Robust Non-Zenoness of Piecewise Affine Systems with Applications to Linear Complementarity Systems

Jinglai Shen*

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Abstract

Piecewise affine systems (PASs) constitute an important class of nonsmooth switching dynamical systems subject to state dependent mode transitions arising from control and dynamic optimization. A fundamental issue in dynamics analysis of switching systems pertains to the possible occurrence of infinitely many switchings in finite time, referred to as the Zeno behavior. There has been a growing interest in characterization of Zeno free switching systems. Different from the recent non-Zeno analysis of switching systems, the present paper studies non-Zeno properties of PASs subject to system parameter and/or initial state perturbations, inspired by sensitivity and uncertainty analysis of PASs. Specifically, by exploiting the geometry of polyhedral subdivisions and dynamical system techniques, this paper establishes a uniform bound on the number of mode switchings for a family of Lipschitz PASs under mild uniform conditions on system parameters and associated polyhedral subdivisions. This result is employed to show robust non-Zenoness of several classes of Lipschitz linear complementarity systems in different switching notions. The paper also develops partial results for robust non-Zenoness of non-Lipschitz PASs, particularly well-posed bimodal non-Lipschitz PASs.

1 Introduction

Nonsmooth switching dynamical systems have received tremendous interest in the last few decades, motivated by analysis, computation, and design of complex systems with multi-modal and hierarchical structure in diverse fields, e.g., contact mechanics, control engineering, operations research, and systems biology. An intricate phenomenon in switching systems subject to state dependent mode switchings is the possible occurrence of infinitely many switchings in finite time, which is referred to as Zeno behavior or Zenoness. Typical examples of Zeno systems are the bouncing ball in contact mechanics and switched engineering systems [18]. Earlier references on the Zeno behavior include [35, 41] in the control field. Recent research focuses on characterization of the Zeno behavior near a Zeno equilibrium using hybrid Lyapunov stability theory [13] and homogeneous or symmetry techniques [14, 27]; see [1, 19, 20] for more results.

Distinct from the Zeno analysis, another important research line is characterization of Zeno free (or non-Zeno) switching systems. The non-Zeno property turns out to be critical to various analytical and numerical issues of finite-time switching dynamics, including scientific computing [18], numerical analysis [10], systems and control analysis [6, 30, 31], and sensitivity analysis [28]. Due to the importance of non-Zenoness, non-Zeno analysis of switching systems has received considerable attention. In the context of constrained optimal control, Brunovsky has exploited some

*Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21250, U.S.A. Email: shenj@umbc.edu. This research was supported by the NSF grants ECCS-0900960 and DMS-1042916.

ODE techniques to establish a bound on the number of switchings of a continuous piecewise linear system [4]. Sussmann extends this result to a continuous piecewise analytic system via advanced algebraic arguments [38]. Similar issues are addressed for linear optimal control problems [24, 25]. Recent effort has been made toward characterizing certain important classes of non-Zeno systems, e.g., complementarity systems and differential variational inequalities (DVIs) that demonstrate inherent nonsmooth and switching behaviors [3, 16, 23, 21]. For a complementarity system with a continuous right-hand side, the paper [32] develops a local solution expansion that yields mode invariance, thus non-Zenoness, at a state. This technique and its extension is applied to the linear complementarity systems (LCSs) with P-property [32] and singleton properties [33], strongly regular nonlinear complementarity systems [22], differential quasi-variational inequalities [15], and Lipschitz piecewise linear/affine systems [6, 39]. Furthermore, Çamlibel shows that a well-posed bimodal piecewise linear/affine system with a discontinuous right-hand side is non-Zeno [8, 40].

The non-Zenoness of a switching system asserts that there exists a (system parameter dependent) bound on the number of switchings in finite time along a trajectory for a given initial state. In sensitivity and uncertainty analysis, however, system parameters and/or an initial state may be unknown. This gives rise to an infinite family of non-Zeno switching systems. The following question naturally arises in uncertainty analysis of switching systems:

Q: Does there exist a uniform bound on the number of mode switchings in a given finite time interval along a trajectory of each switching system from the given family?

If so, we call the family of switching systems *robust non-Zeno*. Despite its fundamental importance, the non-Zeno results in the literature cannot address robust non-Zenoness since they do not take parameter uncertainties into account, although the methods developed by Brunovsky and Sussmann allow initial state variations. In particular, the solution expansion based techniques rely on a local argument, and they do not give an explicit bound on the number of switchings. An exception is robust non-Zenoness of bimodal Lipschitz PASs [29]. However, the extension to general PASs is hindered by the complexity of multiple modes and associated polyhedral geometry.

The goal of the present paper is to give a positive answer to the question **Q** for general Lipschitz PASs, well-posed bimodal non-Lipschitz PASs, and several classes of Lipschitz linear complementarity systems under suitable conditions. In fact, the paper shows an even stronger result by establishing a uniform bound on the number of critical times (cf. Defintion 2.2) along any trajectory of the considered PASs, e.g., Theorem 2.2. Roughly speaking, a critical time of a state trajectory corresponds to a state (i.e., a critical state) such that a small perturbation of the trajectory will cause index (or mode) change near that state [28]. A switching time is a critical time but not vice versa. Clearly, the concept of critical time plays a key role in sensitivity analysis of the PASs (cf. [28]). In this paper, global techniques are employed for robust non-Zeno analysis of PASs, and major contributions of the paper are summarized as follows.

1. Lipschitz PASs. A Lipschitz PAS has a unique (time-)continuously differentiable trajectory for any initial state. In spite of this nice feature, a state trajectory is at best once time differentiable due to a nonsmooth right-hand side. In addition, the presence of multiple (more than two) modes complicates algebraic relations between system matrices. These difficulties as well as parametric and initial state uncertainties pose many challenges in determining a desired uniform bound. To overcome these difficulties, we invoke a technical result by Sussmann that allows us to handle parameter uncertainties; see [38, Lemma A.1] (or Lemma 2.6 of the paper). In order to apply this result, however, it is essential to establish an algebraic relation between system matrices of different modes along a trajectory. In turn, this requires a deep understanding of polyhedral subdivision of a PAS [26]. By exploiting the geometry of polyhedral subdivision, we obtain a useful algebraic result for system matrices (cf. Theorem 2.1). (It is noted that Brunovsky uses a similar result in [4]

but without proof. Nonetheless, the proof of this result is nontrivial, since it heavily relies on the geometry of a polyhedral subdivision (cf. Proposition 2.1), which is overlooked in [4].) Moreover, to deal with a nonsmooth trajectory, we exploit combinatorial arguments (cf. Lemma 2.5) to obtain an approximation of higher order derivatives of a trajectory. These results, along with dynamical systems techniques, pave a way for a uniform bound on the number of critical times. These robust non-Zeno results are then applied to Lipschitz linear complementarity systems.

