proposition 2.2. Therefore, $\rho^* \leq \widehat{\rho}$, and hence $\rho^* = \widehat{\rho}$. □

Theorem 2.4. Let $\nu > 1$. The radius of the domain of weak attraction of the SHS (1) on $\mathcal{C}$ is

$$\rho_* = (a_*)^{-\frac{1}{\nu-1}},$$

where a_* is the GJLSR of the SHS (1). Thus, if $a_* = 0$, then the domain of weak attraction is $\mathcal{C}$.

Proof. Let $\check{\rho} := (a_*)^{-\frac{1}{\nu-1}}$, which is infinity if $a_* = 0$. We first show that the SHS (1) is weakly asymptotically stable on $\mathcal{B}_\rho \cap \mathcal{C}$ for any $\rho \in (0, \check{\rho})$. Indeed, for any given $\rho \in (0, \check{\rho})$, there exists $\eta > 0$ such that $\rho = (a_* + \eta)^{-1/(\nu-1)}$. Since $\lim_{k\to\infty} a_k = a_*$, $a_k \leq a_* + \frac{\eta}{2}, \forall\, k \geq T$ for a sufficiently large $T \in \mathbb{N}$. By similar arguments as in the proof of Theorem 2.2, there exists a constant $\kappa \geq 1$ such that for any $z \in \mathcal{B}_\rho \cap \mathcal{C}$, we can find a switching sequence $\widetilde{\sigma}_z$ (concatenated by sequence segments, each of which is of length T) such that $\|x(t; z/\|z\|, \widetilde{\sigma}_z)\| \leq \kappa(a_* + \eta/2)^{h_t}, \forall\, t \in \mathbb{Z}_+$. Hence,

$$\begin{aligned}\|x(t; z, \widetilde{\sigma}_z)\| &\leq \|x(t; \frac{z}{\|z\|}, \widetilde{\sigma}_z)\| \cdot \|z\|^{\nu^t} \;\leq \kappa \left(a_* + \frac{\eta}{2}\right)^{h_t} \rho^{\nu^t - 1} \|z\| \;=\; \kappa \left(a_* + \frac{\eta}{2}\right)^{h_t} (a_* + \eta)^{-\frac{\nu^t-1}{\nu-1}} \|z\| \\ &\leq \kappa \left(\frac{a_* + \eta/2}{a_* + \eta}\right)^{h_t} \|z\| \;\leq\; \kappa \left(\frac{a_* + \eta/2}{a_* + \eta}\right)^{t} \|z\|, \quad \forall\; t \in \mathbb{Z}_+.\end{aligned}$$

This yields the weak exponential (hence asymptotic) stability on $\mathcal{B}_\rho \cap \mathcal{C}$. In the special case when $a_* = 0$, $\check{\rho} = \infty$ and therefore $\rho_* = \infty$. That is, the domain of weak attraction is the entire $\mathcal{C}$.

We next show that $\rho_* \leq \check{\rho}$, or equivalently, the SHS (1) is not weakly asymptotically stable on $\rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$ for any $\rho > \check{\rho}$. Without loss of generality we assume $a_* > 0$ such that $\check{\rho} < \infty$. For any given $\rho > \check{\rho}$, $\rho = (a_* - \eta)^{-\frac{1}{\nu-1}}$ for some $\eta \in (0, a_*)$. Then for any $z \in \rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$ and under any σ,

$$\|x(t; z, \sigma)\| \;=\; \|z\|^{\nu^t} \|x(t; z/\|z\|, \sigma)\| \;=\; (a_* - \eta)^{-\frac{\nu^t}{(\nu-1)}} \, \|x\,(t; z/\|z\|, \sigma)\| \,.$$

Since $\mathbb{S}^{n-1} \cap \mathcal{C}$ is a compact set and F_i's are continuous maps, we deduce from (9) that at any fixed time $t \in \mathbb{N}$, there exist some $z^*_t \in \rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$ and indices $j_1, \ldots, j_t \in \mathcal{M}$ such that $\inf_{i_1,\ldots,i_t \in \mathcal{M}} \|F_{i_1} \circ \cdots \circ F_{i_t}(z^*_t/\|z^*_t\|)\|^{1/h_t} = \|F_{j_1} \circ \cdots \circ F_{j_t}(z^*_t/\|z^*_t\|)\|^{1/h_t} = a_t$. As a result, under any switching sequence σ, $\big\|x(t; z^*_t/\|z^*_t\|, \sigma)\big\| \geq (a_t)^{h_t} \geq (a_*)^{h_t}$. Hence, under an arbitrary σ,

$$\|x(t; z^*_t, \sigma)\| \geq (a_* - \eta)^{-\frac{\nu^t}{(\nu-1)}} \, (a_*)^{h_t} = (a_* - \eta)^{-\frac{1}{\nu-1}} \left(\frac{a_*}{a_* - \eta}\right)^{h_t} \geq \rho \left(\frac{a_*}{a_* - \eta}\right)^{t}.$$

This shows that the SHS is not weakly (uniformly) asymptotically stable on $\rho\, \mathbb{S}^{n-1} \cap \mathcal{C}$, in light of Proposition 2.2. We thus conclude that $\rho_* \leq \check{\rho}$, and hence $\rho_* = \check{\rho}$. □

3 Stability of SLSs on Cones: A Generating Function Approach

The next two sections perform stability analysis and related numerical studies of the SHSs on cones via the generating function approach. We start our analysis from the simplest case where $\nu = 1$. In this case, the homogeneous dynamics are given by $F_i(x) = A_i x$ for some matrix $A_i \in \mathbb{R}^{n\times n}$ for each $i \in \mathcal{M}$ and the SHS (1) becomes the SLS on the closed cone $\mathcal{C}$:

$$x(t+1) = A_{\sigma(t)}x(t), \quad t \in \mathbb{Z}_+. \tag{11}$$

Here, all the A_i's are positively invariant with respect to $\mathcal{C}$. A particular example of SLSs on cones is *switched positive linear systems*, where $\mathcal{C} = \mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$, and all A_i are positive matrices [6, 9, 10]. The linear structure of the SLSs simplifies stability analysis and enables us to develop global results. For example, different from SHSs with $\nu > 1$, the growth of system trajectories of the SLSs is exponential rather than super-exponential (cf. (5)) and local and global stabilities are equivalent in each sense defined in Section 2. Using similar arguments as in Propositions 2.1 and 2.3, we have the following result.

Proposition 3.1. The following hold for the SLS (11) on the closed cone $\mathcal{C}$:

strong convergence $\Leftrightarrow$ strong asymptotic stability $\Leftrightarrow$ strong exponential stability

weak convergence $\Leftrightarrow$ weak asymptotic stability $\Leftrightarrow$ weak exponential stability

It is worth pointing out that the SLS (11) on $\mathcal{C}$ can be thought of as embedded in a SLS on $\mathbb{R}^n$, with the latter having the same dynamics (11) but in the larger state space $\mathbb{R}^n$. The various stability notions for the SLS on $\mathbb{R}^n$ can be defined similarly as in Definition 2.1 by setting $\nu = 1$ and $\mathcal{C} = \mathbb{R}^n$. Conversely, given a SLS on $\mathbb{R}^n$ and a closed cone $\mathcal{C} \subset \mathbb{R}^n$ invariant under A_i's, the restriction of the SLS on $\mathbb{R}^n$ to $\mathcal{C}$ yields a SLS on the cone $\mathcal{C}$. Stability of the former SLS in any sense implies that of the latter, but not the other way around. Thus, the stability study for SLSs on cones poses new challenges beyond that for SLSs on $\mathbb{R}^n$.

We briefly review some basic notions of cones used later; see [5] for details. A cone $\mathcal{C}$ is *pointed* if the condition that $x_1 + \cdots + x_k = 0$ with $x_i \in \mathcal{C}$, $i = 1, \dots, k$, implies that $x_i = 0$ for all i. A convex cone $\mathcal{C}$ is pointed if and only if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$, or equivalently, $\mathcal{C}$ does not contain a nontrivial subspace. For example, $\mathbb{R}^n_+$ is pointed. A convex cone $\mathcal{C}$ can be decomposed as $\mathcal{C} = \mathcal{K} + \mathcal{V}$, where $\mathcal{K}$ is a pointed cone and $\mathcal{V} = \mathcal{C} \cap (-\mathcal{C})$ is a subspace (i.e., the linearity space of $\mathcal{C}$) orthogonal to $\mathcal{K}$: $\mathcal{K} \perp \mathcal{V}$. A cone $\mathcal{C}$ is *solid* if it has nonempty interior. For example, $\mathbb{R}^n_+$ is solid and hence *proper* (i.e., closed, convex, solid, and pointed.)

3.1 Strong Generating Functions of the SLSs on Cones

In the recent paper [14], the notion of strong generating functions is proposed to study the strong exponential stability of the SLSs on $\mathbb{R}^n$. The strong generating function of the SLS (11) on $\mathbb{R}^n$ is the map $G : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined as follows: for each $z \in \mathbb{R}^n$ and $\lambda \geq 0$,

$$G_\lambda(z) := G(\lambda, z) := \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^q, \tag{12}$$

where the supremum is taken over all switching sequences, q is a positive integer, and $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$. Note that $G_\lambda(z)$ is non-decreasing in $\lambda \geq 0$. The *radius of strong convergence* of the strong generating function on $\mathbb{R}^n$ is defined as $\lambda^*_{\mathbb{R}^n} := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty, \ \forall\, z \in \mathbb{R}^n\}$. The following result, proved in [14, Theorem 2], shows that the radius of strong convergence $\lambda^*_{\mathbb{R}^n}$ completely characterizes the strong exponential stability of the SLS on $\mathbb{R}^n$.

Theorem 3.1. The SLS (11) on $\mathbb{R}^n$ is strongly exponentially stable if and only if $\lambda^*_{\mathbb{R}^n} > 1$.

We now extend the above strong generating function to the SLS (11) on a closed cone $\mathcal{C}$. Define $\mathcal{W}$ to be the smallest *subspace* of $\mathbb{R}^n$ that contains the cone $\mathcal{C}$ and is invariant to $\{A_i\}_{i\in\mathcal{M}}$, or equivalently, the set of all states generated from elements of $\mathcal{C}$ through multiplications by matrices in $\{A_i\}_{i\in\mathcal{M}}$ and linear combinations: $\mathcal{W} := \text{span}\big\{\, \mathcal{C},\, \cup_{i\in\mathcal{M}} A_i\mathcal{C},\, \cup_{i,j\in\mathcal{M}} A_iA_j\mathcal{C},\, \dots\, \big\}$. In particular, if $\mathcal{C}$ is solid, then $\mathcal{W} = \mathbb{R}^n$. If $\mathcal{C}$ is polyhedral, i.e., it is finitely (and positively) generated: $\mathcal{C} = \{\sum_{k=1}^{\ell} \alpha_k\, v^k \,|\, \alpha_i \geq 0\}$ for some vectors $v^k \in \mathbb{R}^n, k = 1, \cdots, \ell$, then

$$\mathcal{W} = \text{span}\Big\{ \{v^k\}_{k=1}^{\ell},\ \cup_{i\in\mathcal{M}} A_i \{v^k\}_{k=1}^{\ell},\ \cup_{i,j\in\mathcal{M}} A_iA_j\{v^k\}_{k=1}^{\ell},\ \dots \Big\}.$$

Note that $\mathcal{C} \subseteq \mathcal{W} \subseteq \mathbb{R}^n$ form a cascade of sets invariant to $\{A_i\}_{i\in\mathcal{M}}$. Hence, the SLS (11) restricted to each set is well defined and $G_\lambda(z)$ in (12) defined on $\mathbb{R}^n$ can be extended to that on $\mathcal{C}$ and $\mathcal{W}$ as well. Specifically, the strong generating function of the SLS (11) on $\mathcal{C}$ is defined as

$$G_\lambda(z) := \sup_{\sigma} \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^q, \quad \forall z \in \mathcal{C},\ \lambda \geq 0. \tag{13}$$

Here, the same notation $G_\lambda(\cdot)$ is used since $G_\lambda(\cdot)$ in (13) is exactly the restriction of $G_\lambda(\cdot)$ in (12) on the cone $\mathcal{C}$. For this reason, we simply refer to (13) as the strong generating function on $\mathcal{C}$. Similarly, we can define $G_\lambda(\cdot)$ on $\mathcal{W}$ as the restriction of (12) on $\mathcal{W}$.

Define the radii of strong convergence on $\mathcal{C}$ and $\mathcal{W}$ respectively as

$$\lambda^*_{\mathcal{C}} := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty,\ \forall z \in \mathcal{C}\}, \quad \lambda^*_{\mathcal{W}} := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty,\ \forall z \in \mathcal{W}\}.$$

For each $\lambda \geq 0$, define the three subsets: $\mathcal{G}_\lambda(\mathcal{C}) := \{z \in \mathcal{C} \,|\, G_\lambda(z) < \infty\} \subseteq \mathcal{C}$, $\mathcal{G}_\lambda(\mathcal{W}) := \{z \in \mathcal{W} \,|\, G_\lambda(z) < \infty\} \subseteq \mathcal{W}$, and $\mathcal{G}_\lambda(\mathbb{R}^n) := \{z \in \mathbb{R}^n \,|\, G_\lambda(z) < \infty\} \subseteq \mathbb{R}^n$, which satisfy $\mathcal{G}_\lambda(\mathcal{C}) \subseteq \mathcal{G}_\lambda(\mathcal{W}) \subseteq \mathcal{G}_\lambda(\mathbb{R}^n)$, and $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{C}$, $\mathcal{G}_\lambda(\mathcal{W}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{W}$.

Obtained through the above restriction, the strong generating functions on $\mathcal{C}$ and $\mathcal{W}$ inherit many of the properties of their counterpart on $\mathbb{R}^n$ established in [14], as listed below.

Proposition 3.2. For any $q \in \mathbb{N}$ and any vector norm $\|\cdot\|$, the strong generating function $G_\lambda(z)$ of the SLS (11) on $\mathcal{C}$ (or on $\mathcal{W}$) has the following properties.

1. (Bellman Equation): For all $\lambda \geq 0$ and $z \in \mathcal{C}$ (or $\mathcal{W}$), $G_\lambda(z) = \|z\|^q + \lambda \cdot \max_{i\in\mathcal{M}} G_\lambda(A_i z)$.
2. (Sub-additivity): If $\mathcal{C}$ is convex (in addition to being closed), then for each $\lambda \geq 0$,
$$\big(G_\lambda(z_1 + z_2)\big)^{1/q} \leq \big(G_\lambda(z_1)\big)^{1/q} + \big(G_\lambda(z_2)\big)^{1/q}, \quad \forall\, z_1, z_2 \in \mathcal{C} \text{ (or } \mathcal{W}).$$
3. (Convexity): If $\mathcal{C}$ is convex, then for each $\lambda \geq 0$, $\big(G_\lambda(\cdot)\big)^{1/q}$ is convex on $\mathcal{C}$ (or $\mathcal{W}$).
4. (Invariant Cone): Let $\lambda \geq 0$ be arbitrary. If $\mathcal{C}$ is convex, then the set $\mathcal{G}_\lambda(\mathcal{C})$ is a closed convex cone in $\mathcal{C}$ invariant to $\{A_i\}_{i\in\mathcal{M}}$. Further, if $\mathcal{C}$ is polyhedral, so is $\mathcal{G}_\lambda(\mathcal{C})$.
5. (Invariant Subspace): For any $\lambda \geq 0$, the set $\mathcal{G}_\lambda(\mathcal{W})$ is a subspace of $\mathcal{W}$ invariant to $\{A_i\}_{i\in\mathcal{M}}$.
6. $G_\lambda(\cdot)$ is finite everywhere on $\mathcal{C}$ for $0 \leq \lambda < (\max_{i\in\mathcal{M}} \|A_i\|^q)^{-1}$, where the matrix norm is induced from the vector norm $\|\cdot\|$.

Proof. 1. This follows directly from the dynamic programming principle.

2. By the convexity of $\mathcal{C}$, for any $z_1, z_2 \in \mathcal{C}$, $z_1 + z_2 \in \mathcal{C}$. Further, due to the linearity property of the SLS, $x(t; z_1 + z_2, \sigma) = x(t; z_1, \sigma) + x(t; z_2, \sigma)$ under any σ. Then, by the definition of G_λ,

$$\begin{aligned} G_\lambda(z_1 + z_2) &= \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t; z_1, \sigma) + x(t; z_2, \sigma)\|^q \ \le\ \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \Big(\|x(t; z_1, \sigma)\| + \|x(t; z_2, \sigma)\| \Big)^q \\ &\le \sup_\sigma \left[\left(\sum_{t=0}^{\infty} \lambda^t \|x(t; z_1, \sigma)\|^q \right)^{1/q} + \left(\sum_{t=0}^{\infty} \lambda^t \|x(t; z_2, \sigma)\|^q \right)^{1/q} \right]^q \\ &\le \left[\big(G_\lambda(z_1)\big)^{1/q} + \big(G_\lambda(z_2)\big)^{1/q} \right]^q, \end{aligned}$$

where the second inequality is due to the Minkowski inequality. The case for $\mathcal{W}$ is entirely similar.

3. This is due to the sub-additivity and the positive homogeneity of $\big(G_\lambda(\cdot)\big)^{1/q}$.

4. The conic property and the convexity of $\mathcal{G}_\lambda(\mathcal{C})$ follow from the positive homogeneity and the convexity of $\big(G_\lambda(\cdot)\big)^{1/q}$, respectively. The invariance to $\{A_i\}_{i\in\mathcal{M}}$ is a consequence of the Bellman equation. To show that $\mathcal{G}_\lambda(\mathcal{C})$ is a closed cone, note that $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{C}$, where $\mathcal{G}_\lambda(\mathbb{R}^n)$ is a subspace due to the sub-additivity property on $\mathbb{R}^n$. Thus, $\mathcal{G}_\lambda(\mathcal{C})$ as the intersection of the closed cone $\mathcal{C}$ and a subspace must be closed itself. Further, it is polyhedral whenever $\mathcal{C}$ is so.

5. $\mathcal{G}_\lambda(\mathcal{W}) = \mathcal{G}_\lambda(\mathbb{R}^n) \cap \mathcal{W}$ is clearly a subspace, and the invariance is due to the definition of $\mathcal{W}$.

6. This follows from (13) and $\|x(t; z, \sigma)\|^q \le (\max_{i\in\mathcal{M}} \|A_i\|^q)^t \|z\|^q$ for all $t \in \mathbb{Z}_+$. ☐

In addition to the above inherited properties, the strong generating functions on $\mathcal{C}$ and on $\mathcal{W}$ also have some other shared properties. Obviously, the former is the restriction of the latter on the cone $\mathcal{C}$. Less obviously, we have the following.

