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Robust Non-Zenoness of Piecewise Analytic Systems with Applications to Complementarity Systems

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Abstract— **This paper addresses non-Zenoness of a class of Lipschitz piecewise analytic systems subject to state perturbations and parameter uncertainties, motivated by sensitivity analysis of such systems. Specifically, the existence of uniform bounds on the number of mode switchings on a finite time interval is established for perturbed systems. For general parameterized piecewise analytic systems, this is achieved locally by extending Sussmann's result; this result is applied to a class of nonlinear complementarity systems arising from contact mechanics and constrained dynamical optimization. Furthermore, the existence of a global uniform bound on the number of switchings is established for bimodal piecewise affine systems by exploiting affine structure, under mild conditions on system parameters. It is shown that this bound is independent of initial state perturbations.**

I. INTRODUCTION

An intricate behavior in hybrid dynamics is the possible occurrence of infinitely many mode transitions on a finite time interval, which is referred to as *Zenoness* or *Zeno behavior* in the hybrid system literature. Typical Zeno hybrid systems include the bouncing-ball example in contact mechanics and switched engineering systems [14]. The Zeno behavior is a unique phenomenon of hybrid systems and plays a crucial role in analysis, simulation and control of hybrid systems [2], [3], [23]. For a Zeno hybrid system, it is impossible to simulate all mode transitions near a Zeno state, which leads to serious problems in computation and convergence analysis. Besides, the Zeno behavior prevents one from applying smooth ODE theory for dynamic and control analysis, even for local analysis.

The past few years have witnessed growing interest in characterization of the (non-)existence of Zeno behavior in hybrid systems. Initial attempts, among many other results, include geometric and topological approaches [1], [20] and dynamical system method [23]. Efforts have been made lately toward understanding (non-)Zenoness of several important classes of hybrid systems, e.g., complementarity systems. Roughly speaking, a complementarity system is a dynamical system coupled with a complementarity problem [5]. Complementarity systems have inherent nonsmooth and hybrid behaviors. By exploiting complementarity and piecewise affine structure, it is shown that a large class of complementarity systems and related piecewise smooth systems do not exhibit Zeno behavior [7], [8], [13], [16], [18], [19]. A highly interesting new perspective in Zeno analysis, originally due to [15], is characterization of Zeno behavior near an equilibrium via Lyapunov-like conditions. This approach is substantially extended to a wide range of hybrid systems possessing switchings and jumps in both continuous-time and discrete-time dynamics, with the aid of the recently developed hybrid stability theory [10] and homogeneous or symmetry techniques [11], [12], [17]. It is worth mentioning that most of the above Zeno results do not consider perturbations and uncertainties.

Inspired by sensitivity and robustness analysis of hybrid systems, the present paper performs Zeno analysis for a family of trajectories due to state perturbations and/or parameter variations. Specifically, we establish robust non-Zenoness, namely, the existence of a uniform bound on the number of mode switchings, for a class of Lipschitz piecewise analytic systems subject to both initial state perturbations and parameter uncertainties.

The rest of the paper is organized as follows. In Section II, we show local robust non-Zenoness for a class of perturbed piecewise analytic systems, by extending Sussmann's result [21]; this result is applied to a class of nonlinear complementarity systems (NCSs) arising from contact mechanics and constrained dynamic optimization. By employing affine structure, Section III addresses global robust non-Zenoness for the bimodal piecewise affine system. It is shown that if two dynamic matrices are bounded, then there exists a uniform bound on the number of mode transitions, regardless of initial states and other system parameters.

II. ROBUST NON-ZENONESS OF PIECEWISE ANALYTIC AND COMPLEMENTARITY SYSTEMS

In this section, we review basic concepts for piecewise analytic systems and introduce Sussmann's result. This result is extended to a class of piecewise analytic systems and NCSs under perturbations and uncertainties.

A. Boundedness of the number of switchings of piecewise analytic systems

In the paper [21], Sussmann considered Lipschitz piecewise analytic systems on real analytic manifolds and obtained a general result on boundedness of the number of mode switchings in a finite time interval. For the purpose of this paper, we assume that the real analytic manifold is the n-dimensional Euclidean space without losing much generality.

Given a nonempty set $E \subseteq \mathbb{R}^n$. Let $\mathcal{O}(E)$ denote the ring of real analytic functions on E, and let $S(\mathcal{O}(E))$ be the smallest family of subsets of E, which contains all the sets of the form $\{x \in E \,|\, f(x) > 0\}$ for $f \in \mathcal{O}(E)$ and which is closed under finite union, finite intersection,

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and complement (relative to $\mathbb{R}^n$). A set $A \subseteq \mathbb{R}^n$ is called *semianalytic* if, for each $x \in \mathbb{R}^n$, there exists a neighborhood U_x of x such that $A \cap U_x \in S(\mathcal{O}(U_x))$ [4]. Equivalently, a set A is semianalytic if and only if for any $x \in \mathbb{R}^n$, there exist a neighborhood U_x of x and finitely many real analytic functions $f_{ij} : U_x \to \mathbb{R}$ where $i = 1, \cdots, p$ and $j = 1, \cdots, q$ such that $A \cap U_x = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} B_{ij}$, where, for each i, j, the set B_{ij} is either $\{x \in U_x \,|\, f_{ij}(x) = 0\}$ or $\{x \in U_x \,|\, f_{ij}(x) > 0\}$ [4], [22]. The family of *subanalytic* sets on $\mathbb{R}^n$ is the smallest collection of subsets of $\mathbb{R}^n$, which contains all the semianalytic sets on $\mathbb{R}^n$ and is closed under the following operations: (i) locally finite union and finite intersection; (ii) complement; (iii) inverse image under a real analytic function; and (iv) direct proper image under a real analytic function [4], [21], [22]. A semianalytic set is subanalytic but not vice versa. The following lemma shows an example of a closed semianalytic, thus subanalytic, set. In fact, the converse of the lemma also holds true locally, i.e., each closed semianalytic set is locally a finite union of the sets stated in the lemma [4, Corollary 2.8].