2. Non-Lipschitz PASs. A non-Lipschitz PAS has a discontinuous right-hand side and can be described by a differential inclusion. For such a PAS, there are multiple solution concepts, and each solution is absolutely continuous and is not (time-)differentiable everywhere. Moreover, well-posedness (i.e., solution existence and uniqueness) becomes a crucial issue; see more in Section 3. Due to the lack of solution uniqueness results for a general non-Lipschitz PAS, we focus on bimodal non-Lipschitz PASs whose well-posedness is established by Çamlibel et al. [8, 40]. Applying the well-posed conditions in [40] and a technical lemma of Sussmann [37], we show robust non-Zenoness of a family of well-posed bimodal non-Lipschitz PASs under mild conditions on their system matrices.

The organization of the paper is as follows. Section 2 is devoted to robust non-Zeno analysis of Lipschitz PASs, and Section 3 to that of non-Lipschitz PASs, particularly bimodal non-Lipschitz PASs. Section 4 extends this analysis to Lipschitz linear complementarity systems under different switching concepts. Finally, concluding remarks are given in Section 5.

2 Robust Non-Zenoness of Lipschitz Piecewise Affine Systems

This section concentrates on robust non-Zeno analysis of Lipschitz PASs.

2.1 Lipschitz Piecewise Affine Function and Its Geometry

A continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise affine (PA) if there exists a finite family of affine functions $\{f_i\}_{i=1}^{\ell}$ such that $f(x) \in \{f_i(x)\}_{i=1}^{\ell}$ for each $x \in \mathbb{R}^n$ [11, 26]. A PA function is globally Lipschitz, and we thus call it a *Lipschitz PA* function in order to distinguish it from a discontinuous PA mapping to be discussed later. A Lipschitz PA function possesses an appealing geometric structure for its domain, which provides an alternative representation for the function. To elaborate this, we review basic notions and properties of face lattice of a polyhedron as follows.

Consider the polyhedron $\mathcal{X}_i := \{\, x \in \mathbb{R}^n \,:\, C_i x \,\geq\, h_i \,\}$ for a matrix $C_i \in \mathbb{R}^{m_i \times n}$ and a vector $h_i \in \mathbb{R}^{m_i}$. Define the family of index sets for $\mathcal{X}_i$:

$$\aleph(C_i, h_i) := \big\{\alpha \subseteq \{1, \ldots, m_i\} \,:\, \text{there exists } z \in \mathbb{R}^n \text{ such that } (C_i z - h_i)_\alpha = 0 \text{ and } (C_i z - h_i)_{\bar{\alpha}} > 0\big\},$$

where $\bar{\alpha}$ denotes the complement of α. A *face* of $\mathcal{X}_i$ is given by

$$\mathcal{F}_{i,\alpha} := \big\{x \in \mathbb{R}^n \,:\, (C_i x - h_i)_\alpha = 0, \text{ and } (C_i x - h_i)_{\bar{\alpha}} \geq 0\big\} \tag{1}$$

for some nonempty index set $\alpha \in \aleph(C_i, h_i)$ [26, Proposition 2.1.3]. If α is singleton, e.g., $\alpha = \{s\}$, we simply write $\mathcal{F}_{i,\alpha}$ as $\mathcal{F}_{i,s}$. A face of $\mathcal{X}_i$ is called *proper* if it does not coincide with $\mathcal{X}_i$. Moreover, the relative interior of the face $\mathcal{F}_{i,\alpha}$ is $\{x \in \mathbb{R}^n : (C_i x - h_i)_\alpha = 0, \text{ and } (C_i x - h_i)_{\bar{\alpha}} > 0\}$ (cf. [26, Proposition 2.1.3]). The dimension of the face $\mathcal{F}_{i,\alpha}$ of $\mathcal{X}_i$ is the dimension of the subspace $\{x \in \mathbb{R}^n : (C_i x)_\alpha = 0\}$. We call an $(n-1)$-dimensional face of $\mathcal{X}_i$ a *facet* of $\mathcal{X}_i$ and a zero dimensional face an *extremal point* of $\mathcal{X}_i$. Since two faces $F_{i,\alpha}$ and $F_{i,\beta}$ of $\mathcal{X}_i$ associated with different index sets $\alpha, \beta \in \aleph(C_i, h_i)$ are distinct [26, Proposition 2.1.3], each point in $\mathcal{X}_i$ is either in the interior of $\mathcal{X}_i$ or in the relative interior of a unique q-dimensional face of $\mathcal{X}_i$, where $q \in \{0, 1, \ldots, n-1\}$.

Let Ξ be a finite family of polyhedra $\{\mathcal{X}_i\}_{i=1}^{m}$, where each $\mathcal{X}_i := \{\, x \in \mathbb{R}^n \,:\, C_i x \,\geq\, h_i \,\}$. We call Ξ a *polyhedral subdivision* of $\mathbb{R}^n$ [11, 26] if

(a) the union of all polyhedra in Ξ is equal to $\mathbb{R}^n$, i.e., $\bigcup_{i=1}^m \mathcal{X}_i = \mathbb{R}^n$;

(b) each polyhedron in Ξ has a nonempty interior (thus is of dimension n); and

(c) the intersection of any two polyhedra in Ξ is either empty or a common proper face of both polyhedra, i.e., if $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$, then $\mathcal{X}_i \cap \mathcal{X}_j = \mathcal{X}_i \cap \{x : (C_i x - h_i)_\alpha = 0\} = \mathcal{X}_j \cap \{x : (C_j x - h_j)_\beta = 0\}$ for nonempty index sets α and β with $\{x \in \mathcal{X}_i : (C_i x - h_i)_\alpha = 0\} \neq \mathcal{X}_i$ and $\{x \in \mathcal{X}_j : (C_j x - h_j)_\beta = 0\} \neq \mathcal{X}_j$.

If each polyhedron in Ξ is a cone, then Ξ is called a *conic subdivision*. See Figure 1 for illustration of a polyhedral subdivision. For a Lipschitz PA function f, one can always find a polyhedral subdivision of $\mathbb{R}^n$ and finitely many affine functions $g_i(x) := A_i x + d_i$ such that f coincides with one of g_i's on each polyhedron in Ξ [11, Proposition 4.2.1]. Therefore, an alternative representation of the Lipschitz PA f is $f(x) = A_i x + d_i$, $\forall\, x \in \mathcal{X}_i$, and $x \in \mathcal{X}_i \cap \mathcal{X}_j \Longrightarrow A_i x + d_i = A_j x + d_j$. Similarly, a Lipschitz piecewise linear system admits a conic subdivision.