Proposition 3.3. The strong generating functions $G_\lambda(z)$ of the SLS (11) on $\mathcal{C}$ and on $\mathcal{W}$ satisfy:

$$G_\lambda(z) < \infty,\ \forall\, z \in \mathcal{C} \quad \Longleftrightarrow \quad G_\lambda(z) < \infty,\ \forall\, z \in \mathcal{W}.$$

Proof. As $\mathcal{C} \subset \mathcal{W}$, it suffices to show the "$\Rightarrow$" direction. Suppose $G_\lambda(\cdot)$ is finite on $\mathcal{C}$, i.e., $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{C}$. Since $\mathcal{G}_\lambda(\mathbb{R}^n)$ is a subspace of $\mathbb{R}^n$ invariant to $\{A_i\}_{i\in\mathcal{M}}$ and contains $\mathcal{G}_\lambda(\mathcal{C})$ hence $\mathcal{C}$, it must also contain $\mathcal{W}$, as $\mathcal{W}$ is the smallest subspace containing $\mathcal{C}$ invariant to $\{A_i\}_{i\in\mathcal{M}}$. In other words, $\mathcal{W} \subseteq \mathcal{G}_\lambda(\mathbb{R}^n)$, which implies that $G_\lambda(z)$ is finite for all $z \in \mathcal{W}$. ☐

As a result of Proposition 3.3, the radii of strong convergence on $\mathcal{C}$, $\mathcal{W}$, and $\mathbb{R}^n$ satisfy:

Corollary 3.1. $\lambda^*_{\mathcal{C}} = \lambda^*_{\mathcal{W}} \ge \lambda^*_{\mathbb{R}^n}$. In particular, if $\mathcal{C}$ is solid, then $\lambda^*_{\mathcal{C}} = \lambda^*_{\mathcal{W}} = \lambda^*_{\mathbb{R}^n}$.

We next prove some additional properties of G_λ on its growth and continuity.

Proposition 3.4. The following hold for the strong generating functions $G_\lambda(z)$ on $\mathcal{C}$ and $\mathcal{W}$:

1. If $\lambda \in [0, \lambda^*_{\mathcal{C}})$ (hence $G_\lambda(z) < \infty$ for all $z \in \mathcal{C}$), then a constant $c \in [1, \infty)$ exists such that $\|z\| \le \big(G_\lambda(z)\big)^{1/q} \le c\|z\|, \forall\ z \in \mathcal{W}$.
2. Let $\lambda \in [0, \lambda^*_{\mathcal{C}})$. Then $\big(G_\lambda(\cdot)\big)^{1/q}$ is relatively Lipschitz on $\mathcal{W}$, i.e., there exists $L > 0$ such that for any $x, y \in \mathcal{W}$, $\big|\big(G_\lambda(x)\big)^{1/q} - \big(G_\lambda(y)\big)^{1/q}\big| \le L\|x - y\|$.

Proof. 1. The first inequality is obvious as $G_\lambda(z) \geq \|z\|^q$. To show the second, by homogeneity, it suffices to show that $\big(G_\lambda(z)\big)^{1/q} \leq c$, $\forall z \in \mathcal{W} \cap \mathbb{S}^{n-1}$, for some constant $c \geq 1$. Let $\{u^i\}_{i=1}^{\ell}$ be a basis of $\mathcal{W}$. Since $\mathcal{W} \cap \mathbb{S}^{n-1}$ is bounded, we can find a finite $\gamma > 0$ such that each $z \in \mathcal{W} \cap \mathbb{S}^{n-1}$ admits a unique representation as $z = \sum_{j=1}^{\ell} \alpha_j u^j$ for some real tuple $(\alpha_1, \dots, \alpha_\ell)$ satisfying $\sum_{j=1}^{\ell} |\alpha_j| < \gamma$. Therefore, by virtue of the sub-additivity on $\mathcal{W}$ and positive homogeneity of $(G_\lambda)^{1/q}$, $\big(G_\lambda(z)\big)^{1/q} \leq c := \gamma \sum_{i=1}^{\ell} \big(G_\lambda(u^i)\big)^{1/q}$ for all $z \in \mathcal{W} \cap \mathbb{S}^{n-1}$. This proves the second inequality.

2. It follows from the sub-additivity of $(G_\lambda)^{1/q}$ on $\mathcal{W}$ that for any $x, y \in \mathcal{W}$, $\big(G_\lambda(x)\big)^{1/q} - \big(G_\lambda(y)\big)^{1/q} \leq \big(G_\lambda(x-y)\big)^{1/q}$. This inequality, together with the one obtained by switching x and y, implies that $\big|\big(G_\lambda(x)\big)^{1/q} - \big(G_\lambda(y)\big)^{1/q}\big| \leq \big(G_\lambda(x-y)\big)^{1/q} \leq c\|x-y\|$, where the last step is due to $(x-y) \in \mathcal{W}$ and the first statement of this proposition. □

Remark 3.1. The above results imply that if $\mathcal{C}$ is a closed convex cone, then $\mathcal{G}_\lambda(\mathcal{C})$ is a closed convex sub-cone of $\mathcal{C}$. Suppose that $\mathcal{C}$ admits the decomposition $\mathcal{C} = \mathcal{K} + \mathcal{V}$, where $\mathcal{K}$ is a pointed cone and $\mathcal{V}$ is a subspace, then $\mathcal{G}_\lambda(\mathcal{C}) = \mathcal{K}_\lambda + \mathcal{V}_\lambda$ with $\mathcal{K}_\lambda \subset \mathcal{K}$ a pointed cone and $\mathcal{V}_\lambda \subset \mathcal{V}$ a subspace. As λ increases, G_λ will increase, hence the invariant subsets $\mathcal{G}_\lambda(\mathcal{C})$, $\mathcal{G}_\lambda(\mathcal{W})$, and $\mathcal{G}_\lambda(\mathbb{R}^n)$ will shrink. In particular, if $\mathcal{C}$ is not pointed (i.e., $\mathcal{V} \neq \{0\}$), then as λ increases, $\mathcal{G}_\lambda(\mathcal{C})$ will change from non-pointed to pointed, or equivalently, $\mathcal{V}_\lambda$ will shrink to $\{0\}$, at exactly $\lambda^*_{\mathcal{V}} := \inf\{\lambda \geq 0 \,|\, G_\lambda(z) = \infty, \ \forall z \in \mathcal{V}\}$.

The radius of strong convergence on either $\mathcal{C}$ or $\mathcal{W}$ completely characterizes the strong exponential stability of the SLS on $\mathcal{C}$ as stated below.

Theorem 3.2. The following are equivalent:

1. The SLS (11) is strongly exponentially stable on the closed cone $\mathcal{C}$ (or on the subspace $\mathcal{W}$);
2. $\lambda^*_{\mathcal{C}} = \lambda^*_{\mathcal{W}} > 1$;
3. $G_1(z)$ is finite for all $z \in \mathcal{C}$ (or $\mathcal{W}$).

Proof. By Propositions 3.1 and 3.4, the proof is essentially as same as that of [14, Theorem 1]. □

Remark 3.2. Define the joint spectral radius of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ on the closed cone $\mathcal{C}$ by $\mu^*_{\mathcal{C}} := \lim_{k\to\infty} \big(\sup\{\|A_{i_1}\cdots A_{i_k} z\|^{1/k} \ : \ i_1, \dots, i_k \in \mathcal{M}, \ z \in \mathbb{S}^{n-1} \cap \mathcal{C}\}\big)$; see Section 2.3.1. Similarly the JSR on $\mathcal{W}$, denoted by $\mu^*_{\mathcal{W}}$, can be defined. Theorem 3.2 implies $\mu^*_{\mathcal{C}} = \mu^*_{\mathcal{W}}$. Moreover, $\mu^*_{\mathcal{C}} = (\lambda^*_{\mathcal{C}})^{-1/q} = (\lambda^*_{\mathcal{W}})^{-1/q}$.

3.2 Weak Generating Functions of the SLSs on Cones

Similar to the strong generating functions, weak generating functions can be defined to address weak stability of SLSs on cones. Specifically, for the SLS (11) on the closed cone $\mathcal{C}$, its weak generating function $H : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+ \cup \{\infty\}$ is defined as

$$H(\lambda, z) := H_\lambda(z) := \inf_{\sigma} \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^q, \quad \forall\, \lambda \geq 0, \quad z \in \mathcal{C}, \tag{14}$$

where $q \in \mathbb{N}$ and the infimum is over all switching sequences σ. If in the above definition, z takes values in $\mathbb{R}^n$ rather than $\mathcal{C}$, then the resulting $H_\lambda(z)$ becomes the weak generating function on $\mathbb{R}^n$ introduced in [14] for the study of weak stability of SLS on $\mathbb{R}^n$.

Proposition 3.5. For any $q \in \mathbb{N}$ and any vector norm $\|\cdot\|$, the weak generating function $H_\lambda(z)$ of the SLS (11) on $\mathcal{C}$ has the following properties.

1. (Bellman Equation): For any $\lambda \geq 0$ and $z \in \mathcal{C}$, $H_\lambda(z) = \|z\|^q + \lambda \cdot \min_{i \in \mathcal{M}} H_\lambda(A_i z)$.

2. (Invariant Cone): For any $\lambda \geq 0$, the set $\mathcal{H}_\lambda(\mathcal{C}) := \{z \in \mathcal{C} \,|\, H_\lambda(z) = \infty\}$ is a sub-cone in $\mathcal{C}$ not containing 0, and invariant to $\{A_i\}_{i\in\mathcal{M}}$, i.e., $A_i \mathcal{H}_\lambda(\mathcal{C}) \subseteq \mathcal{H}_\lambda(\mathcal{C})$, $\forall i \in \mathcal{M}$.

3. $H_\lambda(z)$ is finite everywhere on $\mathcal{C}$ for $0 \leq \lambda < (\min_{i\in\mathcal{M}} \|A_i\|^q)^{-1}$, where the matrix norm is induced from the vector norm $\|\cdot\|$.

Proof. The proofs for these properties are similar to those in Proposition 3.2 or in [14, Proposition 7] for the related properties of the weak generating function on $\mathbb{R}^n$; hence they are omitted. □

The radius of weak convergence is defined as $\lambda_*^{\mathcal{C}} := \sup\{\lambda \geq 0 \,|\, H_\lambda(z) < \infty, \ \forall\, z \in \mathcal{C}\}$. The following proposition shows two results: (i) as λ increases, $\lambda_*^{\mathcal{C}}$ is the exact value at which $H_\lambda(\cdot)$ starts to have the infinite value on $\mathcal{C}$; and (ii) if $H_\lambda(\cdot)$ is finite everywhere on $\mathcal{C}$ for some $\lambda \geq 0$, then it is bounded above by the homogeneous function $c\|\cdot\|^q$ on $\mathcal{C}$.

Proposition 3.6. For each $\lambda \geq 0$, the following are equivalent:

(a) $H_\lambda(z) \leq c\|z\|^q$, $\forall\, z \in \mathcal{C}$, for some constant $c > 0$ (generally dependent on λ);

(b) $H_\lambda(z) < \infty$ for all $z \in \mathcal{C}$;

(c) $\lambda \in [0, \lambda_*^{\mathcal{C}})$.

Proof. It follows from the similar argument in [14, Proposition 8] with $\mathbb{R}^n$ replaced by $\mathcal{C}$. □

The next result, whose proof is left out due to its similarity with that of [14, Theorem 4], shows that the radius of weak convergence $\lambda_*^{\mathcal{C}}$ characterizes the weak exponential stability of the SLS on $\mathcal{C}$.

Theorem 3.3. The SLS (11) on $\mathcal{C}$ is weakly exponentially stable if and only if $\lambda_*^{\mathcal{C}} > 1$.

In what follows, we establish a connection between the radius of weak convergence and the GJLSR. Recall that in Section 2.3.2, the GJLSR of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ on the closed cone $\mathcal{C}$ is defined as $\breve{\rho}_{\mathcal{C}} := \inf\{a_k\}$, where

$$a_k := \sup_{z \in \mathbb{S}^{n-1}\cap\mathcal{C}} \left(\inf\{\, \left\|A_{i_1}\cdots A_{i_k} z\right\|^{1/k} \,:\, i_1,\ldots,i_k \in \mathcal{M}\,\}\right).$$

The connection between this quantity and the radius of weak convergence $\lambda_*^{\mathcal{C}}$ is shown below.

Theorem 3.4. The following holds: $\breve{\rho}_{\mathcal{C}} \;=\; (\lambda_*^{\mathcal{C}})^{-1/q}$.

Proof. Without loss of generality, we assume $\breve{\rho}_{\mathcal{C}} > 0$. We first show that $\breve{\rho}_{\mathcal{C}} \cdot (\lambda_*^{\mathcal{C}})^{1/q} \leq 1$. Suppose otherwise. Then there exist $\lambda \in (0, \lambda_*^{\mathcal{C}})$ and $\grave{\rho} \in (0, \breve{\rho}_{\mathcal{C}})$ such that $\grave{\rho} \cdot \lambda^{1/q} > 1$. Since $\grave{\rho} < \breve{\rho}_{\mathcal{C}}$ and $\breve{\rho}_{\mathcal{C}} = \inf_k\{a_k\}$, we have $a_k \geq \grave{\rho}$ for all k. By the compactness of $\mathbb{S}^{n-1}\cap\mathcal{C}$ and the continuity of the linear mappings $A_{i_1}\cdots A_{i_k}$ (for fixed indices i_j), we can find $z_k \in \mathbb{S}^{n-1}\cap\mathcal{C}$ and $i_1^*,\ldots,i_k^* \in \mathcal{M}$ such that $a_k = \|A_{i_1^*}\cdots A_{i_k^*} z_k\|^{1/k}$ for each fixed k. For each z_k, let $\widetilde{\sigma}_{z_k}$ be a switching sequence such that $H_\lambda(z_k) = \sum_{t=0}^\infty \lambda^t \|x(t; z_k, \widetilde{\sigma}_{z_k})\|^q$. Note that

$$\left\|x(k; z_k, \widetilde{\sigma}_{z_k})\right\| = \left\|A_{\widetilde{\sigma}_{z_k}(k-1)}\cdots A_{\widetilde{\sigma}_{z_k}(0)} z_k\right\| \;\geq\; \left\|A_{i_1^*}\cdots A_{i_k^*} z_k\right\| \;=\; (a_k)^k \;\geq\; (\grave{\rho})^k.$$

This shows that $H_\lambda(z_k) \;\geq\; \lambda^k \|x(k; z_k, \widetilde{\sigma}_{z_k})\|^q \;\geq\; \left(\grave{\rho}\cdot\lambda^{1/q}\right)^{q\,k}$. Since $\grave{\rho}\cdot\lambda^{1/q} > 1$, by choosing k large enough, $H_\lambda(z_k)$ can be arbitrarily large. This contradicts $\lambda \in (0, \lambda_*^{\mathcal{C}})$, in view of Proposition 3.6.

We next show that $\breve{\rho}_{\mathcal{C}} \cdot (\lambda_*^{\mathcal{C}})^{1/q} \geq 1$. Toward this end, we firstly prove that

$$\lambda > \lambda_*^{\mathcal{C}} \implies \breve{\rho}_{\mathcal{C}} \cdot \lambda^{1/q} \geq 1. \tag{15}$$

Fix $\lambda > \lambda_*^{\mathcal{C}}$. By Proposition 3.6 and [14, Proposition 8], the scaled SLS with subsystem matrices $\{\widetilde{A}_i\}_{i\in\mathcal{M}}$, where $\widetilde{A}_i := \lambda^{1/q} A_i$, is not weakly exponentially stable, hence not weakly convergent either, on $\mathcal{C}$. Denote by $\widetilde{x}(t; z, \sigma)$ the trajectories of the scaled SLS. It follows from an extension of [30, Theorem 1] to the SLS on a cone that there exists $z^* \in \mathbb{S}^{n-1} \cap \mathcal{C}$ such that under any switching sequence σ, $\|\widetilde{x}(t; z^*, \sigma)\| \geq \|z^*\| = 1, \forall\, t \in \mathbb{Z}_+$. Let $i_1^*, \ldots, i_k^* \in \mathcal{M}$ be such that $\|\widetilde{A}_{i_1^*} \cdots \widetilde{A}_{i_k^*} z^*\| = \inf\{\, \|\widetilde{A}_{i_1} \cdots \widetilde{A}_{i_k} z^*\| \,:\, i_1, \ldots, i_k \in \mathcal{M}\,\}$. Then,

$$\widetilde{a}_k := \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \inf\left\{ \left\|\widetilde{A}_{i_1} \cdots \widetilde{A}_{i_k} z\right\|^{1/k} \,:\, i_1, \ldots, i_k \in \mathcal{M} \right\} \geq \left\|\widetilde{A}_{i_1^*} \cdots \widetilde{A}_{i_k^*} z^*\right\|^{1/k} \geq 1.$$

Therefore, $\inf\{\widetilde{a}_k\} \geq 1$. Noting that $\inf\{\widetilde{a}_k\} = \lambda^{1/q} \cdot \breve{\rho}_{\mathcal{C}}$, we deduce that the implication (15) holds. Letting $\lambda \downarrow \lambda_*$, we obtain $\breve{\rho}_{\mathcal{C}} \cdot (\lambda_*^{\mathcal{C}})^{1/q} \geq 1$ from (15), and in turn, $(\lambda_*^{\mathcal{C}})^{-1/q} = \breve{\rho}_{\mathcal{C}}$. □

Remark 3.3. Another relevant quantity, the *joint spectral subradius* (JSSR) of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ on $\mathcal{C}$, is defined as [15]

$$\check{\rho}_{\mathcal{C}} := \lim_{k\to\infty} \inf\left\{ \left(\sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \left\|A_{i_1} \cdots A_{i_k} z\right\|^{1/k} \right) \,:\, i_1, \ldots, i_k \in \mathcal{M} \right\}.$$

It is shown in [15, Theorem 1.1] that the standard JSSR (with $\mathcal{C} = \mathbb{R}^n$) satisfies

$$\check{\rho} = \lim_{k\to\infty}\inf\left\{ [\rho(A_{i_1} \cdots A_{i_k})]^{1/k} \,:\, i_1, \ldots, i_k \in \mathcal{M} \right\} = \lim_{k\to\infty}\inf\left\{ \left\|A_{i_1} \cdots A_{i_k}\right\|^{1/k} \,:\, i_1, \ldots, i_k \in \mathcal{M} \right\},$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. The JSSR characterizes the convergence rate of the SLS trajectories $x(t; z, \sigma^*)$ under the best fixed switching sequence σ^* that is chosen independently of z. In fact, the SLS is convergent uniformly in z if and only if $\check{\rho} < 1$ [7, 31]. As pointed out in [14], $\check{\rho}_{\mathcal{C}} \geq \breve{\rho}_{\mathcal{C}}$ and a gap between the two spectral radii exists in general.