Lemma 1 A finite union of the sets of the form $\{x \in \mathbb{R}^n \,|\, f_1(x) \geq 0, \cdots, f_q(x) \geq 0\}$ is semianalytic, where each $f_i : \mathbb{R}^n \to \mathbb{R}$ is a real analytic function.

Proof. For each f_i, let f_{ij} denote its jth component which is real analytic. Let the set $A \equiv \bigcup_{i=1}^{q} \{x \in \mathbb{R}^n \,|\, f_{i1}(x) \geq 0, \cdots, f_{ik_i}(x) \geq 0\}$ be nonempty. It is clear that A is closed. For a given $x^* \in \mathbb{R}^n$, if $x^* \notin A$, then there is a neighborhood U_{x^*} of x^* such that $A \cap U_{x^*}$ is empty. Hence it is trivial that $A \cap U_{x^*} \in S(\mathcal{O}(U_{x^*}))$. If $x^* \in A$, then letting U_{x^*} be a neighborhood of x^*, we have $A \cap U_{x^*} = \bigcup_{i=1}^{q} \bigcap_{j=1}^{k_i} \{x \in U_{x^*} \,|\, f_{ij}(x) \geq 0\}$. It easily follows from the equivalent definition of semianalytic sets that A is semianalytic. $\square$

It can be shown in the similar manner that a finite union of the sets of the form $\{x \in \mathbb{R}^n \,|\, f_1(x) \geq 0, \cdots, f_q(x) \geq 0\}$ or $\{x \in \mathbb{R}^n \,|\, g_1(x) > 0, \cdots, g_p(x) > 0\}$ is semianalytic, where each f_i and g_j is a real analytic function.

We introduce the piecewise analytic system treated in [21] as follows. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise analytic function, namely, there exist a finite family of selection functions $\{f^i\}_{i=1}^m$ such that $f(x) \in \{f^i(x)\}_{i=1}^m$ for each $x \in \mathbb{R}^n$, and that the following conditions hold:

(H1) Associated with each f^i, there exists a nonempty subanalytic set $\mathcal{X}_i \subseteq \mathbb{R}^n$ such that $f(x) = f^i(x)$, $\forall\, x \in \mathcal{X}_i$ and $\{\mathcal{X}_i\}_{i=1}^m$ forms a (locally) finite partition of $\mathbb{R}^n$;

(H2) For each $\mathcal{X}_i$, there exists an open set $\Omega_i \subseteq \mathbb{R}^n$ such that $\text{cls}\, \mathcal{X}_i \subseteq \Omega_i$ and f^i is real analytic on Ω_i, where cls denotes the closure of a set;

(H3) The continuity condition holds: $x \in \text{cls}\, \mathcal{X}_i \cap \text{cls}\, \mathcal{X}_j \implies f^i(x) = f^j(x)$ for any $i, j \in \{1, \cdots, m\}$.

Note that the subanalytic sets $\mathcal{X}_i$ may be neither open nor closed. This leads to a difficulty in imposing an appropriate real analytic property on f^i over $\mathcal{X}_i$. To avoid this problem, the open covering Ω_i is introduced for each $\mathcal{X}_i$ in (H2).

Consider the ODE system whose right-hand side f satisfies the conditions (H1-H3):

$$\dot{x} = f(x) \tag{1}$$

Given $T > 0$, let the time interval $I \equiv [0, T]$ and $x(t, x^0)$ denote a (locally unique) solution of (1) on I corresponding to the initial condition x^0. We say that a time instant $t_* \in (0, T)$ is *not* a switching time along $x(t, x^0)$ or $x(t, x^0)$ has no switching at t_* (in the strict sense defined in [21]) if there exist $i \in \{1, \cdots, m\}$ and $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i, \forall\; t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, we say that t_* is a *switching time* along $x(t, x^0)$, and that $x(t, x^0)$ has a *mode switching* or *mode transition* at t_*.

Theorem 2 [21, Theorem II] Consider the system (1) satisfying the conditions (H1-H3). For a compact set $\mathcal{V} \subseteq \mathbb{R}^n$ and a real $T > 0$, there exists $N(\mathcal{V}, T) \in \mathbb{N}$ such that for any time interval $I \subseteq [0, T]$, if a trajectory $x(t, x^0)$ satisfies $\{x(t, x^0) \,|\, t \in I\} \subseteq \mathcal{V}$, then $x(t, x^0)$ has at most $N(\mathcal{V}, T)$ mode switchings on I.

This theorem forms a corner stone for the robust non-Zeno analysis performed in the subsequent sections. As a matter of fact, it already reveals a bound on the number of switchings under initial state perturbations.

B. Robust non-Zenoness of piecewise analytic systems subject to initial state and parameter variations

In this section, we extend Theorem 2 to the following parameterized piecewise analytic system:

$$\begin{aligned} \dot{x} &= f^i(x, z), \qquad\qquad\qquad\qquad\qquad (2) \\ \forall\, x \in \mathcal{X}_i(z) &\equiv \{x \in \mathbb{R}^n \,|\, h^i(x, z) \geq 0,\; w^i(x, z) > 0\}, \end{aligned}$$

where $z \in \mathbb{R}^p$ is the parameter vector, $h^i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{\ell_1}$, $w^i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{\ell_2}$, and for each given z, $\{f^i(\cdot, z)\}$ forms a piecewise analytic function and the family of semianalytic sets $\{\mathcal{X}_i(z)\}$ forms a partition of $\mathbb{R}^n$. We further assume that each of f^i, h^i and w^i is analytic in both x and z. Since each $f^i(\cdot, z)$ is globally analytic for a fixed z, the hypothesis (H2) holds trivially (by choosing $\Omega_i = \mathbb{R}^n$). While the parameterized system (2) is less general than the system (1) for a given z, it represents a rather broad class of piecewise analytic systems in applications. In the following, we use $x_z(t, x^0)$ to denote the trajectory of (2) associated with parameter z and initial condition x^0.