Since each $\mathcal{X}_i$ in Ξ has nonempty interior, we assume, without loss of generality, that each row of C_i corresponds to an $(n-1)$-dim face of $\mathcal{X}_i$, namely, a facet of $\mathcal{X}_i$. In other words, for each s, $\mathcal{F}_{i,s} := \{x \in \mathcal{X}_i : (C_i x - h_i)_s = 0\}$ is a polyhedral set of dimension $(n-1)$, and its relative interior is given by $\{x \in \mathbb{R}^n : (C_i x - h_i)_s = 0,\ (C_i x - h_i)_\ell > 0, \forall\, \ell \neq s\}$. Since $\mathcal{F}_{i,s}$ and $\mathcal{F}_{i,w}$ are distinct whenever $s \neq w$ (cf. [26, Proposition 2.1.3]), each $\mathcal{F}_{i,\ell}$ corresponds to a unique facet of $\mathcal{X}_i$, and each face of $\mathcal{X}_i$ is either a facet or an intersection of some facets of $\mathcal{X}_i$. We assume throughout this paper that the vector norm on $\mathbb{R}^n$ is the Euclidean norm $\|\cdot\|_2$ and the matrix norm $\|\cdot\|_2$ is induced by the Euclidean norm. Moreover, we assume that each vector $(C_i)_{s\bullet}^T$ is unit, i.e. $\|(C_i)_{s\bullet}^T\|_2 = 1$. Hence, each $(C_i)_{s\bullet}^T$ can be viewed as a unit normal vector of the facet $\mathcal{F}_{i,s}$.

A highly instrumental result in dealing with multiple modes in non-Zeno analysis of general PASs is Proposition 2.1 and Theorem 2.1. Roughly speaking, this result states that if two polyhedra are (path-)connected through facets, then the difference of their corresponding system matrices (i.e., A_i's) is the sum of rank-one matrices, each of which is an outer product of a vector and the normal vector of a facet that a connecting path passes through. While this result is geometrically intuitive and is implicitly treated as a fact without proof in [4], its rigorous argument is nontrivial since two neighboring polyhedra may intersect on a face of an arbitrary dimension, instead of a facet of dimension $(n-1)$. To establish this result, we need to exploit the geometry of polyhedral subdivision. Toward this end, we present a few technical lemmas as follows.

The first lemma states that each facet $\mathcal{F}_{i,j}$ is the intersection of $\mathcal{X}_i$ and a unique polyhedron. (Note that the definition of polyhedral subdivision does not specify the uniqueness.) A by-product is that a conic subdivision has two modes only if a facet of one of its polyhedral cones is a hyperplane.

We introduce more notation. Let $\{\mathcal{X}_i\}_{i=1}^p$ be a collection of polyhedra in $\mathbb{R}^n$ (not necessarily to form a polyhedral subdivision). For a given $x^* \in \mathbb{R}^n$, define the index set $\mathcal{I}(x^*) := \{i : x^* \in \mathcal{X}_i\} \subseteq \{1, \ldots, p\}$. Moreover, for each $i \in \mathcal{I}(x^*)$, define $\mathcal{L}(x^*, i) := \{s : (C_i x^* - h_i)_s = 0\}$.

Lemma 2.1. *Consider a polyhedral subdivision $\Xi = \{\mathcal{X}_i\}_{i=1}^m$ of $\mathbb{R}^n$. The following hold:*

(1) Given a facet $\mathcal{F}_{i,s}$ of $\mathcal{X}_i$, there exist a unique polyhedron $\mathcal{X}_j \in \Xi$ with $j \neq i$ and a unique facet $\mathcal{F}_{j,w}$ of $\mathcal{X}_j$ such that $\mathcal{X}_i \cap \mathcal{X}_j = \mathcal{F}_{i,s} = \mathcal{F}_{j,w}$.

(2) Ξ is a bimodal conic subdivision of $\mathbb{R}^n$ if and only if a facet of a polyhedral cone in Ξ is the hyperplane $\{x \in \mathbb{R}^n : c^T x = 0\}$ for a unit vector c.

(3) Given x^* and $x' \in \mathbb{R}^n$. If $\mathcal{I}(x^*) \subseteq \mathcal{I}(x')$, then for each $i \in \mathcal{I}(x^*)$, $\mathcal{L}(x^*, i) \subseteq \mathcal{L}(x', i)$. Furthermore, if $\mathcal{I}(x^*) = \mathcal{I}(x')$, then $\mathcal{L}(x^*, i) = \mathcal{L}(x', i)$ for each $i \in \mathcal{I}(x^*)$.

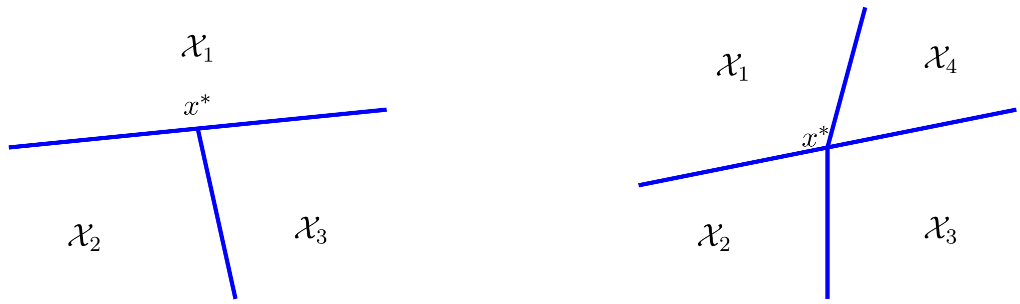

Figure 1: Left: a polyhedral partition of $\mathbb{R}^2$ but not a polyhedral subdivision; Right: a valid polyhedral subdivision of $\mathbb{R}^2$.