3.3 Computation of Generating Functions and Radii of Convergence

One of the key advantages of the generating function approach for the SLS on $\mathbb{R}^n$ is that it yields efficient numerical schemes for computing the maximum exponential growth rates of the SLS under different switching rules; see [14, Sections III.F and IV.E]. These algorithms can be extended to the SLS on a closed convex cone $\mathcal{C}$ with minor modifications. We briefly discuss it as follows.

Consider the strong generating function G_λ first. Define the finite-horizon approximation of G_λ on $\mathcal{C}$ as: $G_\lambda^k(z) := \max_\sigma \sum_{t=0}^k \lambda^t \|x(t; z, \sigma)\|^q$, $\forall\, z \in \mathcal{C}$. It is easy to see that G_λ^k satisfies the Bellman equation: $G_\lambda^k(z) = \|z\|^q + \lambda \max_{i\in\mathcal{M}} G_\lambda^{k-1}(A_i z)$, $\forall\, z \in \mathcal{C}$, with $G_\lambda^0(z) = \|z\|^q$. Thus one can apply this equation to compute G_λ^k on $\mathcal{C} \cap \mathbb{S}^{n-1}$ recursively. In view of Propositions 3.2 and 3.4 and along with similar arguments in [14, Proposition 6], it can be shown that the sequence $\big(G_\lambda^k\big)$ converges uniformly and exponentially fast to G_λ on $\mathcal{C} \cap \mathbb{S}^{n-1}$. Further, an over-approximation can be obtained using the convex and conic structure of $\mathcal{C}$ for an effective implementation; see Algorithm 1 in [14] for details. This procedure thus can be used to compute $\lambda_{\mathcal{C}}^*$.

The similar extension can be made for numerical approximation of the weak generating function H_λ on a closed convex cone $\mathcal{C}$ and $\lambda_*^{\mathcal{C}}$ using the corresponding Bellman equation; see [14, Algorithm 2] for computing H_λ. The details are omitted.

4 Stability of SHSs with $\nu > 1$ on Cones: A Generating Function Approach

In this section, we focus on more nonlinear SHSs on cones, i.e., the SHSs of homogeneous degree $\nu > 1$. A generating function based approach is invoked to characterize domains of attraction. This approach not only serves as a complement to the joint spectral radius approach discussed in Section 2.3 but also offers a numerically efficient way to compute the radii of domains of attraction. Due to the nonlinearity of the vector fields F_i, the generating functions of the SHSs with $\nu > 1$ on cones do not inherit many favorable properties of those for the SLSs, such as sub-additivity and convexity. This leads to new techniques for numerical approximation and convergence analysis of the generating functions as shown in the following sections.

4.1 Strong Generating Functions of the SHSs on Cones

We introduce the strong generating function $G(\cdot, \cdot) : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+ \cup \{\infty\}$ of the SHS (1) on $\mathcal{C}$:

$$G(\lambda, z) := \sup_{\sigma} \sum_{t=0}^{\infty} \lambda^{h_t} \|x(t; z, \sigma)\|^q, \quad \forall \lambda \geq 0, \quad z \in \mathcal{C}, \tag{16}$$

where the supremum is taken over all switching sequences σ, $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $q \in \mathbb{N}$, and recall $h_t := \sum_{i=0}^{t-1} \nu^i$, $t \in \mathbb{N}$ with $h_0 := 0$. We may write $G(\lambda, z)$ as $G_\lambda(z)$ for a fixed $\lambda \geq 0$. Note that if $\nu = 1$, then $h_t = t$ and the above generating function reduces to the one for the SLS on $\mathcal{C}$ defined in Section 3. The following proposition collects some basic properties of G_λ on $\mathcal{C}$; more properties, such as continuity, will be shown via approximation of G_λ in Proposition 4.3.

Proposition 4.1. The strong generating function $G(\lambda, z)$ has the following properties.

1. (Scaling property): For any $\alpha \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$ and $z \in \mathcal{C}$, $G_\lambda(\alpha z) = \alpha^q G_{\lambda \cdot \alpha^{q(\nu-1)}}(z)$.
2. (Bellman Equation): For all $\lambda \geq 0$ and $z \in \mathcal{C}$,
$$G_\lambda(z) = \|z\|^q + \max_{i \in \mathcal{M}} G_\lambda(\lambda^{\frac{1}{q}} F_i(z)). \tag{17}$$
3. (Monotonicity): For any fixed $z \in \mathcal{C}$, $G_{\lambda_1}(z) \leq G_{\lambda_2}(z)$ whenever $0 \leq \lambda_1 \leq \lambda_2$.
4. For $0 \leq \lambda < (\mu_1)^{-q}$, where μ_1 is defined in (3), $\sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G_\lambda(z)$ is finite.
5. (Exponential Stability): If the SHS (1) is strongly exponentially stable on $\mathcal{B}_\rho \cap \mathcal{C}$, then $G_1(z) \leq c$, $\forall\, z \in \mathcal{B}_\rho \cap \mathcal{C}$ for some $c > 0$. Conversely, if the latter holds, then for any $r \in (0, \rho)$, the SHS (1) is strongly exponentially stable on $\mathcal{B}_r \cap \mathcal{C}$.

Proof. Property 1 follows since $\|x(t; \alpha z, \sigma)\|^q = \alpha^{q\nu^t} \|x(t; z, \sigma)\|^q$ and $\nu^t = h_t(\nu - 1) + 1$.

Property 2. It is noted that for any given $\lambda \geq 0$, $z \in \mathcal{C}$ and $\sigma = (\sigma(0), \sigma^*)$, where $\sigma^* = (\sigma(1), \sigma(2), \ldots)$,

$$\begin{aligned}\sum_{t=1}^{\infty} \lambda^{h_t} \big\|x(t; z, \sigma)\big\|^q &= \sum_{s=0}^{\infty} \lambda^{h_{s+1}} \big\|x(s; F_{\sigma(0)}(z), \sigma^*)\big\|^q = \sum_{s=0}^{\infty} \lambda^{h_s} \lambda^{\nu^s} \big\|x(s; F_{\sigma(0)}(z), \sigma^*)\big\|^q \\ &= \sum_{s=0}^{\infty} \lambda^{h_s} \big\|x(s; \lambda^{\frac{1}{q}} F_{\sigma(0)}(z), \sigma^*)\big\|^q. \end{aligned} \tag{18}$$

Therefore, for any fixed $\lambda \geq 0$ and $z \in \mathcal{C}$,

$$G_\lambda(z) = \|z\|^q + \sup_\sigma \sum_{t=1}^{\infty} \lambda^{h_t} \|x(t; z, \sigma)\|^q = \|z\|^q + \sup_{\sigma(0)} \sup_{\sigma^*} \sum_{t=0}^{\infty} \lambda^{h_t} \left\| x(t; \lambda^{\frac{1}{q}} F_{\sigma(0)}(z), \sigma^*) \right\|^q$$
$$= \|z\|^q + \max_{i \in \mathcal{M}} G_\lambda(\lambda^{\frac{1}{q}} F_i(z)).$$

Property 3 is trivial, and Property 4 follows from (5) directly. The first implication in Property 5 is easy to establish; to show the converse, note that $G_1(z) \leq c$, $\forall\, z \in \mathcal{B}_\rho \cap \mathcal{C}$, implies that for any $z \in \mathcal{B}_\rho \cap \mathcal{C}$ and any σ, $\|x(t; z, \sigma)\| \leq c^{1/q}, \forall\, t \in \mathbb{Z}_+$, and $x(t; z, \sigma) \to 0$ as $t \to \infty$. Hence, the SHS is strongly asymptotically stable on $\mathcal{B}_\rho \cap \mathcal{C}$, yielding the exponential stability by Proposition 2.3. □

The radius of strong convergence of the generating function (16) is defined as:

$$\lambda^* := \sup \left\{ \lambda \geq 0 \;\middle|\; \sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G_\lambda(z) < \infty \right\}. \tag{19}$$

In light of Property 4 in Proposition 4.1, we have $\lambda^* \geq (\mu_1)^{-q}$. It is shown below that the radius of the domain of strong attraction can be determined by λ^*.

Theorem 4.1. The following holds: $\rho^* = (\lambda^*)^{\frac{1}{q(\nu-1)}}$.

Proof. We first show $\rho^* \leq (\lambda^*)^{\frac{1}{q(\nu-1)}}$. Let $\rho \in (0, \rho^*)$ be arbitrary. By Proposition 2.3, the SHS is strongly exponentially stable on $\mathcal{B}_{\widetilde{\rho}} \cap \mathcal{C}$ for some $\widetilde{\rho}$ with $\rho < \widetilde{\rho} < \rho^*$, i.e., there exist $\kappa > 0$ and $r \in [0, 1)$ such that for any $z \in \mathcal{B}_{\widetilde{\rho}} \cap \mathcal{C}$ and any σ, $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|, \forall\, t \in \mathbb{Z}_+$. Thus, if $0 < \lambda^{\frac{1}{q(\nu-1)}} < \rho$, then for any $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$,

$$G_\lambda(z) = \sup_\sigma \sum_{t=0}^{\infty} \lambda^{h_t} \left(\rho^{-\nu^t} \|x(t; \rho z, \sigma)\| \right)^q \leq \sum_{t=0}^{\infty} \left(\frac{\lambda^{\frac{1}{q(\nu-1)}}}{\rho} \right)^{q \cdot \nu^t} \lambda^{-\frac{1}{\nu-1}} \left(\kappa r^t \|\rho z\| \right)^q$$
$$\leq \lambda^{-\frac{1}{\nu-1}} (\kappa\rho)^q \sum_{t=0}^{\infty} \left(\frac{r \cdot \lambda^{\frac{1}{q(\nu-1)}}}{\rho} \right)^{q \cdot t}.$$

This implies $\sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G_\lambda(z) < \infty$. Since $\rho \in (0, \rho^*)$ is arbitrary, $\sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G_\lambda(z)$ is finite whenever $\lambda^{\frac{1}{q(\nu-1)}} < \rho^*$. By the definition of λ^*, we conclude $\rho^* \leq (\lambda^*)^{\frac{1}{q(\nu-1)}}$.

We next show $\rho^* \geq (\lambda^*)^{\frac{1}{q(\nu-1)}}$ via contradiction. Suppose $\rho^* < (\lambda^*)^{\frac{1}{q(\nu-1)}}$. Then there exist ρ and λ such that $\rho^* < \rho < \lambda^{\frac{1}{q(\nu-1)}} < (\lambda^*)^{\frac{1}{q(\nu-1)}}$. It is observed from the second part of the proof of Theorem 2.3 that for any $\rho > \rho^*$, there is no uniform bound on all the trajectories starting from $\mathcal{B}_\rho \cap \mathcal{C}$. Indeed, for any $N \in \mathbb{N}$, there exist $z_N \in \mathcal{C}$ with $\|z_N\| = \rho$, a switching sequence σ_N, and a time t_N such that $\|x(t_N; z_N, \sigma_N)\| \geq N$. Hence,

$$G_\lambda(z_N/\rho) \geq \left(\frac{\lambda^{h_{t_N}/q}}{\rho^{\nu^{t_N}}} \right)^q \left\| x(t_N; z_N, \sigma_N) \right\|^q \geq \left(\frac{\lambda^{\frac{1}{q(\nu-1)}}}{\rho} \right)^{q \cdot \nu^{t_N}} \cdot \lambda^{-\frac{1}{\nu-1}} \cdot N^q \geq \lambda^{-\frac{1}{\nu-1}} \cdot N^q, \quad \forall\, N.$$

Therefore, $\sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G_\lambda(z) = \infty$. Since $\lambda < \lambda^*$, we have a contradiction. As a result, $\rho^* \geq (\lambda^*)^{\frac{1}{q(\nu-1)}}$, and thus $\rho^* = (\lambda^*)^{\frac{1}{q(\nu-1)}}$. □

In view of the above result and Theorem 2.3, we see that λ^* and the GJSR μ^* satisfy $\mu^* = (\lambda^*)^{-1/q}$, an analogue to the same relation for the two quantities of the SLS on $\mathcal{C}$ in Remark 3.2.

Remark 4.1. Define a companion of strong generating function $\widetilde{G}(\cdot,\cdot) : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+ \cup \{\infty\}$ as:

$$\widetilde{G}(\lambda, z) := \widetilde{G}_\lambda(z) = \sup_\sigma \sum_{t=0}^{\infty} \|x(t; \lambda^{\frac{1}{q(\nu-1)}} z, \sigma)\|^q, \quad \forall\, \lambda \geq 0, \quad z \in \mathcal{C}. \tag{20}$$

Notice that $\widetilde{G}_\lambda$ is defined via vector scaling (by $\lambda^{\frac{1}{q(\nu-1)}}$), i.e., $\widetilde{G}_\lambda(z) = \widetilde{G}_1(\lambda^{\frac{1}{q(\nu-1)}} z)$, a key feature of the SHS with $\nu > 1$. Despite their different definitions, G_λ and $\widetilde{G}_\lambda$ are closely related. Indeed, for each $\lambda \geq 0$, we have

$$\begin{aligned} \lambda^{\frac{1}{\nu-1}} \sum_{t=0}^{\infty} \lambda^{h_t} \|x(t; z, \sigma)\|^q &= \lambda^{\frac{1}{\nu-1}} \sum_{t=0}^{\infty} \lambda^{\frac{\nu^t - 1}{\nu-1}} \|x(t; z, \sigma)\|^q = \sum_{t=0}^{\infty} \left(\lambda^{\frac{1}{\nu-1}}\right)^{\nu^t} \|x(t; z, \sigma)\|^q \\ &= \sum_{t=0}^{\infty} \left\| x\left(t; \lambda^{\frac{1}{q(\nu-1)}} z, \sigma\right) \right\|^q, \quad \forall\, z \in \mathcal{C}, \quad \forall\, \sigma. \end{aligned} \tag{21}$$

As a result, $\widetilde{G}_\lambda(z) = \lambda^{\frac{1}{\nu-1}} G_\lambda(z)$, $\forall z \in \mathcal{C}$, for any $\lambda \geq 0$. In view of this, we see that $\widetilde{G}_\lambda$ satisfies the Bellman equation $\widetilde{G}_\lambda(z) = \lambda^{\frac{1}{\nu-1}} \|z\|^q + \max_{i \in \mathcal{M}} \widetilde{G}_\lambda(\lambda^{\frac{1}{q}} F_i(z))$ and Properties 3-5 in Proposition 4.1.

We introduce the upper bound of G_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$ that is important to analysis and computation of G_λ and λ^*: for $\lambda \geq 0$,

$$g_\lambda := \sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G_\lambda(z).$$

We also treat $g_\lambda : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ as a function of λ. The next proposition reveals fundamental properties of g_λ. Particularly, it shows that λ^* is exactly the first value of λ where $G_\lambda(\cdot) = +\infty$ at some $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$.

Proposition 4.2. Let ρ^* be finite. The function g_λ is increasing, convex, and semismooth (thus continuous) on $[0, \lambda^*)$ with $g_0 = 1$ and the one-sided derivative of g_λ at $\lambda = 0$ being $g'_\lambda(0_+) = (\mu_1)^q$. Furthermore, there exists $\widetilde{z} \in \mathbb{S}^{n-1} \cap \mathcal{C}$ such that $G_{\lambda^*}(\widetilde{z}) = \infty$ (thus $g_{\lambda^*} = \infty$), and $g_\lambda \to \infty$ as $\lambda \uparrow \lambda^*$.

Proof. The monotonicity, convexity and semismoothness of g_λ follows from similar arguments in the proof of [14, Proposition 4]. To show the rest of the statement, we see from Proposition 2.4 that there exists $z' \in \rho^* \mathbb{S}^{n-1} \cap \mathcal{C}$ such that $x(t; z', \sigma_{z'})$ does not converge to the origin under some switching sequence $\sigma_{z'}$. Recalling $\rho^* = (\lambda^*)^{\frac{1}{q(\nu-1)}}$ and using the companion generating function $\widetilde{G}$ defined in (20), we have $\widetilde{G}_{\lambda^*}(z'/\rho^*) \geq \sum_{t=0}^{\infty} \|x(t; z', \sigma_{z'})\|^q = \infty$, where the last equality is due to the fact that $x(t; z', \sigma_{z'})$ does not converge to the origin. In light of $\widetilde{G}_\lambda(z) = \lambda^{\frac{1}{\nu-1}} G_\lambda(z)$, we have $g_{\lambda^*} \geq G_{\lambda^*}(z'/\rho^*) = \infty$. Further, it follows from a straightforward computation and the definition of μ_1 in (3) that the one-sided derivative $(g_\lambda)'(0_+) = \max_{i \in \mathcal{M}} \max_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} \|F_i(z)\|^q = (\mu_1)^q$.

To show $\lim_{\lambda \uparrow \lambda^*} g_\lambda = \infty$, note that for any $L > 0$, there exists $\widetilde{z} \in \mathbb{S}^{n-1} \cap \mathcal{C}$ such that $G_{\lambda^*}(\widetilde{z}) \geq L$. Let $\sigma_{\widetilde{z}}$ be a switching sequence such that $G_{\lambda^*}(\widetilde{z}) = \sum_{t=0}^{\infty} (\lambda^*)^{h_t} \|x(t; \widetilde{z}, \sigma_{\widetilde{z}})\|^q \geq L$. Hence, there exists $K \in \mathbb{N}$ such that $\widetilde{G}^K_{\lambda^*}(\widetilde{z}) := \sum_{t=0}^{K} (\lambda^*)^{h_t} \|x(t; \widetilde{z}, \sigma_{\widetilde{z}})\|^q \geq L/2$. Since $\widetilde{G}^K_\lambda(\widetilde{z})$ is continuous in λ for the fixed K and $\widetilde{z}$, we deduce the existence of $\eta > 0$ such that for all $\lambda \in (\lambda^* - \eta, \lambda^*]$, $\widetilde{G}^K_\lambda(\widetilde{z}) \geq L/4$. This shows $g_\lambda \geq G_\lambda(\widetilde{z}) \geq \widetilde{G}^K_\lambda(\widetilde{z}) \geq L/4$ for all $\lambda \in (\lambda^* - \eta, \lambda^*]$, and leads to the desired result. □

The above proposition can easily be extended to characterize $1/g_\lambda$ stated below without proof.