Theorem 3 Consider the parameterized piecewise analytic system (2) that satisfies the conditions (H1) and (H3) for a given z. For a compact set $\mathcal{V} \times \mathcal{W} \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a real $T > 0$, there exists $N(\mathcal{V}, \mathcal{W}, T) \in \mathbb{N}$ such that for each $z \in \mathcal{W}$ and any time interval $I \subseteq [0, T]$, if a trajectory $x_z(t, x^0)$ satisfies $\{x_z(t, x^0) \,|\, t \in I\} \subseteq \mathcal{V}$, then $x_z(t, x^0)$ has at most $N(\mathcal{V}, \mathcal{W}, T)$ mode switchings on I.

Proof. Define $\mathbf{z} \equiv \begin{pmatrix} x \\ z \end{pmatrix}$, $\mathbf{F}^i(\mathbf{z}) \equiv \begin{pmatrix} f^i(x, z) \\ 0 \end{pmatrix}$, and $\mathbf{X}_i \equiv \mathcal{X}_i \times \mathbb{R}^p$. Hence, the parameterized piecewise analytic system (2) can be equivalently written as

$$\dot{\mathbf{z}} = \mathbf{F}^i(\mathbf{z}), \quad \forall\, \mathbf{z} \in \mathbf{X}_i \tag{3}$$

where $\{\mathbf{F}^i(\mathbf{z})\}$ forms a piecewise analytic function on $\mathbb{R}^n \times \mathbb{R}^p$, and $\{\mathbf{X}_i\}$ forms a partition of $\mathbb{R}^n \times \mathbb{R}^p$. It is easy to verify that each $\mathbf{F}^i$ is (globally) analytic and each $\mathbf{X}_i$ is a semianalytic set. Since $\mathrm{cls}\mathbf{X}_i = \mathrm{cls}\mathcal{X}_i \times \mathbb{R}^p$ and the system (2) satisfies (H3) holds for any $z \in \mathbb{R}^p$, we deduce that (H3) also holds true for (3). Furthermore, for the initial condition $\mathbf{z}^0 = (x^0, z)$, we have $\mathbf{z}(t, \mathbf{z}^0) = (x_z(t, x^0), z)$. Hence, $x(t, x^0)$ has a mode switching at t_* with respect to the partition $\{\mathcal{X}_i\}$ if and only if $\mathbf{z}(t, \mathbf{z}_0)$ has a switching at t_* with respect to the partition $\{\mathbf{X}_i\}$. It follows from Theorem 2 that for a compact set $\mathcal{V} \times \mathcal{W} \subseteq \mathbb{R}^n \times \mathbb{R}^p$ and a real $T > 0$, there exists $N(\mathcal{V}, \mathcal{W}, T) \in \mathbb{N}$ such that for any time interval $I \subseteq [0, T]$, if a trajectory $\mathbf{z}(t, \mathbf{z}^0)$ satisfies $\{\mathbf{z}(t, \mathbf{z}^0) \,|\, t \in I\} \subseteq \mathcal{V} \times \mathcal{W}$, then $\mathbf{z}(t, \mathbf{z}^0)$ has at most $N(\mathcal{V}, \mathcal{W}, T)$ mode switchings on I. Hence, the same bound holds true for each $x_z(t, x^0)$ on any $I \subseteq [0, T]$ as long as $z \in \mathcal{W}$ and $x_z(t, x^0) \in \mathcal{V}$ for all $t \in I$. $\square$

C. Application to nonlinear complementarity systems

Consider the following nonlinear complementarity system:

$$
\begin{aligned}
\dot{x} &= F(x, y, u) && \text{(4a)}\\
0 \le y &\perp G(x, y) \ge 0 && \text{(4b)}\\
0 &= D(x, y) + N(x, y)u - E^T\lambda && \text{(4c)}\\
0 \le \lambda &\perp H(x, y) + Eu \ge 0 && \text{(4d)}
\end{aligned}
$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^p, \lambda \in \mathbb{R}^\ell$, $F : \mathbb{R}^{n+m+p} \to \mathbb{R}^n$, $G : \mathbb{R}^{n+m} \to \mathbb{R}^m$, $D : \mathbb{R}^{n+m} \to \mathbb{R}^p$, $N : \mathbb{R}^{n+m} \to \mathbb{R}^{p\times p}$, $E \in \mathbb{R}^{p\times\ell}$, and $H : \mathbb{R}^{n+m} \to \mathbb{R}^\ell$. Here $\perp$ means that two vectors are orthogonal. Without loss of generality, we assume that E has no zero rows. The model (4), obtained from differential quasi-variational inequalities [13], represents many applied systems in constrained dynamic optimization [19] and frictional contact systems subject to polygonal frictional laws and with local elastic compliance; see [13] and the references therein for more details and applications. The following assumptions are imposed for a given pair $(x^*, y^*) \in \mathbb{R}^{n+m}$:

(c1) the functions G, D, N, H are analytic in a neighborhood of (x^*, y^*);

(c2) y^* is a strongly regular solution of the the complementarity problem (4b) [16];

(c3) $N(x^*, y^*)$ is positive definite;

(c4) for any (x, y) in a neighborhood of (x^*, y^*) with $y \ge 0$, the set $\{v \in \mathbb{R}^\ell \,|\, H(x, y) + Ev \ge 0\}$ is nonempty.