Proof. (1) We first show that for a given facet $\mathcal{F}_{i,s}$, there exists a unique $\mathcal{X}_j \in \Xi$ such that $\mathcal{X}_j \cap \mathcal{X}_i = \mathcal{F}_{i,s}$. To show the existence, consider x' in the relative interior of $\mathcal{F}_{i,s}$. Clearly, x' is on the boundary of $\mathcal{X}_i$. Hence, there exists $\mathcal{X}_j$ with $j \neq i$ such that $x' \in \mathcal{X}_i \cap \mathcal{X}_j$. By (c) of the definition of polyhedral subdivision, there exists an index set α such that $\mathcal{X}_i \cap \mathcal{X}_j = \{x \in \mathcal{X}_i : (C_i x - h_i)_\alpha = 0\}$. Since x' is in the relative interior of $\mathcal{F}_{i,s}$, we have $(C_i x' - h_i)_s = 0$ and $(C_i x' - h_i)_\ell > 0$ for all $\ell \neq s$. This shows that $\alpha = \{s\}$. In other words, $\mathcal{X}_i \cap \mathcal{X}_j = \mathcal{F}_{i,s}$.

We show the uniqueness of $\mathcal{X}_j$ via contradiction. Suppose not. Then there exists another $\mathcal{X}_k \in \Xi$ with $k \neq i$ and $k \neq j$ such that $\mathcal{X}_i \cap \mathcal{X}_k = \mathcal{F}_{i,s}$. Without loss of generality, we label $\mathcal{F}_{i,s}$ as the first facets of $\mathcal{X}_i$, $\mathcal{X}_j$ and $\mathcal{X}_k$ respectively, i.e., $\mathcal{F}_{i,s} = \{x \in \mathcal{X}_i : (C_i x - h_i)_1 = 0\} = \{x \in \mathcal{X}_j : (C_j x - h_j)_1 = 0\} = \{x \in \mathcal{X}_k : (C_k x - h_k)_1 = 0\}$. Since $\mathcal{F}_{i,s}$ is of dimension $(n-1)$, there exists a unit normal vector $c \in \mathbb{R}^n$ such that $(C_j)_{1\bullet} = (C_k)_{1\bullet} = c^T$ and $(C_i)_{1\bullet} = -c^T$. Let x^* be in the relative interior of $\mathcal{F}_{i,s}$. Therefore, $(C_j x^* - h_j)_p > 0$ for all $p \neq 1$ and $(C_k x^* - h_k)_q > 0$ for all $q \neq 1$. This shows that there exists a real number $\varepsilon > 0$ such that $C_j(x^* + \varepsilon c) - h_j > 0$ and $C_k(x^* + \varepsilon c) - h_k > 0$. Hence $\mathcal{X}_j \cap \mathcal{X}_k$ has nonempty interior, a contradiction to the definition of polyhedral subdivision.

Let $\mathcal{X}_j$ be the unique polyhedron that intersects $\mathcal{X}_i$ on $\mathcal{F}_{i,s}$. We show that there exists a unique facet $\mathcal{F}_{j,w}$ such that $\mathcal{X}_i \cap \mathcal{X}_j = \mathcal{F}_{j,w}$. Let an index set β be such that $\{x \in \mathcal{X}_j : (C_j x - h_j)_\beta = 0\} = \mathcal{F}_{i,s} = \mathcal{X}_i \cap \mathcal{X}_j$. Since $\mathcal{F}_{i,s}$ is of dimension $(n-1)$, β is singleton (otherwise, $\{x \in \mathcal{X}_j : (C_j x - h_j)_\beta = 0\}$ has dimension strictly less than $(n-1)$). This shows that $\mathcal{F}_{i,s}$ is also a facet of $\mathcal{X}_j$. Since all the facets of $\mathcal{X}_j$ are distinct, there must be a unique one denoted by $\mathcal{F}_{j,w}$ that yields $\mathcal{F}_{i,s}$.

(2) The "only if" part follows directly from [6, Example 2.1]. To see the "if" part, consider a conic subdivision Ξ, and let a facet of $\mathcal{X}_i \in \Xi$ be the hyperplane $\mathcal{F} := \{x \in \mathbb{R}^n : c^T x = 0\}$. By (c) of the definition of polyhedral subdivision, $\mathcal{F} = \mathcal{X}_i \cap \{x \in \mathbb{R}^n : (C_i x)_\alpha = 0\}$ for a nonempty index set α. Since $\mathcal{F}$ has dimension $(n-1)$, α is singleton. Furthermore, $\mathcal{X}_i$ must be defined by a single linear inequality, since otherwise, $\mathcal{X}_i \cap \mathcal{F}$ would be a proper subset of $\mathcal{F}$. Hence, it is easy to show $\mathcal{X}_i = \{x \in \mathbb{R}^n : c^T x \geq 0\}$ without loss of generality. By statement (1), there exists a unique $\mathcal{X}_j$ with $j \neq i$ such that $\mathcal{X}_j \cap \mathcal{X}_i = \mathcal{F}$, i.e., $\mathcal{F}$ is a facet of $\mathcal{X}_j$. By a similar argument as before, $\mathcal{X}_j$ is defined by a single linear inequality. Since the boundary of $\mathcal{X}_j$ is $\mathcal{F}$, it is easy to see $\mathcal{X}_j = \{x : c^T x \leq 0\}$. This shows that $\mathcal{X}_i \cup \mathcal{X}_j = \mathbb{R}^n$. Hence Ξ contains $\mathcal{X}_i$ and $\mathcal{X}_j$ only, and thus is bimodal.

(3) Suppose $\mathcal{I}(x^*) \subseteq \mathcal{I}(x')$ and fix $i \in \mathcal{I}(x^*)$. It is easy to see that $\mathcal{I}(x^*)$ is singleton if and only if x^* is in the interior of a polyhedron of the polyhedral subdivision. In this case, $\mathcal{L}(x^*, i)$ is empty so that $\mathcal{L}(x^*, i) \subseteq \mathcal{L}(x', i)$ holds trivially. Without loss of generality, we assume that $\mathcal{L}(x^*, i)$ is nonempty. Let $s \in \mathcal{L}(x^*, i)$. Hence, $(C_i x^* - h_i)_s = 0$. This implies that x^* is in the facet $\mathcal{F}_{i,s} := \{x \in \mathcal{X}_i : (C_i x - h_i)_s = 0\}$. By statement (1), there exists a unique polyhedron $\mathcal{X}_j \in \Xi$ such that $x^* \in \mathcal{F}_{i,s} = \mathcal{X}_i \cap \mathcal{X}_j$. Therefore, $j \in \mathcal{I}(x^*) \subseteq \mathcal{I}(x')$ so that $x' \in \mathcal{X}_i \cap \mathcal{X}_j$. By the uniqueness of an intersecting facet shown in statement (1), $x' \in \mathcal{F}_{i,s}$. This shows that $(C_i x' - h_i)_s = 0$ and $s \in \mathcal{L}(x', i)$. Consequently, $\mathcal{L}(x^*, i) \subseteq \mathcal{L}(x', i)$. The rest of statement (3) follows readily as well. $\square$

Remark 2.1. An immediate consequence of statement (1) of Lemma 2.1 is that if a polyhedral subdivision Ξ contains m polyhedra, then each polyhedron has at most $(m-1)$ facets and the total number of facets of Ξ is not greater than $(m-1)m/2$.