Corollary 4.1. The function $1/g_\lambda : \mathbb{R}_+ \to [0, 1]$ is decreasing and semismooth (thus continuous) on $[0, \lambda^*)$ and $(1/g_\lambda)'(0_+) = -(\mu_1)^q$. Further, $1/g_\lambda = 0$ on $[\lambda^*, \infty)$, and $1/g_\lambda$ is continuous on $\mathbb{R}_+$.

It can be further shown using (5) and a similar argument as in the proof of [14, Lamma 2] that $1/g_\lambda \geq 1 - (\mu_1)^q \cdot \lambda, \forall\, \lambda \in [0, \lambda^*)$. Since $1 - (\mu_1)^q \cdot \lambda$ represents the tangent line of $1/g_\lambda$ at the point $(0, 1)$, the graph of $1/g_\lambda$ is always above its tangent line at $(0, 1)$; see Figure 1 for an example.

4.1.1 Computation of Strong Generating Functions and Radius of Convergence

We addresses numerical approximation of the strong generating function and its convergence analysis. The underlying principle is similar to that shown in Section 3.3: the strong generating function G_λ is approximated by a sequence of similar functions defined on finite but increasingly larger horizons. Particularly, for a given $\lambda \geq 0$, let $G^k_\lambda(z) := \max_\sigma \sum_{t=0}^k \lambda^{h_t} \|x(t; z, \sigma)\|^q$ for all $z \in \mathcal{C}$, where $k \in \mathbb{Z}_+$. The proposition below shows that for any $\lambda \in [0, \lambda^*)$, the sequence $\big(G^k_\lambda\big)$ converges to G_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$ uniformly and (sup-)exponentially fast when k is sufficiently large.

Proposition 4.3. Let $\lambda \in [0, \lambda^*)$. The following hold:

(1) each G^k_λ is continuous on $\mathcal{C}$ and $G^0_\lambda(z) \leq G^1_\lambda(z) \leq \cdots \leq G_\lambda(z)$, $\forall z \in \mathcal{C}$.

(2) there exist $\theta_\lambda \in (0, 1)$ and $N_\lambda \in \mathbb{N}$ (both dependent on λ only) such that for all $k \geq N_\lambda$, $|G^k_\lambda(z) - G_\lambda(z)| \leq (\theta_\lambda)^{h_{k+1}}/(1 - (\theta_\lambda)^{\nu^{k+1}})$ for all $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$.

(3) $G_\lambda(\cdot)$ is continuous on $\mathbb{S}^{n-1} \cap \mathcal{C}$.

Proof. The first statement is trivial. To prove the second, it is noted from $\mu^* = (\lambda^*)^{-1/q}$ that $\mu^* < \lambda^{-1/q}$ for any given $\lambda \in (0, \lambda^*)$. Hence, there exists $\varepsilon > 0$ such that $\mu^* + \varepsilon < \lambda^{-1/q}$, or equivalently $\theta_\lambda := \lambda \cdot (\mu^* + \varepsilon)^q \in (0, 1)$. It follows from the definition of μ^* that there exists $N_\lambda \in \mathbb{N}$ such that $\mu_k \leq \mu^* + \varepsilon$ for all $k \geq N_\lambda$. Hence, by (8) and Theorem 2.1, we have, for any $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$ and under any σ, $\|x(t; z, \sigma)\| \leq (\mu^* + \varepsilon)^{h_t}, \forall\, t \geq N_\lambda$. For a given $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, let $\widehat{x}(t, z)$ denote the trajectory that achieves the supremum in (16) for the given z and λ, i.e., $G_\lambda(z) = \sum_{t=0}^\infty \lambda^{h_t} \|\widehat{x}(t, z)\|^q$. Then, we have, for all $k \geq N_\lambda$,

$$\sum_{t=k+1}^{\infty} \lambda^{h_t} \|\widehat{x}(t, z)\|^q \leq \sum_{t=k+1}^{\infty} \lambda^{h_t} \Big((\mu^* + \varepsilon)^{h_t}\Big)^q = \sum_{t=k+1}^{\infty} (\theta_\lambda)^{h_t} \leq \frac{(\theta_\lambda)^{h_{k+1}}}{1 - (\theta_\lambda)^{\nu^{k+1}}}.$$

As a result, $G_\lambda(z) \geq G^k_\lambda(z) \geq \sum_{t=0}^k \lambda^{h_t} \|\widehat{x}(t, z)\|^q = G_\lambda(z) - \sum_{t=k+1}^\infty \lambda^{h_t} \|\widehat{x}(t, z)\|^q \geq G_\lambda(z) - (\theta_\lambda)^{h_{k+1}}/(1 - (\theta_\lambda)^{\nu^{k+1}})$ for all $k \geq N_\lambda$ and all $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$. This yields (2). Finally, Statement (3) follows from the continuity of G^k_λ and the uniform convergence of $\big(G^k_\lambda\big)$ to G_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$. □

Define $g_{\lambda,k} := \max\limits_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} G^k_\lambda(z)$ as an approximation of g_λ. Clearly, $(g_{\lambda,k})$ is increasing at any λ.

Proposition 4.4. The following hold:

(1) for each $\lambda \in [0, \lambda^*)$, $g_\lambda = \lim_{k\to\infty} g_{\lambda,k}$. Further, there exists a convergent sequence (z_k) in $\mathbb{S}^{n-1} \cap \mathcal{C}$ with $G^{\ell_k}_\lambda(z_k) = g_{\lambda,\ell_k}$ and $(\ell_k) \uparrow \infty$ such that $z^* := \lim_{k\to\infty} z_k$ satisfies $G_\lambda(z^*) = g_\lambda$.

(2) for $\lambda \geq \lambda^*$, $(g_{\lambda,k}) \uparrow \infty$. Further, there exists a convergent sequence (z_k) in $\mathbb{S}^{n-1} \cap \mathcal{C}$ with $G^{\ell_k}_\lambda(z_k) = g_{\lambda,\ell_k}$ and $(\ell_k) \uparrow \infty$ such that the limit z^* of (z_k) satisfies $\limsup_{z\to z^*} G_\lambda(z) = \infty$.

Proof. (1) Note that g_λ is finite and $\{G^k_\lambda(z) \,:\, k \in \mathbb{Z}_+,\ z \in \mathbb{S}^{n-1} \cap \mathcal{C}\}$ is bounded above by g_λ. Hence, by the principle of the iterated suprema, $\sup_k \big(g_{\lambda,k}\big) = \sup_k \big(\sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} G^k_\lambda(z)\big) = \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \big(\sup_k G^k_\lambda(z)\big) = \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} G_\lambda(z) = g_\lambda$. Since $(g_{\lambda,k})$ is an increasing sequence, it converges to its supremum g_λ. Furthermore, since each G^k_λ is continuous on the compact set $\mathbb{S}^{n-1} \cap \mathcal{C}$, it has a maximizer $z_k \in \mathbb{S}^{n-1} \cap \mathcal{C}$. Without loss of generality, we assume that (z_k) converges to $z^* \in \mathbb{S}^{n-1} \cap \mathcal{C}$ with $g_{\lambda,\ell_k} = G^{\ell_k}_\lambda(z_k)$ and $(\ell_k) \uparrow \infty$. Hence, for any fixed $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, we have $G_\lambda(z) = \lim_{k\to\infty} G^{\ell_k}_\lambda(z) \leq \lim_{k\to\infty} G^{\ell_k}_\lambda(z_k) = G_\lambda(z^*)$, where the last identity is due to the continuity of G_λ and the uniform convergence of $\big(G^k_\lambda\big)$. This shows $G_\lambda(z^*) = g_\lambda$.

(2) Given $\lambda \geq \lambda^*$, there exists $z' \in \lambda^{\frac{1}{q(\nu-1)}} \mathbb{S}^{n-1} \cap \mathcal{C}$ such that $x(t; z', \sigma)$ does not converge to 0 under some σ, by observing the argument in Proposition 2.4. Therefore $G_\lambda(z') = \infty$ and $\big(G^k_\lambda(z')\big)$ is an increasing sequence that tends to the infinity. In view of $G^k_\lambda(z') \leq g_{\lambda,k}$, we have $(g_{\lambda,k}) \uparrow \infty$. Further, let (z_k) be a convergent sequence in $\mathbb{S}^{n-1} \cap \mathcal{C}$ with $G^{\ell_k}_\lambda(z_k) = g_{\lambda,\ell_k}$. Noting that $G^{\ell_k}_\lambda(z_k) \leq G_\lambda(z_k)$ for all k, we have $(G_\lambda(z_k)) \to \infty$ as $(z_k) \to z^*$. This shows $\limsup_{z\to z^*} G_\lambda(z) = \infty$. $\square$

A numerical scheme is developed as follows to compute G^k_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$ and λ^*, thus the radius of the domain of strong attraction. The continuous function G^k_λ can be approximated within arbitrary precision by its values on a set of fine grid points on $\mathcal{C}$. The following recursive procedure is used to compute G^k_λ for a given $\lambda \geq 0$ based on the Bellman equation for G^k_λ:

$$G^0_\lambda(z) = \|z\|^q, \qquad G^\ell_\lambda(z) = \|z\|^q + \max_{i\in\mathcal{M}} G^{\ell-1}_\lambda(\lambda^{\frac{1}{q}} F_i(z)), \quad \ell = 1, 2, \ldots. \tag{22}$$

This further leads to $g_{\lambda,k}$, an approximation of g_λ. By Corollary 4.1, we see that λ^* is the first λ where $1/g_\lambda = 0$. Due to the continuity of $1/g_\lambda$ on $[0, \lambda^*)$, the graph of $1/g_\lambda$ on $[0, \lambda^*)$ can be approximated by finitely many points, the rightmost of which will be an approximation of λ^*.

While we are mostly interested in G^ℓ_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$ and $g_{\lambda,\ell}$, the scaling property (i.e., Property 1) in Proposition 4.1 shows that gridding $\mathbb{S}^{n-1} \cap \mathcal{C}$ is not enough to represent G^ℓ_λ. Hence, a larger set $\mathcal{U}$ is needed for gridding, particularly when $\lambda \geq (\mu_1)^{-q}$ (cf. Property 4 in Proposition 4.1). For such λ and a given k, the size of $\mathcal{U}$ can be estimated as follows. To find G^k_λ at $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, it follows from (22), as well as (4) and (7), that we need to know G^{k-1}_λ at $\lambda^{1/q} F_i(z)$, where $\psi := \lambda^{1/q}\zeta \leq \|\lambda^{1/q} F_i(z)\| \leq \theta := \lambda^{1/q}\mu_1$, $\forall z \in \mathbb{S}^{n-1} \cap \mathcal{C}, \forall i \in \mathcal{M}$. Let $\mathcal{A}(\psi, \theta) := \{z : \psi \leq \|z\| \leq \theta\}$ denote the annulus whose radii are between ψ and θ. Thus the required region for G^{k-1}_λ is $\mathcal{A}(\psi, \theta) \cap \mathcal{C}$. Further, to find G^{k-1}_λ on $\overline{\mathcal{B}}_\theta \cap \mathcal{C}$, we need to know G^{k-2}_λ at $\lambda^{1/q} F_i(z)$ with $z \in \overline{\mathcal{B}}_\theta \cap \mathcal{C}$, where $\psi^{1+\nu} = \lambda^{1/q}\zeta\psi^\nu \leq \|\lambda^{1/q} F_i(z)\| \leq \lambda^{1/q}\mu_1\theta^\nu = \theta^{1+\nu}$ in view of (4) and (7). Inductively, for $\ell = 0, \ldots, k-1$, the required region for G^ℓ_λ is $\mathcal{A}(\psi^{h_{k-\ell}}, \theta^{h_{k-\ell}}) \cap \mathcal{C}$ when using (22). Hence, the set $\mathcal{U}$, or equivalently the required region for G^0_λ, can be taken as $\mathcal{A}(\psi^{h_k}, \theta^{h_k}) \cap \mathcal{C}$. A set of gridding points can be obtained from the required region in each iteration. Moreover, we approximate the points outside the grid set by suitable interpolation of known data from the preceding step. This numerical procedure is outlined in Algorithm 1.

For a given k, the proposed algorithm has linear complexity with respect to the number of subsystems and thus is suitable for a SHS with a relatively large number of subsystems but a smaller state dimension. However, its overall complexity is exponential since the size of $\mathcal{U}$ is rapidly enlarged as k increases. Further, as λ is closer to λ^*, a much larger k (thus a larger set $\mathcal{U}$) and much finer grids are needed for a desired accuracy. This agrees with the fact that computing λ^*, or equivalently the GJSR μ^*, is an NP-hard problem, even for the SLS on $\mathbb{R}^n$ [4, 33].

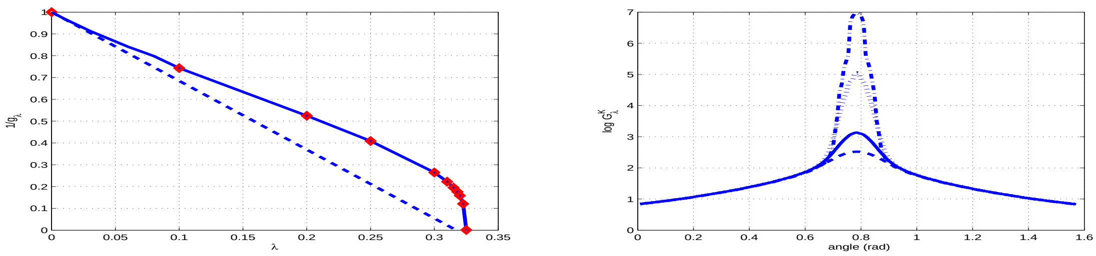

Figure 1: Left: Plots of $1/g_\lambda$ (solid) and the tangent line $1 - (\mu_1)^2 \cdot \lambda$ (dash). Right: Plots of $\log\big(G^k_\lambda([\cos\phi \ \sin\phi])\big)$ with $k = 8$ (dash), $k = 9$ (solid), $k = 10$ (dot), and $k = 11$ (dash-dot) when $\lambda = 0.325$.

Algorithm 1 Computing G_λ^k on grid points of $\mathbb{S}^{n-1}\cap\mathcal{C}$ and $g_{\lambda,k}$ with $\theta := \lambda^{\frac{1}{q}}\mu_1$ and $\psi := \lambda^{\frac{1}{q}}\zeta$

```
Let 𝒱 = {z_j}_{j=1}^N be a set of grid points of 𝒜(ψ^{h_k}, θ^{h_k}) ∩ 𝒞;
Initialize ℓ := 0, and Ĝ_λ^0(z_j) = ‖z_j‖^q for all z_j ∈ 𝒱;
repeat
  ℓ ← ℓ + 1;
  for each z_j ∈ 𝒱 ∩ 𝒜(ψ^{h_{k−ℓ}}, θ^{h_{k−ℓ}}) do
    for each i ∈ ℳ do
      Use interpolation to find g_ij := Ĝ_λ^{ℓ−1}(λ^{1/q} F_i(z_j)) based on the values of Ĝ_λ^{ℓ−1} on 𝒱
    end for
    Set Ĝ_λ^ℓ(z_j) = ‖z_j‖^q + max_{i∈ℳ} g_ij;
    𝒱 ← 𝒱 ∩ 𝒜(ψ^{h_{k−ℓ}}, θ^{h_{k−ℓ}})
  end for
until ℓ = k
ĝ_{λ,k} = max_{z_j ∈ 𝕊^{n−1}∩𝒞} Ĝ_λ^k(z_j)
return Ĝ_λ^k(z_j) for all z_j ∈ 𝕊^{n−1} ∩ 𝒞 and ĝ_{λ,k}
```

Example 4.1. Consider the planar SHS of homogeneous degree $\nu = 2$ on the nonnegative orthant $\mathcal{C} = \mathbb{R}_+^2$, where

$$F_1(x_1, x_2) = \begin{bmatrix} x_1^2 + x_1x_2 + x_2^2 \\ x_1^2 - 0.1x_1x_2 + x_2^2 \end{bmatrix}, \quad F_2(x_1, x_2) = \begin{bmatrix} x_1^2 + 0.5x_1x_2 \\ 0.3x_1x_2 + x_2^2 \end{bmatrix}.$$

We use the Euclidean norm $\|\cdot\|_2$ and $q = 2$. It can be shown that $\mu_1 = 1.776$ such that $\lambda^* \geq 0.3172$. We apply Algorithm 1 to compute $g_{\lambda,k}$ and $1/g_\lambda$ at various $\lambda = 0.1, 0.2, 0.25, 0.3, 0.315, 0.318, 0.32, 0.3225$, and 0.325. The plot of $1/g_\lambda$ versus λ is displayed in Figure 1. Numerical results of $(g_{\lambda,k})$ demonstrate a diverging behavior at $\lambda = 0.325$ which hints $\lambda^* \leq 0.325$. Therefore, denser λ's are chosen on between 0.3 and 0.325. When $\lambda = 0.325$, the plots of $\log(G_\lambda^k)$ for different k on the unit circle in $\mathbb{R}_+^2$ parameterized by the angle ϕ on $[0, \frac{\pi}{2}]$ are also given. It is seen that for each k, the maximizer z_k defined in Proposition 4.4 corresponds to the same $\phi = \frac{\pi}{4}$. This hints that $G_{\lambda=0.325}(z^*) = \infty$ at the point $z^* = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Indeed, a simulation shows that the trajectory $x(t; \sqrt{\lambda}\cdot z^*, \sigma)$ diverges under $\sigma = (1, 1, \dots)$ for $\lambda \geq 0.3237$. We thus deduce from Corollary 4.1 that $\lambda^* \in (0.3225, 0.3237)$ and the radius of the domain of strong attraction $\rho^* = \sqrt{\lambda^*} \in (0.5679, 0.5689)$. This result can be easily extended to the same SHS on the *non-convex* cone $\mathbb{R}_+^2 \cup (-\mathbb{R}_+^2)$ using the observation that any trajectory will enter $\mathbb{R}_+^2$ in at most one step and remain in $\mathbb{R}_+^2$ thereafter.

4.2 Weak Generating Functions of the SHSs on Cones

In this section, we determine the domain of weak attraction of the SHS (1) on $\mathcal{C}$ via its weak generating function $H(\cdot,\cdot) : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+ \cup \{\infty\}$ defined as:

$$H(\lambda, z) \;:=\; H_\lambda(z) \;=\; \inf_\sigma \sum_{t=0}^{\infty} \lambda^{h_t} \|x(t; z, \sigma)\|^q, \quad \forall\, \lambda \geq 0, \quad z \in \mathcal{C}, \tag{23}$$

where the infimum is taken over all switching sequences σ. Similarly, we define the companion of H_λ as $\widetilde{H}_\lambda(z) := \inf_\sigma \sum_{t=0}^{\infty} \|x(t; \lambda^{\frac{1}{q(\nu-1)}} z, \sigma)\|^q$. It follows from (21) that $\widetilde{H}_\lambda(z) = \lambda^{\frac{1}{\nu-1}} H_\lambda(z)$ for all $z \in \mathcal{C}$ and all $\lambda \geq 0$. A few basic properties of H_λ, and thus $\widetilde{H}_\lambda$, are given as follows.