It follows from complementarity theory [9] that there exist neighborhoods $\mathcal{N}_0$ of x^* and $\mathcal{V}_0$ of y^* such that (i) the complementarity problem (4b) has a unique solution $y(x) \in \mathcal{V}_0$ for each $x \in \mathcal{N}_0$ with $y(x^*) = y^*$ and $y(x)$ is Lipschitz continuous and piecewise analytic on $\mathcal{N}_0$; (ii) the mixed complementarity problem (4c-4d) has a unique solution $u(x, y)$ for each $(x, y) \in \mathcal{N}_0 \times \mathcal{V}_0$ and $u(x, y)$ is a Lipschitz continuous and piecewise analytic function on $\mathcal{N}_0 \times \mathcal{V}_0$. Letting $\mathcal{U}_0$ be a neighborhood of $u^* \equiv u(x^*, y^*)$, we also assume

(c5) the function $F(x, y, u)$ is analytic on $\mathcal{N}_0 \times \mathcal{V}_0 \times \mathcal{U}_0$.

Therefore, $\widetilde{F}(x) \equiv F\big(x, y(x), u(x, y(x))\big)$ is continuous on $\mathcal{N}_0$ if we restrict $y(x) \in \mathcal{V}_0$ and $u(x, y) \in \mathcal{U}_0$. Moreover, we show that $\widetilde{F}(x)$ is piecewise analytic on $\mathcal{N}_0$ such that $\dot{x} = \widetilde{F}(x)$ is a (locally) piecewise analytic system as follows:

Proposition 4 Under the conditions (c1-c5), if we restrict $y(x) \in \mathcal{V}_0$ and $u(x, y) \in \mathcal{U}_0$, then the ODE system $\dot{x} = \widetilde{F}(x)$ satisfies the conditions (H1-H3) on a small neighborhood $\mathcal{N}_0$ of x^*.

Proof. The condition (H3) is easy to verify; we focus on (H1-H2). Specifically, we show that corresponding to each analytic selection function of $\widetilde{F}(x)$, there is a semianalytic subset in $\mathcal{N}_0$ and the union of these subsets forms a finite partition of $\mathcal{N}_0$. Consider the three fundamental index sets associated with the complementarity problem (4b) at x^*:

$$
\begin{aligned}
\alpha_* &\equiv \{\, i \,|\, y_i^* > 0 = G_i(x^*, y^*) \,\},\\
\beta_* &\equiv \{\, i \,|\, y_i^* = 0 = G_i(x^*, y^*) \,\},\\
\gamma_* &\equiv \{\, i \,|\, y_i^* = 0 < G_i(x^*, y^*) \,\}.
\end{aligned} \tag{5}
$$

Note that the strong regularity condition in (c2) implies that the Jacobian matrix $J_{y_{\alpha_*}} G_{\alpha_*}(x^*, y^*)$ is a P-matrix and thus is nonsingular [16]. Hece we deduce via the implicit function theorem, $G_{\alpha_*}(x^*, y^*) = 0$, and the continuity of $y(x)$ and $G(x, y)$ near (x^*, y^*) that there exist neighborhoods $\mathcal{N}_0$ of x^*, $\mathcal{V}'_{\alpha_*}$ of $y^*_{\alpha_*}$, and $\mathcal{V}'_{\beta_*}$ of $y^*_{\beta_*} = 0$, and an analytic function $y_{\alpha_*} : \mathcal{N}_0 \times \mathcal{V}'_{\beta_*} \to \mathcal{V}'_{\alpha_*}$ such that for each $(x, y_{\beta_*}) \in \mathcal{N}_0 \times \mathcal{V}'_{\beta_*}$, $y(x, y_{\beta_*}) \equiv \big(y_{\alpha_*}(x, y_{\beta_*}), y_{\beta_*}, y_{\gamma_*}\big)$ with $y_{\gamma_*} \equiv 0$ is the unique solution of the nonlinear complementarity problem (NCP) (4b). Letting $\widehat{G}_{\beta_*}(x, y_{\beta_*}) \equiv G_{\beta_*}(x, y_{\alpha_*}(x, y_{\beta_*}), y_{\beta_*}, 0)$ which remains analytic on $\mathcal{N}_0$, it suffices to consider the reduced NCP for $x \in \mathcal{N}_0$:

$$
0 \le y_{\beta_*} \perp \widehat{G}_{\beta_*}(x, y_{\beta_*}) \ge 0 \tag{6}
$$