The following lemma states that under the common face condition (c) specified in the definition of polyhedral subdivision, the affine hulls of relative interiors of intersecting faces are identical.

Lemma 2.2. *The following hold:*

(1) Let $\{\mathcal{X}_i\}_{i=1}^p$ be a family of polyhedra in $\mathbb{R}^n$ (not necessarily to form a polyhedral subdivision) Assume that the family $\{\mathcal{X}_i\}_{i=1}^p$ satisfies (c) of the definition of polyhedral subdivision. Then for any $x^* \in \cup_{i=1}^p \mathcal{X}_i$ and any $i, j \in \mathcal{I}(x^*)$, $\{x \in \mathbb{R}^n : (C_i x - h_i)_s = 0, \forall\, s \in \mathcal{L}(x^*, i)\} = \{x \in \mathbb{R}^n : (C_j x - h_j)_\ell = 0, \forall\, \ell \in \mathcal{L}(x^*, j)\}$.

(2) Let $\Xi = \{\mathcal{X}_i\}$ be a polyhedral subdivision of $\mathbb{R}^n$. Given $x^* \in \mathbb{R}^n$ and $i, j \in \mathcal{I}(x^*)$. Then $\{x \in \mathbb{R}^n : (C_i x - h_i)_s = 0, \forall\, s \in \mathcal{L}(x^*, i)\} = \{x \in \mathbb{R}^n : (C_j x - h_j)_\ell = 0, \forall\, \ell \in \mathcal{L}(x^*, j)\}$.

Proof. (1) For the ease of notation, define the affine sets $\mathcal{S}_i := \{x \in \mathbb{R}^n : (C_i x - h_i)_s = 0, \forall\, s \in \mathcal{L}(x^*, i)\}$ and $\mathcal{S}_j := \{x \in \mathbb{R}^n : (C_j x - h_j)_\ell = 0, \forall\, \ell \in \mathcal{L}(x^*, j)\}$. Since $x^* \in \mathcal{S}_i$ and $x^* \in \mathcal{S}_j$, $\mathcal{V}_i := \mathcal{S}_i - x^*$ and $\mathcal{V}_j := \mathcal{S}_j - x^*$ are linear subspaces of $\mathbb{R}^n$. To prove $\mathcal{S}_i = \mathcal{S}_j$, it suffices to show $\mathcal{V}_i = \mathcal{V}_j$. Suppose not. Then, without loss of generality, there exists a nonzero vector $v \in \mathcal{V}_i$ but $v \notin \mathcal{V}_j$, i.e., $(C_j v)_w \neq 0$ for some $w \in \mathcal{L}(x^*, j)$. By possibly switching the sign of v, we also assume that $(C_j v)_w < 0$. Since $(C_i x^* - h_i)_s > 0, \forall s \notin \mathcal{L}(x^*, i)$, there exists a constant $\varepsilon > 0$ such that $z := x^* + \varepsilon v$ satisfies $(C_i z - h_i)_s > 0, \forall s \notin \mathcal{L}(x^*, i)$. Since $x^* \in \mathcal{X}_i \cap \mathcal{X}_j$, $\mathcal{X}_i$ and $\mathcal{X}_j$ intersect on a common face, i.e., there exist index sets $\alpha \subseteq \mathcal{L}(x^*, i)$ and $\beta \subseteq \mathcal{L}(x^*, j)$ such that $\mathcal{X}_i \cap \mathcal{X}_j = \mathcal{F}_{i,\alpha} = \mathcal{F}_{j,\beta}$. Since $v \in \mathcal{V}_i$, it is easy to verify $z \in \mathcal{F}_{i,\alpha}$. Therefore, $z \in \mathcal{X}_j$. However, $(C_j z - h_j)_w = \varepsilon (C_j v)_w < 0$, implying $z \notin \mathcal{X}_j$. This yields a contradiction. Consequently, $\mathcal{S}_i = \mathcal{S}_j$.

(2) This is a direct consequence of statement (1). □

Remark 2.2. Note that statement (1) may be invalid if a family of polyhedra fails to satisfy the common face condition (c). To see this, consider the example in the left display of Figure 1. Clearly, $\mathcal{I}(x^*) = \{1, 2, 3\}$. However, $\{x \in \mathbb{R}^2 : (C_1 x - h_1)_s = 0, \ \forall s \in \mathcal{L}(x^*, 1)\}$ is one dimensional affine space containing x^*, but $\{x \in \mathbb{R}^2 : (C_2 x - h_2)_s = 0, \ \forall s \in \mathcal{L}(x^*, 2)\} = \{x \in \mathbb{R}^2 : (C_3 x - h_3)_s = 0, \ \forall s \in \mathcal{L}(x^*, 3)\} = \{x^*\}$. This discrepancy is due to the failure of the common face condition.

Recall that Lemma 2.1 shows that under the polyhedral subdivision assumption, each facet of a polyhedron in a polyhedral subdivision is a unique intersection with another polyhedron in the subdivision, which we call the *unique intersection property* below. Informally speaking, the following proposition shows a converse result, namely, if condition (a) of the definition of polyhedral subdivision is replaced by the unique intersection property, then the family of polyhedra in consideration will cover the entire space and thus become a polyhedral subdivision. This result forms a cornerstone for a critical algebraic relation of system matrices in Theorem 2.1.