Proposition 4.5. The weak generating function $H_\lambda(z)$ has the following properties.

1. (Bellman Equation): For all $\lambda \geq 0$ and $z \in \mathcal{C}$, $H_\lambda(z) = \|z\|^q + \min_{i\in\mathcal{M}} H_\lambda(\lambda^{\frac{1}{q}} F_i(z))$.

2. (Monotonicity): For any fixed $z \in \mathcal{C}$, $H_{\lambda_1}(z) \leq H_{\lambda_2}(z)$ whenever $0 \leq \lambda_1 \leq \lambda_2$.

3. For $0 \leq \lambda < (a_1)^{-q}$, where a_1 is defined in (9), $\sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} H_\lambda(z)$ is finite.

4. (Exponential Stability): If the SHS (1) is weakly exponentially stable on $\mathcal{B}_\rho \cap \mathcal{C}$, then $H_1(z) \leq c$, $\forall\, z \in \mathcal{B}_\rho \cap \mathcal{C}$, for some finite $c > 0$. Conversely, if the latter holds, then for any $\varepsilon \in (0, \rho)$, the SHS (1) is weakly exponentially stable on $\mathcal{B}_{\rho-\varepsilon} \cap \mathcal{C}$.

Proof. Properties 1, 2, and 4 can be shown in a similar way as those in Proposition 4.1; particularly, Property 1 follows from (18) and the dynamic programming principle. Consider Property 3 now. Recall that $a_1 = \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \big(\min_{i\in\mathcal{M}} \|F_i(z)\|\big)$. Therefore, for any $z \in \mathcal{C}$, there exists $i_z \in \mathcal{M}$ such that $\|F_{i_z}(z)\| \leq a_1\|z\|^\nu$. By induction, we can find a switching sequence σ_z (dependent on z) such that $\|x(t; z, \sigma)\| \leq (a_1)^{h_t}\|z\|^{\nu^t}, \forall\, t \in \mathbb{Z}_+$. Thus, $H_\lambda(z) \leq \sum_{t=0}^\infty \lambda^{h_t}\|x(t; z, \sigma_z)\|^q \leq \sum_{t=0}^\infty (\lambda \cdot a_1^q)^{h_t} < \infty$, which proves Property 3. ☐

The radius of weak convergence of the generating function H_λ is defined as:

$$\lambda_* := \sup\left\{\lambda \geq 0 \;\middle|\; \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} H_\lambda(z) < \infty\right\}. \tag{24}$$

It is clear from Property 3 in Proposition 4.5 that $\lambda_* \geq (a_1)^{-q}$. The radius of weak convergence characterizes the domain of weak attraction as shown below.

Theorem 4.2. The following holds: $\rho_* = (\lambda_*)^{\frac{1}{q(\nu-1)}}$. Further, $a_* = (\lambda_*)^{-1/q}$.

Proof. The argument is similar to that of Theorem 4.1 except that the strong stability (resp. μ_k, μ_*) is replaced by the weak stability (resp. a_k, a_*). We briefly present its key steps as follows.

To show $\rho_* \leq (\lambda_*)^{\frac{1}{q(\nu-1)}}$, choose $\rho \in (0, \rho_*)$. By the weak exponential stability on $\mathcal{B}_{\widetilde{\rho}} \cap \mathcal{C}$ with $\widetilde{\rho} \in (\rho, \rho_*)$, the constants $\kappa > 0$ and $r \in [0, 1)$ exist such that for any $z \in \mathcal{B}_{\widetilde{\rho}}\cap\mathcal{C}$, $\|x(t; z, \sigma_z)\| \leq \kappa r^t\|z\|$ under some switching sequence σ_z. Hence, for $\lambda^{\frac{1}{q(\nu-1)}} < \rho$ and each $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, we have $H_\lambda(z) \leq \sum_{t=0}^\infty \lambda^{h_t}\|x(t; z, \sigma_z)\|^q \leq \lambda^{-\frac{1}{\nu-1}}(\kappa\rho)^q \sum_{t=0}^\infty \left(\frac{r\cdot\lambda^{\frac{1}{q(\nu-1)}}}{\rho}\right)^{q\cdot t} < \infty$. This yields $\rho_* \leq (\lambda_*)^{\frac{1}{q(\nu-1)}}$.

Suppose $\rho_* < (\lambda_*)^{\frac{1}{q(\nu-1)}}$. Then ρ and λ exist such that $\rho_* < \rho < \lambda^{\frac{1}{q(\nu-1)}} < (\lambda_*)^{\frac{1}{q(\nu-1)}}$. It follows from the second part of the proof of Theorem 2.4 that for any $N \in \mathbb{N}$, there exists $z_N \in \rho\mathbb{S}^{n-1} \cap \mathcal{C}$ such that for any σ, $\|x(N; z_N, \sigma)\| \geq \rho(\frac{a_*}{a_*-\eta})^N$ for some $\eta > 0$. Thus $H_\lambda(z_N/\rho) \geq \lambda^{-\frac{1}{\nu-1}}(\frac{a_*}{a_*-\eta})^{N\cdot q}$ for all N, implying $\sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} H_\lambda(z) = \infty$, a contradiction. This shows $\rho_* = (\lambda_*)^{\frac{1}{q(\nu-1)}}$. ☐

The next proposition asserts the continuity of $H_\lambda(\cdot)$ on $\mathbb{S}^{n-1}\cap\mathcal{C}$ for $\lambda \in [0, \lambda_*)$. Similar to that of G_λ (cf. Proposition 4.3), its proof relies on uniform convergence of the finite-horizon approximation of H_λ defined as $H_\lambda^k(z) := \min_\sigma \sum_{t=0}^k \lambda^{h_t}\|x(t; z, \sigma)\|^q$ for all $z \in \mathcal{C}$, where $k \in \mathbb{Z}_+$. However, weak stability complicates the convergence analysis. For example, unlike $G_\lambda^k(z)$ shown in Proposition 4.3, $H_\lambda^k(z)$ cannot be bounded below by the k-tail of $H_\lambda(z)$. Further, the linear technique used for $H_\lambda^k(z)$ of the SLS in [14] fails because of the nonlinear F_i's. To overcome these difficulties, we use different arguments via new techniques, e.g., the semicontinuity and Dini's Theorem. This result will be exploited later for numerical approximation of H_λ and λ_* and its convergence analysis.

Proposition 4.6. Given $\lambda \in [0, \lambda_*)$. The following hold:

(1) each H_λ^k is continuous on $\mathcal{C}$ and $H_\lambda^0(z) \leq H_\lambda^1(z) \leq \cdots \leq H_\lambda(z)$, $\lim_{k\to\infty} H_\lambda^k(z) = H_\lambda(z)$, $\forall z \in \mathbb{S}^{n-1} \cap \mathcal{C}$.

(2) $\big(H_\lambda^k\big)$ converges uniformly to H_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$.

(3) $H_\lambda(\cdot)$ is continuous on $\mathbb{S}^{n-1} \cap \mathcal{C}$.

Proof. We introduce some notation first. For a given $\lambda \in [0, \lambda_*)$ and $z \in \mathcal{C}$, let $\widehat{x}_\lambda^k(t, z)$ denote the trajectory that achieves the minimum in $H_\lambda^k(z)$, i.e., $H_\lambda^k(z) = \sum_{t=0}^k \lambda^{h_t} \|\widehat{x}_\lambda^k(t, z)\|^q$. Similarly, let $\widehat{x}_\lambda(t, z)$ denote the trajectory that achieves the infimum in $H_\lambda(z)$.

(1) The continuity of each H_λ^k is obvious. For a given $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, the monotonicity of $\big(H_\lambda^k(z)\big)$ is due to the observation that $H_\lambda^{k+1}(z) = \sum_{t=0}^{k+1} \lambda^{h_t} \|\widehat{x}_\lambda^k(t, z)\|^q \geq \sum_{t=0}^{k} \lambda^{h_t} \|\widehat{x}_\lambda^k(t, z)\|^q \geq H_\lambda^k(z)$. Similarly, we have $H_\lambda^k(z) \leq H_\lambda(z)$ for all k. To show the pointwise convergence of $\big(H_\lambda^k\big)$ on $\mathbb{S}^{n-1} \cap \mathcal{C}$, it suffices to prove $H_\lambda(z) = \sup_k \{H_\lambda^k(z)\}$ for any fixed $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, namely, at each $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$ and for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $H_\lambda^k(z) > H_\lambda(z) - \varepsilon$. Suppose not. Then for some $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$ and $\varepsilon > 0$, $H_\lambda^k(z) \leq H_\lambda(z) - \varepsilon$ for all k. Let $\widehat{\sigma}^k$ denote the corresponding switching sequence for $\widehat{x}_\lambda^k(t, z)$, i.e., $\widehat{x}_\lambda^k(t, z) = x(t; z, \widehat{\sigma}^k)$. Since $\mathcal{M}$ is finite, $\{\widehat{\sigma}^k(0)\}$ has a subsequence of a constant index denoted by $\sigma^*(0) \in \mathcal{M}$, i.e., $\{\widehat{\sigma}^{k'}(0)\} = \{\sigma^*(0)\}$. Similarly, the corresponding subsequence $\{\widehat{\sigma}^{k'}(1)\}$ has another subsequence of a constant index denoted by $\sigma^*(1) \in \mathcal{M}$. Repeating this argument via induction, we construct a switching sequence $\sigma^* := \{\sigma^*(0), \sigma^*(1), \ldots\}$. It is easy to see from this construction that for any $k \in \mathbb{N}$, $\sum_{t=0}^k \lambda^{h_t} \|x(t; z, \sigma^*)\|^q \leq H_\lambda^s(z)$ for some $s \geq k$. This implies that $\sum_{t=0}^\infty \lambda^{h_t} \|x(t; z, \sigma^*)\|^q \leq H_\lambda(z) - \varepsilon$, contradicting $H_\lambda(z) \leq \sum_{t=0}^\infty \lambda^{h_t} \|x(t; z, \sigma^*)\|^q$. Consequently, the pointwise convergence holds.

(2) For each H_λ^k, define its companion function $\widetilde{H}_\lambda^k(z) := \min_\sigma \sum_{t=0}^k \|x(t; \lambda^{\frac{1}{q(\nu-1)}} z, \sigma)\|^q$. In light of (21), $\widetilde{H}_\lambda^k(z) = \lambda^{\frac{1}{(\nu-1)}} H_\lambda^k(z)$. Hence, $\big(\widetilde{H}_\lambda^k\big)$ is monotone and pointwise convergent to $\widetilde{H}_\lambda$ on $\mathbb{S}^{n-1} \cap \mathcal{C}$, and we only need to show the uniform convergence property for $\big(\widetilde{H}_\lambda^k\big)$. For a given $\lambda \in (0, \lambda_*)$ (the trivial case $\lambda = 0$ is omitted), let $\rho_\lambda := \lambda^{\frac{1}{q(\nu-1)}}$. We break the proof into two steps.

(2.1) We first show a weaker continuity property for $\widetilde{H}_\lambda$ stated as follows:

Claim 1: $\widetilde{H}_\lambda$ is upper semicontinuous on $\mathbb{S}^{n-1} \cap \mathcal{C}$.

To prove this property, it suffices to show that at each $z_* \in \mathbb{S}^{n-1} \cap \mathcal{C}$, for any $\varepsilon > 0$, there exists an open set $\mathcal{U}$ of z_* such that $\widetilde{H}_\lambda(z) \leq \widetilde{H}_\lambda(z_*) + \varepsilon$ for all $z \in \mathcal{U} \cap (\mathbb{S}^{n-1} \cap \mathcal{C})$. It follows from Theorem 4.2 that $\rho_\lambda \in (0, \rho_*)$ such that the SHS is weakly exponentially stable on $\mathcal{B}_{\rho_\lambda} \cap \mathcal{C}$. Therefore, there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that for any $z_0 \in \mathcal{B}_{\rho_\lambda} \cap \mathcal{C}$, $\|x(t; z_0, \sigma_{z_0})\| \leq \kappa r^t \|z_0\|$ under some switching sequence σ_{z_0}. For the given z_*, let σ_* be a switching sequence that achieves $\widetilde{H}_\lambda(z_*)$, i.e., $\widetilde{H}_\lambda(z_*) = \sum_{t=0}^\infty \|x(t; \rho_\lambda z_*, \sigma_*)\|^q$. Since the series $\sum_{t=0}^\infty \|x(t; \rho_\lambda z_*, \sigma_*)\|^q$ converges, $\|x(t; \rho_\lambda z_*, \sigma_*)\| \to 0$ as $t \to \infty$. Thus there exists a sufficiently large $T \in \mathbb{N}$ such that $\sum_{t=0}^T \|x(t; \rho_\lambda z_*, \sigma_*)\|^q \leq \widetilde{H}_\lambda(z_*)$, $\|x(T; \rho_\lambda z_*, \sigma_*)\| < \rho_\lambda$, and $\|x(T; \rho_\lambda z_*, \sigma_*)\|^q \cdot \kappa^q/(1 - r^q) \leq \varepsilon/3$. Since $x(t; \rho_\lambda z, \sigma_*)$ is continuous in z for any fixed $t \in [0, T]$, there exists an open set $\mathcal{U}$ of z_* such that $\sum_{t=0}^T \|x(t; \rho_\lambda z, \sigma_*)\|^q \leq \widetilde{H}_\lambda(z_*) + \varepsilon/2$, $\|x(T; \rho_\lambda z, \sigma_*)\| < \rho_\lambda$, and $\|x(T; \rho_\lambda z, \sigma_*)\|^q \cdot \kappa^q/(1 - r^q) \leq \varepsilon/2$ for all $z \in \mathcal{U} \cap (\mathbb{S}^{n-1} \cap \mathcal{C})$. For any $z \in \mathcal{U} \cap (\mathbb{S}^{n-1} \cap \mathcal{C})$, let $\widetilde{\sigma}_z$ be a switching sequence such that $\|x(s; x(T; \rho_\lambda z, \sigma_*), \widetilde{\sigma}_z)\| \leq \kappa r^s \|x(T; \rho_\lambda z, \sigma_*)\|, \forall\, s \in \mathbb{Z}_+$. Consequently, under the switching sequence $\sigma_z := (\sigma_*(0), \ldots, \sigma_*(T), \widetilde{\sigma}_z)$, we have

$$\begin{aligned}\widetilde{H}_\lambda(z) &\leq \sum_{t=0}^\infty \|x(t; \rho_\lambda z, \sigma_z)\|^q = \sum_{t=0}^{T-1} \|x(t; \rho_\lambda z, \sigma_*)\|^q + \sum_{t=T}^\infty \|x(t - T; x(T; \rho_\lambda z, \sigma_*), \widetilde{\sigma}_z)\|^q \\ &\leq \Big(\widetilde{H}_\lambda(z_*) + \frac{\varepsilon}{2}\Big) + \|x(T; \rho_\lambda z, \sigma_*)\|^q \frac{\kappa^q}{1 - r^q} \;\leq\; \widetilde{H}_\lambda(z_*) + \varepsilon.\end{aligned}$$

This shows $\widetilde{H}_\lambda(z) \le \widetilde{H}_\lambda(z_*) + \varepsilon$ for all $z \in \mathcal{U} \cap (\mathbb{S}^{n-1} \cap \mathcal{C})$, and thus the upper semicontinuity of $\widetilde{H}_\lambda$ on $\mathbb{S}^{n-1} \cap \mathcal{C}$.

(2.2) We then show, via Dini's Theorem, that

$$\textit{Claim 2:} \quad (\widetilde{H}^k_\lambda) \text{ uniformly converges to } \widetilde{H}_\lambda \text{ on } \mathbb{S}^{n-1} \cap \mathcal{C}.$$

To this end, define $E^k_\lambda(z) := \widetilde{H}_\lambda(z) - \widetilde{H}^k_\lambda(z)$. Since each $\widetilde{H}^k_\lambda$ is continuous on $\mathbb{S}^{n-1} \cap \mathcal{C}$, so is $-\widetilde{H}^k_\lambda$ which is also upper semicontinuous. Hence, in view of Claim 1, each E^k_λ is upper semicontinous on $\mathbb{S}^{n-1} \cap \mathcal{C}$. Further, by Statement (1) shown above, $\big(E^k_\lambda\big)$ is pointwise monotonically decreasing to zero on the compact set $\mathbb{S}^{n-1} \cap \mathcal{C}$. In light of Dini's Theorem [24, Chapter 9, Proposition 11], $\big(E^k_\lambda\big)$ converges uniformly to zero on $\mathbb{S}^{n-1} \cap \mathcal{C}$. This yields the uniform convergence of $\big(\widetilde{H}^k_\lambda\big)$ to $\widetilde{H}_\lambda$.

(3) This directly follows from the continuity of H^k_λ and Statement (2). □

Equipped with the above proposition, we study the upper bound of H_λ on $\mathbb{S}^{n-1} \cap \mathcal{C}$ defined as

$$h_\lambda := \sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} H_\lambda(z).$$

Its approximation is given by $h_{\lambda,k} := \max_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} H^k_\lambda(z)$. As before, h_λ and $h_{\lambda,k}$ are treated as functions of λ. Similar to g_λ treated in Section 4.1, the function h_λ plays a critical role in computation of λ_*. Fundamental properties of h_λ via approximation of $\big(h_{\lambda,k}\big)$ are shown below.

Proposition 4.7. The following hold:

(1) each function $h_{\lambda,k}$ is continuous in λ on $\mathbb{R}_+$.

(2) at each $\lambda \in [0, \lambda_*)$, $\big(h_{\lambda,k}\big)$ is an increasing sequence that converges to h_λ. Furthermore, there exists a convergent sequence (z_k) in $\mathbb{S}^{n-1} \cap \mathcal{C}$ with $H^{\ell_k}_\lambda(z_k) = h_{\lambda,\ell_k}$ and $(\ell_k) \uparrow \infty$ such that the limit z_* of (z_k) satisfies $H_\lambda(z_*) = h_\lambda$.

(3) for any $\widetilde{\lambda} \in (0, \lambda_*)$, $\big(h_{\lambda,k}\big)$ is an increasing sequence that uniformly converges to h_λ on $[0, \widetilde{\lambda}]$.