Furthermore, the above NCP has a strongly regular solution $y^*_{\beta_*} = 0$ at x^* and thus has a unique solution in $\mathcal{V}'_{\beta_*}$ for all $x \in \mathcal{N}_0$ (possibly by restricting $\mathcal{N}_0$). For notational convenience, let $\widehat{y}$ denote y_{β_*}. For each $x \in \mathcal{N}_0$, there is a unique fundamental index triple $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ for (6) (which forms a partition of β_*) such that the unique solution $\widehat{y}(x) \equiv (\widehat{y}_{\widehat{\alpha}}(x), \widehat{y}_{\widehat{\beta}}(x), \widehat{y}_{\widehat{\gamma}}(x))$ in $\mathcal{V}'_{\beta_*}$ satisfies $\widehat{y}_{\widehat{\beta}}(x) \equiv 0$ and $\widehat{y}_{\widehat{\gamma}}(x) \equiv 0$ for all $x \in \mathcal{N}_0$. It follows from the implicit function theorem that there exists an analytic function $h_{\widehat{\alpha}} : \mathcal{N}_0 \to \mathbb{R}^{|\widehat{\alpha}|}$ such that for each $x \in \mathcal{N}_0$, $h_{\widehat{\alpha}}(x)$ is the unique vector satisfying $\widehat{G}_{\widehat{\alpha}}(x, h_{\widehat{\alpha}}(x), 0, 0) = 0$ and $\widehat{y}_{\widehat{\alpha}}(x) = h_{\widehat{\alpha}}(x)$. For each index triple $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$, define the set $\mathcal{X}_{(\widehat{\alpha},\widehat{\beta},\widehat{\gamma})} \equiv \{x \in \mathcal{N}_0 \,|\, h_{\widehat{\alpha}}(x) > 0,\ \widehat{G}_{\widehat{\alpha}}(x, h_{\widehat{\alpha}}(x), 0, 0) = 0,\ \widehat{G}_{\widehat{\beta}}(x, h_{\widehat{\alpha}}(x), 0, 0) = 0,\ \widehat{G}_{\widehat{\gamma}}(x, h_{\widehat{\alpha}}(x), 0, 0) > 0\}$, which is clearly semianalytic (cf. Lemma 1). By the (local) solution existence and uniqueness of the NCP (6), we see that the collection of the sets $\mathcal{X}_{(\widehat{\alpha},\widehat{\beta},\widehat{\gamma})}$ forms a finite partition of $\mathcal{N}_0$. Moreover, its corresponding y, denoted by $y_{(\widehat{\alpha},\widehat{\beta},\widehat{\gamma})}$, is $y_{(\widehat{\alpha},\widehat{\beta},\widehat{\gamma})}(x) = \big(y_{\alpha_*}(x, \widehat{y}(x)), \widehat{y}(x), 0\big), \forall\ x \in \mathcal{X}_{(\widehat{\alpha},\widehat{\beta},\widehat{\gamma})}$, where $\widehat{y}(x) = (h_{\widehat{\alpha}}(x), 0, 0)$ is an analytic function on $\mathcal{N}_0$. We next consider the mixed complementarity problem (4c-4d). For the given (x^*, y^*, u^*), there exists an index pair $(\theta_*, \overline{\theta}_*)$, where $\overline{\theta}_*$ is the complement of θ_*, such that $\big(H(x^*, y^*) + Eu^*\big)_{\theta_*} = 0$ and $\big(H(x^*, y^*) + Eu^*\big)_{\overline{\theta}_*} > 0$. The continuity of $y(x)$ and $u(x, y)$ shows that for all $x \in \mathcal{N}_0$, $\big(H(x, y(x)) + Eu(x, y(x))\big)_{\overline{\theta}_*} > 0$ such that $\lambda_{\overline{\theta}_*} = 0$. Using local positive definiteness of $N(x, y)$ near (x^*, y^*), we obtain the following reduced complementarity problem

$$0 \leq \lambda_{\theta_*} \perp H_{\theta_*}(x, y) + E_{\theta_* \bullet} u \geq 0 \tag{7}$$

where $u = N^{-1}(x, y)[(E_{\theta_* \bullet})^T \lambda_{\theta_*} - D(x, y)]$. Substituting u into (7) gives rise to a PSD-plus complementarity problem [19]. We thus deduce via the uniqueness of $H_{\theta_*}(x, y) + E_{\theta_* \bullet} u$ that for each $x \in \mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}$, there exists a unique index set $\widehat{\theta} \subseteq \theta_*$ such that $H_{\widehat{\theta}}(x, y) + E_{\widehat{\theta} \bullet} u(x, y) = 0$ and $\big(H(x, y) + Eu(x, y)\big)_{\theta_* \setminus \widehat{\theta}} > 0 = \lambda_{\theta_* \setminus \widehat{\theta}}$. The two latter equations and the expression for u below (7) yield a unique analytic u in term of x, denoted by $u_{\widehat{\theta}}(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x))$ [13, Lemma 1]. For the index tuple $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})$, define the set

$$\begin{aligned} \mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})} \equiv \mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})} \bigcap & \\ \{x \in \mathcal{N}_0 \,|\, \big(H(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x)) + Eu_{\widehat{\theta}}(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x))\big)_{\widehat{\theta}} = 0, & \\ \big(H(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x)) + Eu_{\widehat{\theta}}(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x))\big)_{\theta_* \setminus \widehat{\theta}} > 0\} & \end{aligned}$$

Clearly $\mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})}$ is a semianalytic set and the collection of $\mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})}$ forms a finite partition of $\mathcal{N}_0$. Moreover, for $x \in \mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})}$, $\widetilde{F}(x) = F(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x), u_{\widehat{\theta}}(x, y_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})}(x)))$ is an analytic function on $\mathcal{N}_0$. Hence (H1-H2) hold. $\square$

Let $\big(x(t), y(t), u(t)\big)$ be a trajectory on a time interval $[0, T]$ (where the initial state x^0 is dropped for notational simplicity). Define the following index sets along this trajectory for each $t \in [0, T]$:

$$\begin{aligned} \alpha(t) &\equiv \{\, i \,|\, y_i(t) > 0 = G_i(x(t), y(t)) \,\}, \\ \beta(t) &\equiv \{\, i \,|\, y_i(t) = 0 = G_i(x(t), y(t)) \,\}, \\ \gamma(t) &\equiv \{\, i \,|\, y_i(t) = 0 < G_i(x(t), y(t)) \,\}, \\ \theta(t) &\equiv \{\, j \,|\, \big(H(x(t), y(t)) + Eu(t)\big)_j = 0 \,\}, \\ \zeta(t) &\equiv \{\, j \,|\, \big(H(x(t), y(t)) + Eu(t)\big)_j > 0 \,\}. \end{aligned} \tag{8}$$

We say that $t_* \in (0, T)$ is *not* a switching time along $(x(t), y(t), u(t))$ if there exist $\varepsilon > 0$ and an index tuple $(\alpha', \beta', \gamma', \theta', \zeta')$ such that $\big(\alpha(t), \beta(t), \gamma(t), \theta(t), \zeta(t)\big) = \big(\alpha', \beta', \gamma', \theta', \zeta'\big)$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, we say that the trajectory has a *mode switching/transition* at t_*. It follows from Proposition 4 that t_* is not a switching time if and only if there exist $\varepsilon > 0$ and a set $\mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})}$ defined in Proposition 4 such that $x(t) \in \mathcal{X}_{(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})}, \forall\, t \in [t_* - \varepsilon, t_* + \varepsilon]$ as long as $y(t) \in \mathcal{V}_0, u(t) \in \mathcal{U}_0$ on the time interval, where $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\theta})$ uniquely corresponds to some $(\alpha', \beta', \gamma', \theta', \zeta')$. In view of this and Theorem 3, we have:

Theorem 5 Consider the complementarity system (4) satisfying the conditions (c1-c5). For a real $T > 0$, there exists $N(T) \in \mathbb{N}$ such that for any time interval $I \subseteq [0, T]$, if a trajectory $(x(t), y(t), u(t)) \in \mathcal{N}_0 \times \mathcal{V}_0 \times \mathcal{U}_0$ for all $t \in I$, then the trajectory has at most $N(T)$ mode switchings on I.