Proposition 2.1. *Let $\{\mathcal{C}_i\}_{i=1}^r$ be a family of polyhedral cones in $\mathbb{R}^n$ such that*

(i) *Each $\mathcal{C}_i := \{x \in \mathbb{R}^n : C_i x \geq 0\}$ has nonempty interior, where $C_i \in \mathbb{R}^{m_i \times n}$, and $\{x \in \mathbb{R}^n : C_i x = 0\} = \{0\}$ for any i;*

(ii) *For any $i, j \in \{1, \ldots, r\}$ with $i \neq j$, $\mathcal{C}_i \cap \mathcal{C}_j$ is a common proper face of $\mathcal{C}_i$ and $\mathcal{C}_j$ (in the sense of (c) of the definition of polyhedral subdivision);*

(iii) *For each $s \in \{1, \ldots, m_i\}$, $\{x \in \mathcal{C}_i : (C_i x)_s = 0\}$ is an $(n-1)$-dimensional facet of $\mathcal{C}_i$, and each facet of $\mathcal{C}_i$ is the unique intersection of $\mathcal{C}_i$ and some $\mathcal{C}_j$ with $i \neq j$ in the given family.*

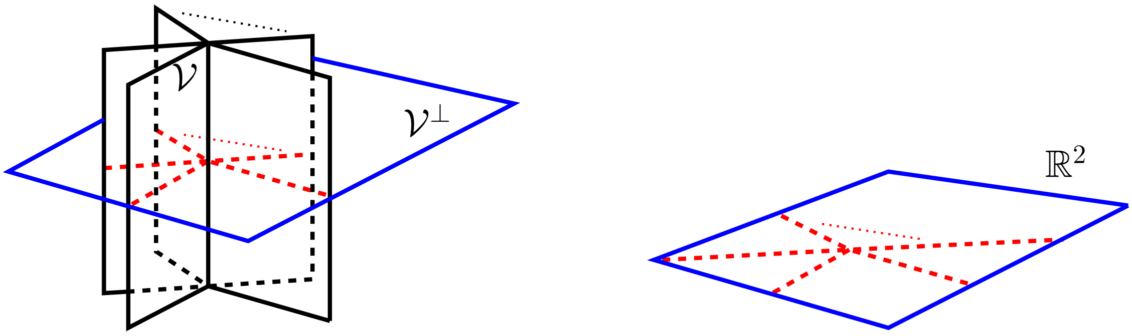

Figure 2: Left: the illustration of $\mathcal{V}$ and $\mathcal{V}^{\perp}$ in $\mathbb{R}^3$; Right: the isomorphic of $\mathcal{V}^{\perp}$.

Then the following statements hold:

(1) *for any relative interior point x^* of any q-dimensional face $\mathcal{F}$ of each $\mathcal{C}_i$, where $q \in \{0, 1, \ldots, n-1\}$, there exists a neighborhood $\mathcal{N}(x^*)$ such that $\mathcal{N}(x^*) \subseteq \cup_{i=1}^r \mathcal{C}_i$;*

(2) $\cup_{i=1}^r \mathcal{C}_i = \mathbb{R}^n$.

Proof. Let $\mathcal{U} := \cup_{i=1}^r \mathcal{C}_i$. We prove statement (1) via induction on $n \in \mathbb{N}$. Consider $n = 1$ first. In this case, it is easy to see that the only family of polyhedral cones satisfying (i)–(iii) is two rays of $\mathbb{R}$, i.e., $\mathcal{C}_1 = \mathbb{R}_+$ and $\mathcal{C}_2 = \mathbb{R}_-$. Clearly, the only face of each $\mathcal{C}_i, i = 1, 2$ is $\mathcal{F} = \{0\}$ and the relative interior point $x^* = 0$ of $\mathcal{F}$ is in the interior of $\mathcal{U} = \mathcal{C}_1 \cup \mathcal{C}_2$. Hence, statement (1) holds.

Assume that statement (1) holds for $n = 1, \ldots, p$, and consider $n = p + 1$. Consider two cases as follows: (a) x^* is a relative interior point of a (unique) q-dimensional face $\mathcal{F}$ of some $\mathcal{C}_i$, where $q \in \{1, \ldots, n-1\}$; and (b) x^* is an extremal point of some $\mathcal{C}_i$, i.e., $\{x^*\}$ is a face of dimension zero.

Case (a). Since x^* is in the relative interior of a q-dimensional face $\mathcal{F}$ of some $\mathcal{C}_i$ with $q \geq 1$, it follows from condition (ii) and Lemma 2.2 that for any $i, j \in \mathcal{I}(x^*) \subseteq \{1, \ldots, r\}$, $\{x \in \mathbb{R}^n : (C_i x)_s = 0, \forall s \in \mathcal{L}(x^*, i)\} = \{x \in \mathbb{R}^n : (C_j x)_s = 0, \forall s \in \mathcal{L}(x^*, j)\}$. Let $\mathcal{V} \subseteq \mathbb{R}^n$ denote this common linear subspace which is of dimension q, i.e., $\mathcal{V} := \{x \in \mathbb{R}^n : (C_i x)_s = 0, \forall s \in \mathcal{L}(x^*, i)\}$, and let $\mathcal{V}^{\perp}$ be the orthogonal complement of $\mathcal{V}$ in $\mathbb{R}^n$. Furthermore, for each $i \in \mathcal{I}(x^*)$, let the polyhedral cone $\mathcal{S}_i := \{x \in \mathbb{R}^n : (C_i x)_s \geq 0, \forall s \in \mathcal{L}(x^*, i)\}$ and $\mathcal{K}'_i$ denote the projection of $\mathcal{S}_i$ onto $\mathcal{V}^{\perp}$. Let $P \in \mathbb{R}^{n \times n}$ be the projection matrix corresponding to the projection onto $\mathcal{V}^{\perp}$, i.e., $Px \in \mathcal{V}^{\perp}$ for $x \in \mathbb{R}^n$. Since $Px^* = 0$ and $(I - P)x \in \mathcal{V}, \forall x$, we have

$$\begin{aligned}
\mathcal{K}'_i &= \{Px : (C_i x)_s \geq 0, \forall s \in \mathcal{L}(x^*, i)\} = \{Px : (C_i([I-P]x + Px)_s \geq 0, \forall s \in \mathcal{L}(x^*, i)\} \\
&= \{Px : (C_i Px)_s \geq 0, \forall s \in \mathcal{L}(x^*, i)\} = \{v \in \mathcal{V}^{\perp} : (C_i v)_s \geq 0, \forall s \in \mathcal{L}(x^*, i)\} = \mathcal{S}_i \cap \mathcal{V}^{\perp}.
\end{aligned}$$

We show the following claims for $\mathcal{K}'_i, \forall\, i \in \mathcal{I}(x^*)$ that are illustrated in Figure 2:

(a.1) *Claim 1: Each $\mathcal{K}'_i$ has nonempty relative interior in $\mathcal{V}^{\perp}$ and $\{v \in \mathcal{V}^{\perp} : (C_i v)_s = 0, \forall s \in \mathcal{L}(x^*, i)\} = \{0\}$.*