(4) the function h_λ is increasing and continuous on $[0, \lambda_*)$ with $h_0 = 1$ and the one-sided derivative of h_λ at $\lambda = 0$ being $h'_\lambda(0_+) = (a_1)^q$. Furthermore, there exists $\widetilde{z} \in \mathbb{S}^{n-1} \cap \mathcal{C}$ such that $H_{\lambda_*}(\widetilde{z}) = \infty$ (thus $h_{\lambda_*} = \infty$), and $h_\lambda \to \infty$ as $\lambda \uparrow \lambda_*$.

Proof. (1) Fix $k \in \mathbb{N}$. Clearly, $H^k_\lambda(z)$ is continuous in (λ, z). Noting that $h_{\lambda,k}$ is pointwise supreme of $H^k_\lambda(z)$, i.e., $h_{\lambda,k} = \sup_{z \in \mathbb{S}^{n-1} \cap \mathcal{C}} H^k_\lambda(z)$, we conclude that $h_{\lambda,k}$ is lower semicontinuous in λ.

Next, we show that h^k_λ is upper semicontinuous, and thus continuous, in λ. To see this, choose $\lambda_0 \ge 0$ and for each $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, let σ^*_{z,λ_0} be a switching sequence that achieves $H^k_{\lambda_0}(z)$. Notice that there exists $c > 0$ such that $\|x(t; z, \sigma)\| \le c$ for all $t \in [0, k]$ and all z in the compact set $\mathbb{S}^{n-1} \cap \mathcal{C}$ under any switching sequence σ. Hence, for any given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\lambda - \lambda_0| \le \eta \;\Rightarrow\; \Big| \sum_{t=0}^{k} \lambda^{h_t} \|x(t; z, \sigma)\|^q - \sum_{t=0}^{k} (\lambda_0)^{h_t} \|x(t; z, \sigma)\|^q \Big| \le \varepsilon, \;\forall \sigma, \;\forall z \in \mathbb{S}^{n-1} \cap \mathcal{C}. \tag{25}$$

As a result, for any $\lambda \ge 0$ with $|\lambda - \lambda_0| \le \eta$, we have, for each $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$,

$$H^k_\lambda(z) \le \sum_{t=0}^{k} \lambda^{h_t} \|x(t; z, \sigma^*_{z,\lambda_0})\|^q \le H^k_{\lambda_0}(z) + \Big| \sum_{t=0}^{k} \lambda^{h_t} \|x(t; z, \sigma^*_{z,\lambda_0})\|^q - \sum_{t=0}^{k} (\lambda_0)^{h_t} \|x(t; z, \sigma^*_{z,\lambda_0})\|^q \Big| \le H^k_{\lambda_0}(z) + \varepsilon.$$

Therefore, $h_{\lambda,k} \le h_{\lambda_0,k} + \varepsilon$ in a neighborhood of λ_0, implying the upper semicontinuity of $h_{\lambda,k}$.

(2) Fix $\lambda \in [0, \lambda_*)$. The monotonicity is clear; it suffices to show the convergence. Since each $H^k_\lambda(\cdot)$ is continuous on the compact set $\mathbb{S}^{n-1} \cap \mathcal{C}$, there exists a maximizer $z_k \in \mathbb{S}^{n-1} \cap \mathcal{C}$ such that $h_{\lambda,k} = H^k_\lambda(z_k)$. Without loss of generality, we assume that (z_k) converges to $z_* \in \mathbb{S}^{n-1} \cap \mathcal{C}$ with $H^{\ell_k}_\lambda(z_k) = h_{\lambda,\ell_k}$ and $(\ell_k) \uparrow \infty$. By the uniform convergence of $\big(H^k_\lambda\big)$ to the continuous function H_λ proven in Proposition 4.6, we see that $\big(H^{\ell_k}_\lambda(z_k)\big)$ converges to $H_\lambda(z_*)$. Moreover, for any $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, $H_\lambda(z) = \lim_{k\to\infty} H^{\ell_k}_\lambda(z) \le \lim_{k\to\infty} H^{\ell_k}_\lambda(z_k) = H_\lambda(z_*)$. Hence, $h_\lambda = H_\lambda(z_*)$ and $\big(h_{\lambda,\ell_k}\big) \uparrow h_\lambda$. Since (h_{λ,ℓ_k}) is a subsequence of the increasing sequence $(h_{\lambda,k})$, we have $(h_{\lambda,k}) \uparrow h_\lambda$. The above argument also constructs a desired sequence (z_k) for the second part of Statement (2).

(3) The main idea of this proof is similar to that for Proposition 4.6. Specifically, the upper semicontinuity of h_λ on $[0, \lambda_*)$ is established first. Based on this and Statements (1)-(2) shown above, the uniform convergence follows from Dini's Theorem.

In what follows, we prove the upper semicontinuity of h_λ at each $\lambda_0 \in [0, \lambda_*)$. When $\lambda_0 = 0$, the continuity of h_λ follows from the Bellman equation in Proposition 4.5. Now consider $\lambda_0 \in (0, \lambda_*)$. Choose $\widehat{\lambda} \in (\lambda_0, \lambda_*)$ and let $\rho_{\widehat{\lambda}} := \widehat{\lambda}^{\frac{1}{q(\nu-1)}}$. The SHS is weakly exponentially stable on $\mathcal{B}_{\rho_{\widehat{\lambda}}} \cap \mathcal{C}$ with the constants $\widehat{\kappa} \ge 1$ and $\widehat{r} \in [0, 1)$ (dependent on $\widehat{\lambda}$ only). For any given $\varepsilon > 0$, thanks to the uniform convergence of $\big(\widetilde{H}^k_{\lambda_0}\big)$ to $\widetilde{H}_{\lambda_0}$ on $\mathbb{S}^{n-1} \cap \mathcal{C}$, we obtain $K \in \mathbb{N}$ such that for all $k \ge K$, $\widetilde{H}_{\lambda_0}(z) - \widetilde{H}^k_{\lambda_0}(z) \le \min\big([\varepsilon/3 \cdot (1 - \widehat{r}^q)/\widehat{\kappa}^q],\ (\rho_{\widehat{\lambda}}/3)^q\big)$ for all $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$. For a given $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, recall that σ^*_{z,λ_0} is a switching sequence that achieves $\widetilde{H}_{\lambda_0}(z)$. Hence, $\|x(K+1; \rho_{\lambda_0} z, \sigma^*_{z,\lambda_0})\| \le \big(\widetilde{H}_{\lambda_0}(z) - \sum_{t=0}^K \|x(t; \rho_{\lambda_0} z, \sigma^*_{z,\lambda_0}\|^q\big)^{1/q} \le \big(\widetilde{H}_{\lambda_0}(z) - \widetilde{H}^k_{\lambda_0}(z)\big)^{1/q} \le \min\big((\varepsilon/3 \cdot (1 - \widehat{r}^q)/\widehat{\kappa}^q)^{1/q},\ \rho_{\widehat{\lambda}}/3\big)$, for all $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$. It follows from (25) that there exists $\eta_1 \in (0, \lambda_0/2)$ such that for any λ with $|\lambda - \lambda_0| < \eta_1$, $\|x(K+1; \rho_\lambda z, \sigma^*_{z,\lambda_0})\| \le \min\big([\varepsilon/2 \cdot (1 - \widehat{r}^q)/\widehat{\kappa}^q]^{1/q},\ \rho_{\widehat{\lambda}}/2\big)$ and $\big|\sum_{t=0}^K \|x(t; \rho_\lambda z, \sigma^*_{z,\lambda_0})\|^q - \sum_{t=0}^K \|x(t; \rho_{\lambda_0} z, \sigma^*_{z,\lambda_0})\|^q\big| \le \varepsilon/2$, for all $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$. Consequently, for any $\lambda \in (\lambda_0 - \eta_1, \lambda_0 + \eta_1)$ and each $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$, letting $\widetilde{\sigma}_z$ be a switching sequence such that $\|x(s; x(K+1; \rho_\lambda z, \sigma^*_{z,\lambda_0}), \widetilde{\sigma}_z)\| \le \widehat{\kappa}\widehat{r}^s \|x(K+1; \rho_\lambda z, \sigma^*_{z,\lambda_0})\|, \forall\, s \in \mathbb{Z}_+$, we obtain

$$
\begin{aligned}
\widetilde{H}_\lambda(z) - \widetilde{H}_{\lambda_0}(z) &\le \sum_{t=0}^K \big\|x(t; \rho_\lambda z, \sigma^*_{z,\lambda_0})\big\|^q + \sum_{s=0}^\infty \big\|x(s; x(K+1; \rho_\lambda z, \sigma^*_{z,\lambda_0}), \widetilde{\sigma}_z)\big\|^q - \sum_{t=0}^K \big\|x(t; \rho_{\lambda_0} z, \sigma^*_{z,\lambda_0})\big\|^q \\
&\le \Big|\sum_{t=0}^K \big\|x(t; \rho_\lambda z, \sigma^*_{z,\lambda})\big\|^q - \sum_{t=0}^K \big\|x(t; \rho_{\lambda_0} z, \sigma^*_{z,\lambda_0})\big\|^q\Big| + \sum_{s=0}^\infty \big\|x(t; x(K+1; \rho_\lambda z, \sigma^*_{z,\lambda_0}), \widetilde{\sigma}_z)\big\|^q \\
&\le \frac{\varepsilon}{2} + \big\|x(K+1; \rho_\lambda z, \sigma^*_{z,\lambda_0})\big\|^q \frac{\widehat{\kappa}^q}{1 - \widehat{r}^q} \ \le\ \varepsilon.
\end{aligned}
$$

Moreover, there exists $\eta_2 > 0$ such that $|\lambda - \lambda_0| \le \eta_2 \ \Rightarrow\ |\lambda^{-\frac{1}{\nu-1}} - \lambda_0^{-\frac{1}{\nu-1}}| \le \beta \cdot |\lambda - \lambda_0|$ for some constant $\beta > 0$ that depends on λ_0 only. Let $\eta := \min\big(\eta_1, \eta_2, \varepsilon/(\beta \cdot \lambda_0^{-\frac{1}{\nu-1}} \cdot h_{\lambda_0})\big)$. For any $\lambda \in (\lambda_0 - \eta, \lambda_0 + \eta)$ and any $z \in \mathbb{S}^{n-1} \cap \mathcal{C}$,

$$
\begin{aligned}
H_\lambda(z) - H_{\lambda_0}(z) = \lambda^{-\frac{1}{\nu-1}} \widetilde{H}_\lambda(z) - \lambda_0^{-\frac{1}{\nu-1}} \widetilde{H}_{\lambda_0}(z) &\le \lambda^{-\frac{1}{\nu-1}} \Big[\widetilde{H}_\lambda(z) - \widetilde{H}_{\lambda_0}(z)\Big] + |\lambda^{-\frac{1}{\nu-1}} - \lambda_0^{-\frac{1}{\nu-1}}| \cdot \widetilde{H}_{\lambda_0}(z) \\
&\le (\lambda_0/2)^{-\frac{1}{\nu-1}} \Big[\widetilde{H}_\lambda(z) - \widetilde{H}_{\lambda_0}(z)\Big] + \beta \cdot |\lambda - \lambda_0| \cdot \lambda_0^{-\frac{1}{\nu-1}} \cdot h_{\lambda_0} \\
&\le \underbrace{\Big[(\lambda_0/2)^{-\frac{1}{\nu-1}} + 1\Big]}_{\widetilde{c}} \varepsilon.
\end{aligned}
$$

Hence, $h_\lambda \le h_{\lambda_0} + \widetilde{c} \cdot \varepsilon, \forall\, \lambda \in (\lambda_0 - \eta, \lambda + \eta)$. This yields the upper semicontinuity of h_λ at λ_0.

Finally, since $h_{\lambda,k}$ is continuous in λ and $\big(h_{\lambda,k}\big)$ monotonically converges to h_λ pointwise on $[0, \lambda_*)$, it follows from Dini's Theorem that $\big(h_{\lambda,k}\big)$ uniformly converges to h_λ on the compact set $[0, \widetilde{\lambda}]$ for any $\widetilde{\lambda} \in (0, \lambda_*)$.

(4) The increasing property of h_λ is trivial, and it follows from the continuity of $h_{\lambda,k}$ on $[0,\lambda_*)$ and the uniform convergence that for any $\widetilde{\lambda} \in (0,\lambda_*)$, h_λ is continuous on $[0,\widetilde{\lambda}]$, leading to the continuity of h_λ at each $\lambda \in [0,\lambda_*)$. We further deduce via the Bellman equation for $H_\lambda(\cdot)$ and the definition of a_1 in (9) that the directional derivative $(h_\lambda)'(0_+) = \sup_{z\in\mathbb{S}^{n-1}\cap\mathcal{C}} \big(\min_{i\in\mathcal{M}} \|F_i(z)\|^q\big) = (a_1)^q$. Moreover, by Proposition 2.4, we deduce that there exists $z' \in \rho_*\mathbb{S}^{n-1}\cap\mathcal{C}$ such that $x(t;z',\sigma)$ does not converge to the origin under any σ. Hence, $H_{\lambda_*}(z'/\rho_*) = (\lambda_*)^{\frac{1}{\nu-1}} \cdot \widetilde{H}_{\lambda_*}(z'/\rho_*) = (\lambda_*)^{\frac{1}{\nu-1}} \cdot \sum_{t=0}^{\infty} \|x(t;z',\sigma_*)\|^q \geq \infty$, where $\rho_* = (\lambda_*)^{\frac{1}{q(\nu-1)}}$ is used and σ_* is the switching sequence that achieves the infimum in $\widetilde{H}_{\lambda_*}(z'/\rho_*)$. Letting $\widetilde{z} := z'/\rho^*$, we have $H_{\lambda_*}(\widetilde{z}) = \infty$. It also follows from the similar argument in Proposition 4.2 that $h_\lambda \to \infty$ as $\lambda \uparrow \lambda_*$. $\square$

As a counterpart to Corollary 4.1, we obtain the next result on $1/h_\lambda$ without proof.

Corollary 4.2. The function $1/h_\lambda : \mathbb{R}_+ \to [0,1]$ is decreasing and continuous on $[0,\lambda^*)$ and $(1/h_\lambda)'(0_+) = -(a_1)^q$. Moreover, $1/h_\lambda = 0$ on $[\lambda^*,\infty)$, and $1/h_\lambda$ is continuous on $\mathbb{R}_+$.

Using the argument for Property 3 of Proposition 4.5 and a similar one in the proof of [14, Lamma 2], it can be shown that $1/h_\lambda \geq 1-(a_1)^q\cdot\lambda, \forall\, \lambda \in [0,\lambda_*)$. This observation implies that the graph of $1/h_\lambda$ is always above its tangent line at the point $(0,1)$.

4.2.1 Computation of Weak Generating Functions and Radius of Weak Convergence

We discuss numerical approximation of H_λ and λ_*. Recall that $H^k_\lambda(z) := \min_\sigma \sum_{t=0}^k \lambda^{h_t}\|x(t;z,\sigma)\|^q$ uniformly converges to $H_\lambda(z)$ on $\mathbb{S}^{n-1}\cap\mathcal{C}$; see Proposition 4.6. The following recursive procedure is used to compute H^k_λ for a given $\lambda \geq 0$ based on the Bellman equation for H^k_λ:

$$H^0_\lambda(z) = \|z\|^q, \qquad H^\ell_\lambda(z) = \|z\|^q + \min_{i\in\mathcal{M}} H^{\ell-1}_\lambda\big(\lambda^{\frac{1}{q}}F_i(z)\big), \quad \ell = 1,2,\ldots. \tag{26}$$

This also yields $h_{\lambda,k}$ that approximates h_λ. Corollary 4.2 shows that λ_* is the first λ where $1/h_\lambda = 0$. By the continuity of $1/h_\lambda$ on $[0,\lambda_*)$, the graph of $1/h_\lambda$ on $[0,\lambda_*)$ can be approximated by finitely many points, the rightmost of which will be an approximation of λ_*.

It follows from the discussions below (22) that in order to compute H^k_λ, the gridding set needs to be taken as $\mathcal{A}(\psi^{h_k},\theta^{h_k})\cap\mathcal{C}$. Based on this observation, a numerical procedure is developed to compute H^k_λ and $h_{\lambda,k}$ given in Algorithm 2. For a given k, this algorithm has linear complexity in the number of subsystems but has overall exponential complexity. Note that unlike the sup-exponential convergence of $\big(G^k_\lambda\big)$ displayed in Proposition 4.3, Proposition 4.6 does not specify a convergence rate of $\big(H^k_\lambda\big)$. In particular, as λ is getting closer to λ_*, the convergence behavior may deteriorate rapidly, i.e., a very large k is needed to attain suitable convergence which may be slow. In turn, the large k requests a much bigger gridding set and finer grids to maintain a desired accuracy. Computational experience also demonstrates numerical sensitivity when λ is close to λ_*, e.g., a small variation of z may lead to a large change of the obtained H^k_λ. This is due to the instability of some z on $\mathbb{S}^{n-1}\cap\mathcal{C}$ under almost all σ when λ is sufficiently close to λ_*; see Statement (4) of Proposition 4.7. On the other hand, this numerical challenge can be understood from the fact that even for the SLS on $\mathbb{R}^n$, computing the joint spectral sub-radius, a numerically less difficult quantity, is known to be an NP-hard problem [4, 33].