This theorem can be easily extended to the case with suitable parameter variations.

III. Global Robust Non-Zenoness of Bimodal Piecewise Affine Systems

We consider a class of piecewise analytic systems, namely, piecewise affine systems. These systems represent a broad class of affine hybrid systems, e.g., affine complementarity systems with singleton properties [19]. A bimodal piecewise affine system is a piecewise affine system with two modes only. The bimodal property considerably reduces analytic complexity while still illustrates nontrivial dynamic behaviors of the piecewise affine systems. This section is devoted to global robust non-Zenoness in the presence of both parameter variations and state perturbations. Particularly, a global uniform bound on the number of switchings is established, under mild boundedness conditions on system parameters. To our best knowledge, Sussmann's result (i.e. Theorem 2) does not directly yield this global bound. Instead, this bound is established by exploiting affine structure of the bimodal system.

The bimodal piecewise affine system is described by

$$\dot{x} \,=\, Ax + d + b \max(0, -c^T x - \gamma) \tag{9}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c, d \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$. We denote this system by the bimodal PAS(A, b, c, d, γ). To avoid triviality, we assume $c \neq 0$ and may further assume $\|c\|_2 = 1$. The bimodal piecewise affine system can be written as

$$\dot{x} = \begin{cases} Ax + d, & x \in \mathcal{X}_1 \equiv \{x \,|\, c^T x + \gamma \geq 0\} \\ (A - bc^T)x + d - \gamma b, & x \in \mathcal{X}_2 \equiv \{x \,|\, c^T x + \gamma \leq 0\} \end{cases}$$

The mode switching is defined with respect to the affine spaces $\mathcal{X}_1$ and $\mathcal{X}_2$. Our main non-Zeno result asserts that as long as the A and $A - bc^T$ are bounded, there is a uniform bound on the number of switchings on a given time interval, regardless of initial states, d and γ. Specifically, we have:

Theorem 6 Let $\big\{(A_\alpha, b_\alpha, c_\alpha, d_\alpha, \gamma_\alpha)\big\}$ be a family of indexed matrix, vector and real number tuples. Suppose that a positive real ρ exists such that $\|A_\alpha\| \leq \rho$ and $\|A_\alpha - b_\alpha c_\alpha^T\| \leq \rho$ for all α. Then for any $T > 0$, there exists $N(T, \rho) \in \mathbb{N}$ such that for any α and any $x^0 \in \mathbb{R}^n$, the trajectory $x(t, x^0)$ of the bimodal PAS$(A_\alpha, b_\alpha, c_\alpha, d_\alpha, \gamma_\alpha)$ has at most $N(T, \rho)$ mode switchings on $[0, T]$.

To prove this theorem, we first consider a special case with $d = 0$ and $\gamma = 0$. In this case, the bimodal piecewise affine system becomes a bimodal conewise linear system (denoted by CLS(A, b, c)). We shall show that Theorem 6 holds for the bimodal CLS and then turn to the general case with possibly nonzero d and/or γ. To reach this goal, we present several technical lemmas as follows.

Lemma 7 Under the specified boundedness condition on A_α and $A_\alpha - b_\alpha c_\alpha^T$, there exists $\rho_1 > 0$ such that $\big|c_\alpha^T A_\alpha^k b_\alpha\big| \leq \rho_1$ for all α and each $k = 0, 1, \cdots, n-1$.

Proof. For each $k \in \{0, 1, \cdots, n-1\}$, note that $c_\alpha^T A_\alpha^k (A_\alpha - b_\alpha c_\alpha^T) = c_\alpha^T A_\alpha^{k+1} - c_\alpha^T A_\alpha^k b_\alpha c_\alpha^T$. Hence $\big|c_\alpha^T A_\alpha^k b_\alpha\big| \|c_\alpha\| \leq$

$\|c_\alpha^T A_\alpha^{k+1}\| + \|c_\alpha^T A_\alpha^k (A_\alpha - b_\alpha c_\alpha^T)\|$. In view of the boundedness of A_α and $(A_\alpha - b_\alpha c_\alpha^T)$ as well as $\|c_\alpha\| = 1$, the lemma holds true. □

A trajectory $x(t, x^0)$ is generally only known to be once differentiable with respect to t due to possible switching at t. In spite of this, the next lemma states a combinatorial property of zeros on the boundary of two half spaces.

Lemma 8 Let $x(t, x^0)$ be a trajectory of a bimodal CLS(A, b, c), and $[t_1, t_2]$ be a time interval. If $c^T x(t, x^0)$ has $\prod_{i=1}^k (2^{i-1} + 1)$ zeros on $[t_1, t_2]$ for some $k \in \mathbb{N}$, then there exist $t_* \in (t_1, t_2)$ and $S_i \in \{A, A - bc^T\}$, $i = 1, \cdots, k$ such that $c^T \{ \prod_{i=1}^k S_i \} x(t_*, x^0) = 0$.