To show claim 1, note that for each $s \in \mathcal{L}(x^*, i)$, there exists a vector $x_s \in \mathbb{R}^n$ such that $(C_i x_s)_s = 0$ and $(C_i x_s)_\ell > 0$ for any $\ell \in \mathcal{L}(x^*, i)$ with $s \neq \ell$. Letting $z := \sum_{s \in \mathcal{L}(x^*, i)} x_s$, we have $(C_i z)_w > 0$ for all $w \in \mathcal{L}(x^*, i)$. Let u_z and v_z be the (unique) projections of z onto $\mathcal{V}$ and $\mathcal{V}^{\perp}$, respectively. Noting $z = u_z + v_z$, we have $(C_i v_z)_s = (C_i z)_s > 0$ for all $s \in \mathcal{L}(x^*, i)$. Hence v_z is in the relative interior of $\mathcal{K}'_i$ such that $\mathcal{K}'$ has nonempty relative interior in $\mathcal{V}^{\perp}$. Finally, since $\{v \in \mathcal{V}^{\perp} : (C_i v)_s = 0, \forall s \in \mathcal{L}(x^*, i)\} \subseteq \mathcal{V}^{\perp} \cap \mathcal{V} = \{0\}$, we deduce that $\{v \in \mathcal{V}^{\perp} : (C_i v)_s = 0, \forall s \in \mathcal{L}(x^*, i)\} = \{0\}$.

(a.2) *Claim 2: For any $i,j \in \mathcal{I}(x^*)$ with $i \neq j$, $\mathcal{K}'_i \cap \mathcal{K}'_j$ is a common proper face of $\mathcal{K}'_i$ and $\mathcal{K}'_j$.*

To see this claim, observe that for any $i,j \in \mathcal{I}(x^*)$, $x^* \in \mathcal{C}_i \cap \mathcal{C}_j$ such that $\mathcal{C}_i \cap \mathcal{C}_j = \mathcal{F}_{i,\alpha} = \mathcal{F}_{j,\beta}$, where $\mathcal{F}_{i,\alpha}, \mathcal{F}_{j,\beta}$ are respective faces of $\mathcal{C}_i$ and $\mathcal{C}_j$ defined in (1) with $h_i = 0$ and $h_j = 0$ for some (nonempty) index sets $\alpha \subseteq \mathcal{L}(x^*, i)$ and $\beta \subseteq \mathcal{L}(x^*, j)$. Since $(C_i x^*)_s > 0$ for all $s \notin \mathcal{L}(x^*, i)$ and $(C_j x^*)_\ell > 0$ for all $\ell \notin \mathcal{L}(x^*, j)$, there exists a neighborhood $\mathcal{N}(x^*)$ of x^* such that for each $x \in \mathcal{N}(x^*)$, $(C_i x)_s > 0$ for all $s \notin \mathcal{L}(x^*, i)$ and $(C_j x)_\ell > 0$ for all $\ell \notin \mathcal{L}(x^*, j)$. This shows that for each $k \in \{i,j\}$, $\mathcal{X}_k \cap \mathcal{N}(x^*) = \mathcal{S}_k \cap \mathcal{N}(x^*)$ and $\mathcal{F}_{i,\alpha} \cap \mathcal{N}(x^*) = \mathcal{G}_{i,\alpha} \cap \mathcal{N}(x^*)$, $\mathcal{F}_{j,\beta} \cap \mathcal{N}(x^*) = \mathcal{G}_{j,\beta} \cap \mathcal{N}(x^*)$, where $\mathcal{G}_{i,\alpha} := \{x \in \mathbb{R}^n : (C_i x)_\alpha = 0, \text{ and } (C_i x)_s \geq 0, \forall s \in \mathcal{L}(x^*, i) \setminus \alpha\}$ and $\mathcal{G}_{j,\beta} := \{x \in \mathbb{R}^n : (C_j x)_\beta = 0, \text{ and } (C_j x)_\ell \geq 0, \forall \ell \in \mathcal{L}(x^*, j) \setminus \beta\}$. Hence, we have $\mathcal{S}_i \cap \mathcal{S}_j \cap \mathcal{N}(x^*) = \mathcal{G}_{i,\alpha} \cap \mathcal{N}(x^*) = \mathcal{G}_{j,\beta} \cap \mathcal{N}(x^*)$. Next we show that $\mathcal{S}_i \cap \mathcal{S}_j = \mathcal{G}_{i,\alpha} = \mathcal{G}_{j,\beta}$ via the fact that for each $k \in \{i,j\}$, $(C_k x^*)_s = 0$ for any $s \in \mathcal{L}(x^*, k)$ in two directions:

(a.2.1) $\mathcal{S}_i \cap \mathcal{S}_j \subseteq \mathcal{G}_{i,\alpha}$ and $\mathcal{S}_i \cap \mathcal{S}_j \subseteq \mathcal{G}_{j,\beta}$. It suffices to show the first inclusion. Let $x' \in \mathcal{S}_i \cap \mathcal{S}_j$. Suppose $x' \notin \mathcal{G}_{i,\alpha}$. Then either $(C_i x')_{s'} \neq 0$ for some $s' \in \alpha$ or $(C_i x')_w < 0$ for some $w \in \mathcal{L}(x^*, i) \setminus \alpha$. Since $(C_i x^*)_s = 0$ for any $s \in \mathcal{L}(x^*, i)$, it is easy to see that for any sufficiently small $\varepsilon > 0$, $(1-\varepsilon)x^* + \varepsilon x' \notin \mathcal{G}_{i,\alpha}$. In view of $(1-\varepsilon)x^* + \varepsilon x' \in \mathcal{S}_i \cap \mathcal{S}_j \cap \mathcal{N}(x^*)$ for all small $\varepsilon > 0$, this yields a contradiction to $\mathcal{S}_i \cap \mathcal{S}_j \cap \mathcal{N}(x^*) = \mathcal{G}_{i,\alpha} \cap \mathcal{N}(x^*)$.

(a.2.2) $\mathcal{G}_{i,\alpha} \subseteq \mathcal{S}_i \cap \mathcal{S}_j$ and $\mathcal{G}_{j,\beta} \subseteq \mathcal{S}_i \cap \mathcal{S}_j$. Again, we show the first inclusion only. Let $x' \in \mathcal{G}_{i,\alpha}$. Suppose $x' \notin \mathcal{S}_i \cap \mathcal{S}_j$. This means that there exist $k \in \{i,j\}$ and $w' \in \mathcal{L}(x^*, k)$ such that $(C_k x')_{w'} < 0$. Similarly as in (a.2.1), it is easy to see that for any sufficiently small $\varepsilon > 0$, $\big(C_k[(1-\varepsilon)x^* + \varepsilon x']\big)_{w'} < 0$. This shows $(1-\varepsilon)x^* + \varepsilon x' \notin \mathcal{S}_k \cap \mathcal{N}(x^*)$, where $k \in \{i,j\}$. Since $(1-\varepsilon)x^* + \varepsilon x' \in \mathcal{G}_{i,\alpha} \cap \mathcal{N}(x^*)$, a contradiction is reached.