Example 4.2. Consider the same planar SHS in Example 4.1 with the Euclidean norm $\|\cdot\|_2$ and $q=2$. In this case, $a_1 = 1.0679$ such that $\lambda_* \geq 0.8769$. We apply Algorithm 2 to compute $h_{\lambda,k}$ and $1/h_\lambda$ at various λ displayed in Figure 2. Numerical results of $(h_{\lambda,k})$ demonstrate an unstable behavior around $\lambda = 0.88$, hinting that λ_* is close to 0.88 where denser λ's are chosen. While a more accurate approximation $h_{\lambda,k}$ at $\lambda \geq 0.88$ can be obtained by choosing a large k,

Algorithm 2 Computing H_λ^k on grid points of $\mathbb{S}^{n-1} \cap \mathcal{C}$ and $h_{\lambda,k}$ with $\theta := \lambda^{\frac{1}{q}}\mu_1$ and $\psi := \lambda^{\frac{1}{q}}\zeta$

Let $\mathcal{V} = \{z_j\}_{j=1}^N$ be a set of grid points of $\mathcal{A}(\psi^{h_k}, \theta^{h_k}) \cap \mathcal{C}$;
Initialize $\ell := 0$, and $\widehat{H}_\lambda^0(z_j) = \|z_j\|^q$ for all $z_j \in \mathcal{V}$;
repeat
  $\ell \leftarrow \ell + 1$;
  for each $z_j \in \mathcal{V} \cap \mathcal{A}(\psi^{h_{k-\ell}}, \theta^{h_{k-\ell}})$ **do**
    for each $i \in \mathcal{M}$ **do**
      Use interpolation to find $h_{ij} := \widehat{H}_\lambda^{\ell-1}(\lambda^{\frac{1}{q}} F_i(z_j))$ based on the values of $\widehat{H}_\lambda^{\ell-1}$ on $\mathcal{V}$
    end for
    Set $\widehat{H}_\lambda^\ell(z_j) = \|z_j\|^q + \min_{i\in\mathcal{M}} h_{ij}$;
    $\mathcal{V} \leftarrow \mathcal{V} \cap \mathcal{A}(\psi^{h_{k-\ell}}, \theta^{h_{k-\ell}})$
  end for
until $\ell = k$
$\widehat{h}_{\lambda,k} = \max_{z_j \in \mathbb{S}^{n-1}\cap\mathcal{C}} \widehat{H}_\lambda^k(z_j)$
return $\widehat{H}_\lambda^k(z_j)$ for all $z_j \in \mathbb{S}^{n-1} \cap \mathcal{C}$ and $\widehat{h}_{\lambda,k}$

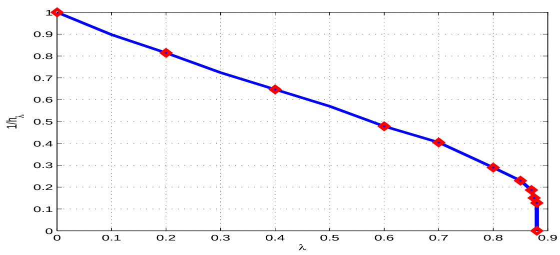

Figure 2: Plot of $1/h_\lambda$

practical computational complexity makes it highly difficult because gridding sets are very large and fine gridding is needed for numerical accuracy and stability. This significantly increases the computational complexity. Therefore, we conclude that a (sharp) lower bound or an under-estimate of λ_* is 0.88, and the radius of the domain of weak attraction $\rho_* = \sqrt{\lambda_*} \approx 0.9381$.

5 Discrete-Time Conewise Homogenous Inclusions

In the rest of the paper, we extend stability analysis of the switched homogeneous (resp. linear) systems on cones to that of conewise homogeneous (resp. linear) inclusions (CHIs). The CHIs form a class of discrete-time, switched dynamical systems subject to state-dependent switchings. Such a system partitions a close cone into finitely many sub-cones, and the system dynamics is homogeneous (and continuous) on each cone. Switching occurs as a state trajectory exits from one cone and enters another. Note that the state dynamics may attain multiple values on the intersection of two cones, thus making the system a class of homogeneous inclusions.

In specific, let $\mathcal{C} \subset \mathbb{R}^n$ be a closed cone containing the origin. Assume that $\Xi := \{\mathcal{X}_i\}_{i=1}^m$ is a finite family of nonempty closed sub-cones of $\mathcal{C}$ satisfying $\cup_{i=1}^m \mathcal{X}_i = \mathcal{C}$, with the index set $\mathcal{M} := \{1, \ldots, m\}$ for some $m \in \mathbb{N}$. Each cone $\mathcal{X}_i$ need not be polyhedral or even convex, and $\mathcal{X}_i$ and $\mathcal{X}_j$ may overlap for $i \neq j$. Associated with each cone $\mathcal{X}_i$ is a continuous and homogeneous mapping $x \mapsto F_i(x)$, $\forall x \in \mathcal{X}_i$, where each $F_i : \mathcal{C} \to \mathbb{R}^n$ is assumed to be positively invariant on $\mathcal{C}$, i.e., $F_i(\mathcal{C}) \subseteq \mathcal{C}$. The (discrete-time) conewise homogeneous inclusion (CHI) on $\mathcal{C}$ is then defined as

$$x(t+1) \in f(x(t)), \quad t \in \mathbb{Z}_+. \tag{27}$$

Here, $f : \mathcal{C} \rightrightarrows \mathcal{C}$ is the set-valued map defined by $f(x) := \{F_i(x) \mid \text{ for all } i \text{ such that } x \in \mathcal{X}_i\}$. Thus, at any time t, each cone $\mathcal{X}_i$ where the current state $x(t) = x$ resides provides a possible destination $F_i(x) \in \mathcal{C}$ that the state may evolve to at the next step. An important class of the CHIs is the conewise linear inclusions (CLIs), where each $F_i(x) = A_i x$ for some matrix A_i. The SHSs can also be treated as a special class of the CHIs by setting $\mathcal{X}_i = \mathcal{C}, \forall\, i \in \mathcal{M}$.

For a given initial state $z \in \mathcal{C}$ and a time window of length $T \in \mathbb{Z}_+$, let $\mathcal{W}(z, T) \subseteq \underbrace{\mathcal{M} \times \cdots \times \mathcal{M}}_{(T+1)-\text{copies}}$ denote the set of all admissible switching segments of length T corresponding to z. In particular, $\mathcal{W}(z, 0) = \{i \in \mathcal{M} \mid z \in \mathcal{X}_i\}$. Further, $\mathcal{W}(z, \infty)$ denotes the set of all admissible switching sequences corresponding to z. Note that for the SHSs, the equality $\mathcal{W}(z, T) = \mathcal{M} \times \cdots \times \mathcal{M}$ holds for each $T \in \mathbb{Z}_+$. For each $\sigma_T \in \mathcal{W}(z, T)$, let $x(t; z, \sigma_T)$ denote a trajectory starting from z under the admissible switching segment σ_T over the time horizon $t = 0, 1, \ldots, T + 1$. In a similar manner, for $\sigma \in \mathcal{W}(z, \infty)$, $x(t; z, \sigma)$ denotes a trajectory starting from z under σ over the infinite time horizon. It is easy to show that given a CHI of homogeneous degree ν, for any $z \in \mathcal{C}$ and any scalar $\alpha \geq 0$, $\mathcal{W}(z, T) = \mathcal{W}(\alpha z, T)$ for all T and $x(t; \alpha z, \sigma_T) = \alpha^{\nu^t} x(t; z, \sigma_T), \forall\, t = 0, \ldots, T + 1$ for any $\sigma_T \in \mathcal{W}(z, T)$. The (local) stability notions of the CHI (27) at $x_e = 0$ are defined as follows.

Definition 5.1 (Local Strong Stability of CHI). At $x_e = 0$, the CHI (27) on $\mathcal{C}$ is called

- *locally strongly stable* if, for each $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $\|x(t; z, \sigma)\| < \varepsilon, \forall\, t \in \mathbb{Z}_+$ under any $\sigma \in \mathcal{W}(z, \infty)$ starting from $z \in \mathcal{C}$ with $\|z\| \leq \delta_\varepsilon$;

- *locally strongly convergent* if there exists a neighborhood $\mathcal{N}$ of $x_e = 0$ in $\mathcal{C}$ such that for any $z \in \mathcal{N}$ and any $\sigma \in \mathcal{W}(z, \infty)$, $x(t; z, \sigma) \to 0$ as $t \to \infty$;

- *locally strongly asymptotically stable* if it is locally strongly stable and strongly convergent;

- *locally strongly exponentially stable* if there exist a neighborhood $\mathcal{N}$ of $x_e = 0$ in $\mathcal{C}$ and constants $\kappa \geq 1$ and $r \in [0, 1)$ such that $\|x(t; z, \sigma)\| \leq \kappa r^t \|z\|$, $\forall t \in \mathbb{Z}_+$, for any $\sigma \in \mathcal{W}(z, \infty)$ starting from an arbitrary $z \in \mathcal{N}$.

Definition 5.2 (Local Weak Stability of CHI). The CHI (27) on $\mathcal{C}$ is called *locally weakly stable* (respectively, *locally weakly convergent*, *locally weakly asymptotically stable* and *locally weakly exponentially stable*) at $x_e = 0$ if the corresponding condition in Definition 5.1 holds under at least one (instead of any) switching sequence $\sigma \in \mathcal{W}(z, \infty)$.

The non-local version of the strong and weak stability can be defined for the CHIs in a similar way as in Definition 2.2. Further, the domains of strong and weak attraction can also be defined. The following result shows the equivalence of the strong stability notions.

Theorem 5.1. Let the compact set $\mathcal{S} := \{z \in \mathcal{C} \mid \|z\| \leq \rho\}$ for some $\rho > 0$. The following hold for the CHI (27) on $\mathcal{S}$:

$$\text{strong convergence} \;\Leftrightarrow\; \text{strong asymptotic stability} \;\Leftrightarrow\; \text{strong exponential stability}$$

Proof. It suffices to show that strong convergence implies strong exponential stability on $\mathcal{S}$. We first prove the following claim that characterizes the uniform convergence of the CHI.

Claim: If there exists a finite time $T_* \in \mathbb{Z}_+$ (independent of z) such that for any $z \in \mathcal{S}$ with $\|z\| = \rho$ (the radius of $\mathcal{S}$) and any $\sigma \in \mathcal{W}(z, \infty)$, $\|x(t_*; z, \sigma)\| < 0.5\rho$ at some time $t_* \leq T_*$, then the CHI (27) on $\mathcal{S}$ is strongly exponentially stable.

The proof of the claim follows from a similar argument as in Propositions 2.1. In fact, in light of

$\mathcal{W}(z,T) = \mathcal{W}(\alpha z, T)$ for all T, $\alpha \ge 0$, and $x(t;\alpha z,\sigma) = \alpha^{\nu^t} x(t;z,\sigma), \forall\, t \in \mathbb{Z}_+$ for any $\sigma \in \mathcal{W}(z,\infty)$, as well as the compactness of $\mathcal{S}$ and the continuity of F_i, we deduce the existence of $\kappa > 0$ and $\gamma \in [0,1)$ such that for any $z \in \mathcal{S}$, $\|x(t;z,\sigma)\| \le \kappa\gamma^{\nu^t-1}\|z\|, \forall\, t \in \mathbb{Z}_+$ under any $\sigma \in \mathcal{W}(z,\infty)$. This yields the strong exponential stability.

We then prove by contradiction that strong convergence implies strong exponential stability. Suppose the CHI is strongly convergent but not strongly exponentially stable on $\mathcal{S}$. Then it follows from the contrapositive of the above claim that there exist a sequence of initial states $\{x_k^0\} \subseteq \mathcal{S}$ with $\|x_k^0\| = \rho$, a sequence of corresponding trajectories $\{x(t;x_k^0,\sigma^k)\}$, and a strictly increasing sequence of times $\{t_k\} \subset \mathbb{Z}_+$ with $t_k \uparrow \infty$, such that $\|x(t;x_k^0,\sigma^k)\| \ge 0.5\rho$, $t = 0,1,\ldots,t_k$, for all $k = 1,2,\ldots$. Since the sequence $\{x_k^0\}$ is contained in a compact set, a subsequence of it (which we may assume without loss of generality to be itself) converges to some $x_*^0 \in \mathcal{S}$ with $\|x_*^0\| = \rho$. Define the index set $\mathcal{I}(x_*^0) := \{i \,|\, x_*^0 \in \mathcal{X}_i\}$. Since each cone $\mathcal{X}_i$ is closed and there are a finite number of them, a neighborhood $\mathcal{N}$ of x_*^0 in $\mathcal{S}$ can be found such that $\mathcal{N} = \mathcal{S} \setminus \cup_{i \notin \mathcal{I}(x_*^0)} \mathcal{X}_i \subseteq \cup_{i \in \mathcal{I}(x_*^0)} \mathcal{X}_i$.

We next look at the sequence $\{x(1;x_k^0,\sigma^k)\}$. Our assumption on the times $\{t_k\}$ implies that $0.5\rho \le \|x(1;x_k^0,\sigma^k)\| \le \mu_1$ for all k large enough, where $\mu_1 := \max_{i\in\mathcal{M}, x\in\mathcal{S}, \|x\|=\rho} \|F_i(x)\|$. Thus, a subsequence of $\{x(1;x_k^0,\sigma^k)\}$ (which we again assume to be itself) converges to some $x_*^1 \in \mathcal{S}$ satisfying $0.5\rho \le \|x_*^1\| \le \mu_1$. As $x_k^0 \to x_*^0$, for k sufficiently large, x_k^0 will be inside the neighborhood $\mathcal{N}$ defined above. Thus, due to the continuity of F_i, $x(1;x_k^0,\sigma^k) = F_{j_k}(x_k^0)$ for some index $j_k \in \mathcal{I}(x_*^0)$. By letting $k \to \infty$ and noting that $\mathcal{I}(x_*^0)$ is a finite set, we conclude that $x_*^1 = F_j(x_*^0)$ for some $j \in \mathcal{I}(x_*^0)$, i.e., $x_*^1 \in f(x_*^0)$.

Repeating the above argument and using induction, we obtain a sequence $\{x_*^t\}_{t\in\mathbb{Z}_+} \subseteq \mathcal{S}$ such that (i) $0.5\rho \le \|x_*^t\| \le (\mu_1)^{\nu^t}$ for all $t \in \mathbb{Z}_+$; and (ii) $x_*^{t+1} \in f(x_*^t)$ for each $t \in \mathbb{Z}_+$. This shows that there exists at least one valid trajectory $x(t;x_*^0,\sigma) := x_*^t$, $t \in \mathbb{Z}_+$ of the CHI (27) that does not converge to the origin. This contradicts the strong convergence of the CHI (27) on $\mathcal{S}$. $\square$

The above result can be strengthened for the CLI (i.e., $\nu = 1$). In fact, due to the linearity of the CLI, the local stabilities are equivalent to their respective global ones. Hence, we obtain:

Corollary 5.1. The following hold true for the CLI on the closed cone $\mathcal{C}$:

strong convergence $\Leftrightarrow$ strong asymptotic stability $\Leftrightarrow$ strong exponential stability

It should be emphasized that the closedness of the cones $\mathcal{X}_i$'s is critical in establishing the equivalence. The example below shows that without the closedness, Theorem 5.1 may be invalid.

Example 5.1. Consider the CLI on $\mathcal{C} = \mathbb{R}^2$ with $\Xi = \{\mathcal{X}_i\}_{i=1}^4$, where $\mathcal{X}_1 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_1 > 0,\, x_2 > 0\}$, $\mathcal{X}_2 = \{(0,x_2)^T \in \mathbb{R}^2 \,|\, x_2 > 0\}$, $\mathcal{X}_3 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_1 < 0, x_2 > 0\}$, $\mathcal{X}_4 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_2 \le 0\}$. Note that $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$ are not closed. Let the corresponding dynamics matrices be $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_2 = A_4 = 0$. Since the cones in Ξ are disjoint, the set-valued mapping $f(\cdot)$ in (27) becomes a function on $\mathbb{R}^2$ (albeit a discontinuous one). Hence, the CLI has a unique trajectory for each initial state. It is easy to verify that for any initial state $x^0 = (x_1^0, x_2^0)^T \in \mathcal{X}_1$, the trajectory sequence is $(x_1^0,x_2^0)^T \to (-x_1^0,x_2^0)^T \to (x_1^0, x_2^0 - x_1^0)^T \to \cdots \to (x_1^0, x_2^0 - 2x_1^0)^T \to \cdots$ until the second coordinate becomes negative; after that, the trajectory jumps to the origin and remains there. In particular, we observe that $\|x(t;x^0,\sigma)\|$ is non-increasing in t and converges to zero as $t \to \infty$. This observation can also be verified for any trajectory starting from $\mathcal{X}_2$, $\mathcal{X}_3$, or $\mathcal{X}_4$. This shows that the CLI is strongly asymptotically stable. On the other hand, let $x^0 = (\varepsilon, 1)^T \in \mathcal{X}_1$ for some small $\varepsilon > 0$. From the above argument, we see that the times it takes for $x(t;x^0,\sigma)$ to reach the origin is about $1/(2\varepsilon)$ time steps, which tends to infinity as $\varepsilon \downarrow 0$. Thus the CLI is not strongly exponentially stable.

Theorem 5.1 does apply when we consider the CLI defined on the closure cones $\{\mathrm{cls}\,\mathcal{X}_i\}_{i=1}^4$ where cls denotes set closure, with the same subsystem matrices $\{A_i\}_{i=1}^4$. The same argument as above shows that this CLI is not strongly exponentially stable. By Theorem 5.1, the new CLI must not be strongly asymptotically stable. Indeed, starting from $x^0 = (0,1)^T \in \mathrm{cls}\,\mathcal{X}_1$, $x(t; x^0, \sigma) \equiv x^0$ is a non-convergent trajectory.

The next example shows that the conclusion of Theorem 5.1 is not true if the strong stability notions are replaced by their weak counterparts. Recall that Corollary 2.1 and Proposition 3.1 show that weak asymptotic stability of the SHS and SLS is equivalent to weak exponential stability. The underlying reason for this difference is that solutions to the SHSs on $\mathcal{C}$ under a fixed switching sequence depend continuously on initial states, while it is not the case for the CHIs on $\mathcal{C}$.

Example 5.2. Consider the CLI on $\mathcal{C} = \mathbb{R}^2$ with $\Xi = \{\mathcal{X}_i\}_{i=1}^3$, where $\mathcal{X}_1 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0\}$, $\mathcal{X}_2 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_1 \leq 0, x_2 \geq 0\}$, $\mathcal{X}_3 = \{(x_1,x_2)^T \in \mathbb{R}^2 \,|\, x_2 \leq 0\}$. Let the subsystem matrices be $A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $A_3 = 0$. This CLI is neither strongly asymptotically nor exponentially stable since the trajectory of periodicity two: $(0,1)^T \to (-1,0)^T \to (0,1)^T \to \cdots$ fails to converge to the origin. Nevertheless, it is weakly asymptotically stable. For example, starting from $x^0 = (x_1^0, x_2^0)^T$ in the interior of $\mathcal{X}_1$, the trajectory sequence is $(x_1^0, x_2^0)^T \to (-x_2^0, x_1^0)^T \to (x_1^0, x_2^0 - x_1^0)^T \to \cdots \to (x_1^0, x_2^0 - 2x_1^0)^T \to \cdots$ which eventually reaches $\mathcal{X}_3$; then the sequence can arrive at 0 at the next time step. Similarly, we can verify the existence of at least one convergent trajectory starting from any other initial states. Now consider the initial state $x^0 = (\varepsilon, 1)^T \in \mathcal{X}_1$ for a small $\varepsilon > 0$. For any trajectory $x(t; x^0, \sigma)$ starting from x^0, the number of time steps it takes for $\|x(t; x^0, \sigma)\|$ to decreases to half of its initial value is at least about $1/\varepsilon$, which grows unboundedly as $\varepsilon \to 0$. Thus the CLI is not weakly exponentially stable.