Proof. We prove the lemma by induction on k. Consider $k = 1$. Since $c^T x(t, x^0)$ has two zeros on $[t_1, t_2]$ and $c^T x(t, x^0)$ is (time) differentiable, there is $t_* \in (t_1, t_2)$ such that $c^T \dot{x}(t_*, x^0) = 0$ by the mean-value theorem. Thus we have $c^T A x(t_*, x^0) = 0$ or $c^T (A - bc^T) x(t_*, x^0) = 0$. Hence the lemma holds. Now suppose the lemma holds true for all $k = 1, \cdots, \ell$, where $\ell \in \mathbb{N}$. Consider $k = \ell + 1$. Without loss of generality, we assume that the zeros $\tau_i \in [t_1, t_2]$, $i = 1, \cdots, \prod_{i=1}^{\ell+1} (2^{i-1} + 1)$, of $c^T x(t, x^0)$ are in the (strictly) increasing order. For notational simplicity, let $m \equiv \prod_{i=1}^{\ell} (2^{i-1} + 1)$. Define the time interval $I_j \equiv [\tau_{1+(j-1)m}, \tau_{j\,m}] \subset [t_1, t_2]$, where $j = 1, \cdots, 2^\ell + 1$. Hence $c^T x(t, x^0)$ has m zeros on each I_j. It follows from the induction hypothesis that for each j, there exist $\widetilde{\tau}_j \in \mathrm{int} I_j$ and $S_{j\,i} \in \{A, A - bc^T\}$ such that $c^T \{ \prod_{i=1}^{\ell} S_{j\,i} \} x(\widetilde{\tau}_j, x^0) = 0$, where all the $\widetilde{\tau}_j$'s are distinct. Since the product string $\{ \prod_{i=1}^{\ell} S_{j\,i} \}$ has 2^ℓ combinations but $j = 1, \cdots, 2^\ell + 1$, there must be two identical product strings, denoted by $\{ \prod_{i=1}^{\ell} \bar{S}_i \}$. Therefore, $c^T \{ \prod_{i=1}^{\ell} \bar{S}_i \} x(t, x^0)$ has two distinct zeros on $[t_1, t_2]$. Using the mean-value theorem again, we obtain $t_* \in (t_1, t_2)$ and $\bar{S}_{\ell+1} \in \{A, A - bc^T\}$ such that $c^T \{ \prod_{i=1}^{\ell} \bar{S}_i \} \bar{S}_{\ell+1} x(t_*, x^0) = 0$. This show that the lemma holds for $k = \ell + 1$, and thus for all $k \in \mathbb{N}$. □

The following lemma is due to Sussmann [21]:

Lemma 9 Let $\kappa > 0$ and $n \in \mathbb{N}$. Let $\Delta T > 0$ be such that $\Delta T < \min\Big(1, \dfrac{e^{-\kappa n}}{n^{\frac{3}{2}} \kappa}\Big)$. If $\phi_1(t), \cdots, \phi_n(t)$ are absolutely continuous functions on a time interval I of length ΔT that satisfy a linear system of differential equations: $\dot{\phi}_i(t) = \sum_{j=1}^n \alpha_{ij}(t)\, \phi_j(t)$, $i = 1, \cdots, n$, where the coefficients $\alpha_{ij}(t)$ are measurable real-valued functions on I such that $|\alpha_{ij}(t)| \le \kappa$ for all $1 \le i, j \le n$ and all $t \in I$, then either (i) all the $\phi_i(t)$ vanish identically on I, or (ii) at least one of $\phi_i(t)$ has no zeros on I.

With these technical lemmas in hand, we are ready to prove Theorem 6 for the CLS case:

Proof of Theorem 6 with $d_\alpha = 0$ **and** $\gamma_\alpha = 0$**.** It suffices to show that there exist a time length $\varepsilon_T > 0$ and $N \in \mathbb{N}$ such that the number of mode transitions in any time interval of length less than ε_T along a trajectory of a bimodal CLS with the bounded parameter variations does not exceed N. It is clear that if this is true, then on any given time interval $[0, T]$ with $T > 0$, there are at most $[\frac{T}{\varepsilon_T} + 1] N$ mode transitions.

For a fixed α, let

$$\begin{aligned} q &= \begin{pmatrix} q_1, & q_2, & \cdots, & q_n \end{pmatrix}^T \\ &\equiv \begin{pmatrix} c_\alpha^T x, & c_\alpha^T A_\alpha x, & \cdots, & c_\alpha^T A_\alpha^{n-1} x \end{pmatrix}^T. \end{aligned}$$

Hence, the bimodal CLS becomes

$$\dot{q} = \widetilde{A}_\alpha\, q + \widetilde{b}_\alpha \max(0, -q_1),$$

where

$$\widetilde{A}_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -a_{\alpha,0} & -a_{\alpha,1} & \cdots & \cdots & -a_{\alpha,n-2} & -a_{\alpha,n-1} \end{bmatrix}$$

and $\widetilde{b}_\alpha = \begin{pmatrix} c_\alpha^T b_\alpha, & c_\alpha^T A_\alpha b_\alpha, & \cdots, & c_\alpha^T A_\alpha^{n-1} b_\alpha \end{pmatrix}^T$. Here the real numbers $a_{\alpha,j}$ satisfy $\det(\lambda I - A_\alpha) = \lambda^n + \sum_{j=0}^{n-1} a_{\alpha,j} \lambda^j$. Since $a_{\alpha,j}$'s are continuous in A_α and A_α's are bounded, so are $a_{\alpha,j}$'s for all j. It also follows from Lemma 7 that $\widetilde{b}_\alpha$'s are bounded as well. For the given A_α and $A_\alpha - b_\alpha c_\alpha^T$, define the matrix tuple $\psi = \big(I, S_{11}, S_{21} S_{22}, \cdots, \prod_{j=1}^{n-1} S_{(n-1)j}\big)$, where each $S_{ij} \in \{A_\alpha, A_\alpha - b_\alpha c_\alpha^T\}$. Hence, there are 2^n such tuples. Associated with each tuple ψ, define the n-vector q^ψ as $q_1^\psi \equiv q_1$ and $q_k^\psi \equiv c^T \{\prod_{j=1}^{k-1} S_{(k-1)j}\}\, x$ for $k = 2, \cdots, n$. It is easy to see that for each q^ψ, we have

$$q^\psi = \begin{pmatrix} q_1^\psi \\ q_2^\psi \\ \vdots \\ q_n^\psi \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ \star & 1 & & \\ & & \ddots & \\ \star & \cdots & \star & 1 \end{bmatrix}}_{P_\alpha^\psi} q,$$

where $\star$ denotes the elements that are either zero or are finite products of $c_\alpha^T A_\alpha^j b_\alpha$ for $j = 0, 1, \cdots, n-1$. Therefore, these elements are bounded (cf. Lemma 7). Since P^ψ is invertible, the bimodal CLS$(A_\alpha, b_\alpha, c_\alpha)$ is written in term of q^ψ as

$$\dot{q^\psi} = \widetilde{A}_\alpha^\psi q^\psi + \widetilde{b}_\alpha^\psi \max(0, -q_1^\psi), \tag{10}$$

where $\widetilde{A}_\alpha^\psi = P_\alpha^\psi \widetilde{A}_\alpha (P_\alpha^\psi)^{-1}$ and $\widetilde{b}_\alpha^\psi = P_\alpha^\psi \widetilde{b}_\alpha$. Let $\widetilde{c} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$ and define

$$\kappa \equiv \sup_{\alpha, \psi} \Big(\|\widetilde{A}_\alpha^\psi\|, \ \|\widetilde{A}_\alpha^\psi - \widetilde{b}_\alpha^\psi \widetilde{c}^T\| \Big). \tag{11}$$

Choose $\varepsilon_T \in \Big(0, \min\big(1, \frac{e^{-\kappa n}}{n^{\frac{3}{2}} \kappa}\big)\Big)$. Letting I be a time interval of length less than ε_T, we claim that for any α and any trajectory $x(t, x^0)$ of the bimodal CLS$(A_\alpha, b_\alpha, c_\alpha)$, either $c_\alpha^T x(t, x^0)$ is identically zero on I or $c_\alpha^T x(t, x^0)$ has at most $N \equiv \prod_{i=1}^{n-1} (2^{i-1} + 1) - 1$ zeros on I. To prove the claim, suppose that $c_\alpha^T x(t, x^0)$ has $\prod_{i=1}^{n-1} (2^{i-1} + 1)$ zeros on I for some time interval I and a trajectory $x(t, x^0)$ corresponding to some α. It follows from Lemma 8 that there exists a matrix tuple ψ such that each element of q^ψ has a zero on I. Note that $q^\psi(t)$ satisfies the piecewise linear dynamics (10). Moreover, since the trajectory $q^{\psi}(t)$ is piecewise linear (in t) [7], the coefficients on the right hand side of (10) are piecewise constant and measurable (with respect to t) on I. In view of Lemma 9, we deduce that $q^{\psi}(t)$ is identically zero on I. This shows that $q_1^{\psi}(t) \equiv c_\alpha^T x(t,x^0)$ is identically zero on I. This completes the proof of the claim. Note that if $c_\alpha^T x(t,x^0) \equiv 0$ on I, then there is no switching on I. By observing that any switching time corresponds to a zero of $c_\alpha^T(t,x^0)$, we see that $x(t,x^0)$ has at most N mode switchings on I. □

Remark: For a given $T > 0$, a global uniform bound on the number of switchings can be determined as $\big[2T/\min(1, e^{-\kappa n}/(n^{3/2}\kappa)) + 1\big] \prod_{i=1}^{n-1}(2^{i-1}+1)$, where κ is given in (11).

Finally, we prove Theorem 6 in its general form.

Proof of Theorem 6 (with general d_α and γ_α). Given a bimodal piecewise affine system $\dot{x} = Ax + d + b\max(0, -c^T x - \gamma)$, define $y \equiv x + \gamma c$. Hence, the bimodal PAS(A,b,c,d,γ) is equivalent to

$$\dot{y} = Ay + \widehat{d} + b\max(0, -c^T y), \tag{12}$$

where $\widehat{d} = d - \gamma Ac$. Furthermore, define $z \equiv (y^T, \widehat{d}^T)^T \in \mathbb{R}^{2n}$. Then the bimodal piecewise affine system (12) is equivalent to the bimodal CLS

$$\dot{z} = \widehat{A}z + \widehat{b}\max(0, -\widehat{c}^T z)$$

where $\widehat{A} = \begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}$, $\widehat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$, $\widehat{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}$. It is easy to verify that if $\|A\|$ and $\|A - bc^T\|$ are bounded, so are $\|\widehat{A}\|$ and $\|\widehat{A} - \widehat{b}\widehat{c}^T\|$. Since the two systems have the same definition of switchings, the uniform bound on the number of mode switchings of the indexed bimodal piecewise affine systems on $[0,T]$ follows from that of the bimodal CLSs established before, regardless of initial states, d and γ. □

The next example shows that if the bound on $\|A\|$ and $\|A - bc^T\|$ is dropped, then robust non-Zenoness may fail.

Example 10 Consider a planar bimodal CLS(A,b,c), i.e. $n=2$. Let A and $A - bc^T$ have complex eigenvalues $\mu_1 \pm \imath\omega_1$ and $\mu_2 \pm \imath\omega_2$ respectively, where $\omega_1 > 0$ and $\omega_2 > 0$. It is shown in [6] that if $\frac{\mu_1}{\omega_1} + \frac{\mu_2}{\omega_2} = 0$, then the CLS has a periodic solution from any nonzero initial state with the constant period $\frac{\pi}{\omega_1} + \frac{\pi}{\omega_2}$. Let $A = \begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The eigenvalues of A and $A - bc^T$ are $\pm\imath\omega_1$ and $\pm\imath\sqrt{\omega_1(\omega_1+1)}$ respectively. Clearly for a fixed $T > 0$, the number of mode switchings along a trajectory from a nonzero initial state is roughly proportional to ω_1 for all ω_1 sufficiently large. Consequently, the uniform bound on the number of mode switchings does not exist if $\|A\|$ is unbounded as $\omega_1 \to \infty$. □

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