6 Stability of the CLIs

We briefly mention some general results for stability of the CHI, including the CLI as a special case. Consider the CHI of homogeneous degree $\nu \geq 1$. Define, for each $k \in \mathbb{N}$,

$$\mu_k := \sup\left\{\|x(k; z, \sigma_k)\|^{1/h_k} : \sigma_k \in \mathcal{W}(z,k),\ z \in \mathbb{S}^{n-1} \cap \mathcal{C}\right\}.$$

By the concatenation property of the switching segments, it is easy to show that $(\mu_{p+q})^{h_{p+q}} \leq (\mu_p)^{h_p} (\mu_q)^{h_q \nu^q}$ for all $p, q \in \mathbb{N}$. Hence, a similar argument as in Theorem 2.1 shows that the sequence (μ_k) converges to $\inf\{\mu_k\}$. This limit, denoted by μ^*, is the generalized joint spectral radius of the CHI. Further, it follows from a similar development in Section 2.3.1 that (i) when $\nu = 1$, the CLI is strongly exponentially stable if and only if $\mu^* < 1$; and (ii) when $\nu > 1$, the radius of the domain of strong attraction of the CHI is $(\mu^*)^{-\frac{1}{\nu-1}}$.

Define the strong generating function for the CLI: $G_\lambda(z) := \sup_{\sigma \in \mathcal{W}(z,\infty)} \sum_{t=0}^{\infty} \lambda^t \|x(t; z, \sigma)\|^q$, $\forall\, z \in \mathcal{C}$, $\lambda \geq 0$, and the radius of strong convergence: $\lambda_{\mathcal{C}}^* := \sup\{\lambda \geq 0 \,|\, G_\lambda(z) < \infty,\ \forall\, z \in \mathcal{C}\}$. In view of Corollary 5.1 and [14, Theorem 2], we have the following implications for the CLI on $\mathcal{C}$:

$$\text{strong exponential stability} \quad \Leftrightarrow \quad \lambda_{\mathcal{C}}^* > 1 \quad \Leftrightarrow \quad G_1 \text{ is pointwise bounded on } \mathcal{C}$$

Similar results can be obtained for the weak generating function of the CHI and weak exponential stability of the CHI and CLI. It is worth pointing out that computation of the radii of strong and weak convergence of the CHI or CLI is more difficult than that of the corresponding quantities of the SHS or SLS. This is attributed to the sensitivity of admissible switching segments/sequences with respect to initial states z. Further discussions of this issue are beyond the scope of the present paper. Next we characterize the stability of CLIs from a Lyapunov perspective.

6.1 Stability of the CLIs: A Lyapunov Perspective

A function $V : \mathcal{C} \to \mathbb{R}$ is called *(infinitely) piecewise quadratic* on $\mathcal{C}$ if it is positively homogeneous of degree two along each ray: $V(\lambda x) = \lambda^2 V(x)$ for all $\lambda \geq 0$ and $x \in \mathcal{C}$. In particular, V is called *finitely piecewise quadratic* on $\mathcal{C}$ if it is piecewise quadratic and for each $x \in \mathcal{C}$, $V(x) = x^T P x$ for some matrix $P \in \mathbb{R}^{n \times n}$ taking values in a finite set of positive semidefinite matrices. The following result asserts the equivalence of strong exponential stability of CLI and the existence of a finitely piecewise quadratic Lyapunov function.

Proposition 6.1. The CLI (27) on $\mathcal{C}$ is strongly exponentially stable if and only if there exists a finitely piecewise quadratic Lyapunov function $V : \mathcal{C} \to \mathbb{R}_+$ satisfying

(a) there exist $c_1 > 0$ and $c_2 > 0$ such that $c_1 \|z\|^2 \leq V(z) \leq c_2 \|z\|^2$ for all $z \in \mathcal{C}$;

(b) there exists $c_3 > 0$ such that $V(z') - V(z) \leq -c_3 \|z\|^2, \ \forall\, z' \in f(z)$ for all $z \in \mathcal{C}$.

Proof. The sufficiency follows from the standard argument (even without the finitely piecewise quadratic property for V) and is omitted. We prove the necessity as follows. Suppose the CLI is strongly exponentially stable on $\mathcal{C}$. Then there exist $\kappa \geq 1$ and $r \in [0, 1)$ such that $\|x(t; z, \sigma)\| \leq \kappa\, r^t \|z\|, \forall\ t \in \mathbb{Z}_+$, for all $z \in \mathcal{C}$ and any $\sigma \in \mathcal{W}(z, \infty)$. Find a time T_* large enough such that $\kappa^2\, r^{2(T_*+1)} \leq \frac{1}{2}$. Define the function

$$V(z) := \max_{\sigma_{T_*} \in \mathcal{W}(z, T_*)} \sum_{t=0}^{T_*} \|x(t; z, \sigma_{T_*})\|^2, \quad \forall z \in \mathcal{C}. \tag{28}$$

Then $V(z)$ can be written as $V(z) = \max_{P \in \mathcal{P}} z^T P z$, where $\mathcal{P}$ is the set of all positive definite matrices of the form $I + A_{i_0}^T A_{i_0} + \big(A_{i_1} A_{i_0}\big)^T \big(A_{i_1} A_{i_0}\big) + \cdots + (\prod_{t=T_*-1}^{0} A_{i_t})^T (\prod_{t=T_*-1}^{0} A_{i_t})$ for $i_t \in \mathcal{M}$, $t = 0, \ldots, T_* - 1$. Since $\mathcal{P}$ is a finite set, V is finitely piecewise quadratic.

We shall prove that V is the desired Lyapunov function. It is clear that for each $z \in \mathcal{C}$, $\|z\|^2 \leq V(z) \leq \sum_{t=0}^{T_*} \kappa^2\, r^{2t}\, \|z\|^2 \leq c_2 \|z\|^2$ where $c_2 := \kappa^2/(1 - r^2)$. Therefore (a) holds true. To prove (b), we note that for any $z' \in f(z)$ and any $\sigma_{T_*} \in \mathcal{W}(z', T_*)$, the concatenation of the cone index corresponding to the transition from z to z' and σ_{T_*} is an admissible switching segment $\sigma_{T_*+1} \in \mathcal{W}(z, T_* + 1)$. Therefore,

$$\|z\|^2 + \sum_{t=0}^{T_*} \|x(t; z', \sigma_{T_*})\|^2 \ \leq \max_{\sigma_{T_*+1} \in \mathcal{W}(z, T_*+1)} \sum_{t=0}^{T_*+1} \|x(t; z, \sigma_{T_*+1})\|^2 = \sum_{t=0}^{T_*+1} \|x(t; z, \widetilde{\sigma}_{T_*+1})\|^2,$$

where $\widetilde{\sigma}_{T_*+1} \in \mathcal{W}(z, T_* + 1)$ is the switching segment that achieves the above maximum. Since $\|x(T_* + 1; z, \widetilde{\sigma}_{T_*+1})\|^2 \leq \frac{1}{2} \|z\|^2$ by the choice of T_*, we have

$$\sum_{t=0}^{T_*+1} \|x(t; z, \widetilde{\sigma}_{T_*+1})\|^2 \ = \ \sum_{t=0}^{T_*} \|x(t; z, \widetilde{\sigma}_{T_*+1})\|^2 + \|x(T_* + 1; z, \widetilde{\sigma}_{T_*+1})\|^2 \ \leq \ V(z) + \frac{1}{2} \|z\|^2.$$

Combining the above two inequalities, we obtain $\sum_{t=0}^{T_*} \|x(t; z', \sigma_{T_*})\|^2 \leq V(z) - \frac{1}{2} \|z\|^2$, for any $\sigma_{T_*} \in \mathcal{W}(z', T_*)$. Therefore, $V(z') \leq V(z) - \frac{1}{2} \|z\|^2$, and $V(z)$ indeed satisfies (b). $\square$

Proposition 6.1 can be extended to weak exponential stability with the finiteness of a piecewise quadratic Lyapunov function dropped.

Proposition 6.2. The CLI (27) is weakly exponentially stable on $\mathcal{C}$ if and only if there exists a (in general infinitely) piecewise quadratic Lyapunov function $V : \mathcal{C} \to \mathbb{R}_+$ satisfying

(a) there exist $c_1 > 0$ and $c_2 > 0$ such that $c_1\|z\|^2 \le V(z) \le c_2\|z\|^2$ for all $z \in \mathcal{C}$;

(b) $c_3 > 0$ exists such that for each $z \in \mathcal{C}$, there exists $z' \in f(z)$ such that $V(z') - V(z) \le -c_3\|z\|^2$.

Proof. For sufficiency, suppose there exists a Lyapunov function V satisfying (a) and (b). For any initial state $x^0 \in \mathcal{C}$, it follows from (b) that there exists $x^1 \in V(x^0)$ such that $V(x^1) \le V(x^0) - c_3\|x^0\|^2 \le \eta V(x^0)$, where $\eta := 1 - c_3/c_2$ via (a). To avoid triviality, we may assume $\eta \in (0, 1)$. Letting $x(1; x^0) := x^1$ and repeating the above argument, we obtain a trajectory $x(t; x^0)$ whose V-value is exponentially decaying (with the decay rate determined by η). Hence the CLI is weakly exponentially stable on $\mathcal{C}$.

For necessity, assume the CLI (27) is weakly exponentially stable on $\mathcal{C}$. Then starting from any $z \in \mathcal{C}$, there exists at least one switching sequence $\sigma \in \mathcal{W}(z, \infty)$ satisfying $\|x(t; z, \sigma)\| \le \kappa\, r^t \|z\|$, $\forall\, t \in \mathbb{Z}_+$, for some constants $\kappa > 0$ and $r \in [0, 1)$. Thus, the function $V(z)$ defined by $V(z) := \inf_{\sigma \in \mathcal{W}(z,\infty)} \sum_{t=0}^{\infty} \|x(t; z, \sigma)\|^2$, where the infimum is taken over all admissible switching sequences corresponding to z, is finite for each z. Furthermore, it satisfies $\|z\|^2 \le V(z) \le c\,\|z\|^2$ for some constant $c \ge 1$, where the existence of c is due to the weak exponential stability. Moreover, being the value function of an infinite-horizon optimal control problem, $V(z)$ satisfies the Bellman equation: $V(z) = \min_{z' \in f(z)} \big\{\|z\|^2 + V(z')\big\} = \|z\|^2 + \min_{z' \in f(z)} V(z')$. The minimum in the above equation is achieved by some $z'_* \in f(z)$. Then we have $V(z) = \|z\|^2 + V(z'_*)$, i.e., $V(z'_*) - V(z) = -\|z\|^2$. Thus, $V(z)$ satisfies both (a) and (b). □

6.2 CLIs Obtained from SLSs on Cones

In the previous sections, we have studied the strong and weak stabilities of CLIs and SLSs on cones. We next present results that connect the stability of these two systems. On a closed convex cone $\mathcal{C}$, a function $E : \mathcal{C} \to \mathbb{R}_+$ is called an *energy function* on $\mathcal{C}$ if it is (i) homogeneous of degree two: $E(\lambda z) = \lambda^2 E(z)$, $\forall\, \lambda \in \mathbb{R}$, $\forall\, z \in \mathcal{C}$; and (ii) bounded on the unit sphere $\mathbb{S}^{n-1}$: $c_1\|z\|^2 \le E(z) \le c_2\|z\|^2$ for constants $0 < c_1 \le c_2 < \infty$. For the SLS (11) on $\mathcal{C}$, two CLIs are introduced below based on a conic partition of $\mathcal{C}$ defined by an energy function E.

Definition 6.1. Given the SLS (11) on the closed cone $\mathcal{C}$ and an energy function E, let $\Xi = \{\mathcal{X}_i\}_{i\in\mathcal{M}}$ be a set of closed sub-cones of $\mathcal{C}$ defined as $\mathcal{X}_i := \{x \in \mathcal{C} \,|\, E(A_i x) = \min_{j\in\mathcal{M}} E(A_j x)\}$, $i \in \mathcal{M}$. Then a CLI on $\mathcal{C}$ is defined with the subsystem matrix A_i on the cone $\mathcal{X}_i$, and is called the *descending realization* of the SLS (11) on $\mathcal{C}$ with respect to the energy function E. If in the above definition of $\mathcal{X}_i$, minimum is replaced by maximum, while the matrices A_i are defined in the same way, then the resulting CLI on $\mathcal{C}$ is called the *ascending realization* of the SLS on $\mathcal{C}$ with respect to E.

It is seen from the above definition that the trajectories $x(t; x^0)$ of the descending (resp. ascending) realization CLI are exactly the trajectories $x(t; x^0, \sigma)$ of the SLS (11) under the switching policy σ that tries to decrease (resp. increase) the value of the energy function E as much as possible at the next time step. If for a given $x \in \mathcal{C}$, $\max_{i\in\mathcal{M}} E(A_i x)$ is achieved by multiple $i \in \mathcal{M}$, then the ascending realization CLI has multiple trajectories starting from x: $f(x) = \{A_i x \,|\, E(A_i x) = \max_{j\in\mathcal{M}} E(A_j x)\}$. A similar observation holds for the descending realization.

The following two theorems state that the characterization of exponential stability for SLSs on $\mathcal{C}$ can be reduced to that of weak exponential stability for the ascending/descending CLIs on $\mathcal{C}$ with respect to properly chosen energy functions.

Theorem 6.1. A necessary and sufficient condition for the SLS (11) on $\mathcal{C}$ to be strongly exponentially stable is that its ascending realization CLI on $\mathcal{C}$ with respect to any energy function E on $\mathcal{C}$ is weakly exponentially stable with the uniform parameters $\kappa \ge 0$ and $r \in [0, 1)$.

Proof. The necessity part is trivial once we notice that any trajectory of the ascending realization CLI is also a trajectory of the SLS under some switching sequence σ.

To show sufficiency, suppose that the ascending realization CLI of the SLS (11) on $\mathcal{C}$ with respect to any energy function E is weakly exponentially stable. Consider the strong generating function $G_\lambda(z)$ defined in (13) with $q = 2$ and $\|\cdot\|$ the Euclidean norm. Let $\lambda^*_{\mathcal{C}}$ be its radius of convergence. Then for any $\lambda \in [0, \lambda^*_{\mathcal{C}})$, $E(z) = G_\lambda(z)$ is an energy function. Thus by assumption, the ascending realization CLI, denoted by CLI(λ), of the SLS (11) with respect to the energy function $G_\lambda(z)$ is weakly exponentially stable, i.e., starting from any $z \in \mathcal{C}$, there exists $\sigma^* \in \mathcal{W}(z, \infty)$ such that the trajectory $x(t; z, \sigma^*)$ of CLI(λ) satisfies $\|x(t; z, \sigma^*)\| \le \kappa r^t \|z\|$, $\forall\, t \in \mathbb{Z}_+$. Due to the construction of the ascending realization CLI(λ), $x(t; z, \sigma^*)$ is also a trajectory of the SLS (11) that achieves the supremum in (13). Hence, $G_\lambda(z) = \sum_{t=0}^\infty \lambda^t \|x(t; z, \sigma^*)\|^2 \le \sum_{t=0}^\infty \lambda^t \kappa^2 r^{2t} \|z\|^2$, $\forall z \in \mathcal{C}$, $\lambda \in [0, \lambda^*_{\mathcal{C}})$. We must have $\lambda^*_{\mathcal{C}} \ge r^{-2}$, for otherwise the right most term in the above inequality, hence $G_\lambda(z)$, would be finite at all $z \in \mathcal{C}$ for some $\lambda > \lambda^*_{\mathcal{C}}$, contradicting the definition of $\lambda^*_{\mathcal{C}}$. Since $r \in [0, 1)$, we have $\lambda^*_{\mathcal{C}} \ge r^{-2} > 1$. By Theorem 3.2, the SLS (11) on $\mathcal{C}$ is strongly exponentially stable. □

Similarly, the weak exponential stability of the SLS (11) on $\mathcal{C}$ can be related to that of its descending realization CLIs, as stated in the following theorem.

Theorem 6.2. The SLS (11) on $\mathcal{C}$ is weakly exponentially stable if and only if its descending realization CLI with respect to at least one energy function E is weakly exponentially stable.

Proof. The sufficient part is trivial, as an exponentially convergent trajectory of the descending realization CLI with respect to an arbitrary energy function E is automatically an exponentially convergent trajectory of the SLS under some suitable switching sequence.

To show necessity, assume that the SLS (11) on $\mathcal{C}$ is weakly exponentially stable. By Theorem 3.3, we have $\lambda_*^{\mathcal{C}} > 1$. Hence, the weak generating function $H_\lambda(z)$ at $\lambda = 1$, i.e., $H_1(z) = \inf_\sigma \sum_{t=0}^\infty \|x(t; z, \sigma)\|^2$, is finite everywhere on $\mathcal{C}$. Here we assume $q = 2$. By Propositions 3.5 and 3.6, $H_1(z)$ is an energy function and satisfies the Bellman equation $H_1(z) = \|z\|^2 + \min_{i \in \mathcal{M}} H_1(A_i z)$, $\forall\, z \in \mathcal{C}$. Consider the descending realization CLI of the SLS (11) with respect to the energy function $H_1(z)$. By its definition and the Bellman equation, $H_1(z') - H_1(z) = -\|z\|^2$ for all $z \in \mathcal{C}$ and $z' \in f(z)$ under the CLI dynamics. Thus, $H_1(z)$ is a Lyapunov function of the CLI satisfying the hypotheses of Proposition 6.1. This shows that the CLI is weakly exponentially stable. □

We remark that Theorem 6.2 remains valid even if weak exponential stability is replaced by strong exponential stability for the CLI. We also note that the energy functions E in this section are required to be bounded away from both zero and infinity on the unit sphere. To justify this requirement, assume for example E is identically zero (or identically infinity) on $\mathbb{R}^n$. Then the ascending (or descending) CLI will have the same set of trajectories as the SLS (11), thus making the conclusions of Theorem 6.1 and Theorem 6.2 trivial.

7 Conclusion

The stability of discrete-time switched homogeneous systems on cones are investigated from several interconnected perspectives, including the joint spectral radius approach and the generating function approach. The former approach generalizes the similar notions for the SLSs on the Euclidean space and characterizes the domains of strong and weak attraction. The generating function approach provides a unified and numerically effective framework to determine the stability of the SHS on cones; various analytic and numerical properties of the generating functions and their radii of convergence are derived. Extensions to the CHIs and the CLIs are obtained. Future research includes computation of the generating functions via such techniques as sum-of-squares [23].

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