Thus $\mathcal{S}_i \cap \mathcal{S}_j = \mathcal{G}_{i,\alpha} = \mathcal{G}_{j,\beta}$. In light of $\mathcal{K}'_i \cap \mathcal{K}'_j = (\mathcal{S}_i \cap \mathcal{V}^\perp) \cap (\mathcal{S}_j \cap \mathcal{V}^\perp) = (\mathcal{S}_i \cap \mathcal{S}_j) \cap \mathcal{V}^\perp = \mathcal{G}_{i,\alpha} \cap \mathcal{V}^\perp = \mathcal{G}_{j,\beta} \cap \mathcal{V}^\perp$, we deduce that $\mathcal{K}'_i \cap \mathcal{K}'_j$ is a common proper face of $\mathcal{K}'_i$ and $\mathcal{K}'_j$.

(a.3) *Claim 3: For each $s \in \mathcal{L}(x^*, i)$, the polyhedral cone $\{v \in \mathcal{V}^\perp : (C_i v)_s = 0, \text{ and } (C_i v)_\ell \geq 0, \forall \ell \in \mathcal{L}(x^*, i) \setminus \{s\}\}$ is an $(n-q-1)$-dim face of $\mathcal{K}'_i$. Furthermore, each of such an $(n-q-1)$-dim face of $\mathcal{K}'_i$ is the unique intersection of $\mathcal{K}'_i$ and $\mathcal{K}'_j$ for some $j \in \mathcal{I}(x^*)$ with $i \neq j$.*

The first statement of claim 3 follows directly from the definition of a face and the fact that $\mathcal{V}^\perp$ is of dimension $(n-q)$. To show the second statement, recall from condition (iii) that for each $s \in \mathcal{L}(x^*, i)$, the facet $\mathcal{F}_{i,s} = \{x \in \mathcal{C}_i : (C_i x)_s = 0\}$ is the unique intersection of $\mathcal{C}_i$ and some $\mathcal{C}_j$ with $i \neq j \in \mathcal{I}(x^*)$, i.e., $\mathcal{C}_i \cap \mathcal{C}_j = \mathcal{F}_{i,s}$. By a similar argument in the proof of claim 2 (namely, $(C_i x^*)_w > 0$ for all $w \notin \mathcal{L}(x^*, i)$ and $(C_j x^*)_\ell > 0$ for all $\ell \notin \mathcal{L}(x^*, j)$), we have $\mathcal{S}_i \cap \mathcal{S}_j = \mathcal{G}_{i,s}$, where $\mathcal{G}_{i,s} := \{x \in \mathcal{S}_i : (C_i x)_s = 0\}$. In view of $\mathcal{K}'_k = \mathcal{S}_k \cap \mathcal{V}^\perp$ for each $k = i,j$, we deduce $\mathcal{K}'_i \cap \mathcal{K}'_j = \mathcal{G}_{i,s} \cap \mathcal{V}^\perp$, leading to the second statement.

Note that $\mathcal{V}^\perp$ is isomorphic to $\mathbb{R}^{n-q}$ via an isomorphism $F : \mathcal{V}^\perp \to \mathbb{R}^{n-q}$ defined by an invertible matrix, and let $\mathcal{K}_i = F(\mathcal{K}'_i)$ for each i. In light of the three claims shown above, it can be verified via the isomorphism that the family of the polyhedral cones $\{\mathcal{K}_i, i \in \mathcal{I}(x^*)\}$ in $\mathbb{R}^{n-q}$ satisfies conditions (i)–(iii) of the proposition with respect to the topology of $\mathbb{R}^{n-q}$. Moreover, the vector $x^* \in \mathbb{R}^n$ uniquely corresponds to the zero vector in $\mathbb{R}^{n-q}$ via the projection and the isomorphism. Since the zero vector in $\mathbb{R}^{n-q}$ is an extremal point of each $\mathcal{K}_i$, it follows from the induction hypothesis that there is a neighborhood $\mathcal{N}$ of the zero vector of $\mathbb{R}^{n-q}$ such that $\mathcal{N} \subseteq \cup_{i \in \mathcal{I}(x^*)} \mathcal{K}_i$.

Now suppose that x^* is not an interior point of $\mathcal{U}$ with respect to the topology of $\mathbb{R}^n$. Then there exists a sequence (z_k) in $\mathbb{R}^n$ such that $z_k \to x^*$ as $k \to \infty$ and each $z_k \notin \mathcal{U}$. Since for each i, $(C_i x^*)_s > 0$ for all $s \notin \mathcal{L}(x^*, i)$, we deduce that for all k sufficiently large, there exists some $s_{k,i} \in \mathcal{L}(x^*, i)$ with $(C_i z_k)_{s_{k,i}} < 0$ for each $i \in \mathcal{I}(x^*)$. Recall that $P z_k$ is the projection of z_k onto $\mathcal{V}^\perp$ and it is easy to verify $(C_i P z_k)_{s_{k,i}} < 0$ for each $i \in \mathcal{I}(x^*)$. Letting $v'_k := P z_k \in \mathcal{V}^\perp$, we see that

(v'_k) converges to $Px^* = 0$ in $\mathcal{V}^\perp$ and $v'_k \notin \cup_{i\in\mathcal{I}(x^*)}\mathcal{K}'_i$ for all k sufficiently large. Let $v_k = F(v'_k)$ for each k such that (v_k) is a sequence in $\mathbb{R}^{n-q}$. By the virtue of the above argument, (v_k) converges to the zero vector. But for all k arbitrarily large, $v_k \notin \cup_{i\in\mathcal{I}(x^*)}\mathcal{K}_i$. This leads to a contradiction.

Case (b). By condition (i) and the conic property of $\mathcal{C}_i$, the zero vector is the unique extremal point of each $\mathcal{C}_i$. Hence, $x^* = 0$. We show that $x^* = 0$ is in the interior of $\mathcal{U}$ by contradiction. Suppose not. Then there exists a sequence (z_k) such that $z_k \to 0$ as $k \to \infty$ with each $z_k \notin \mathcal{U}$. Clearly, each $z_k \neq 0$ as $x^* = 0 \in \mathcal{U}$. This shows that x^* is a boundary point of $\mathcal{U}$, i.e., $x^* \in \mathcal{U} \cap \mathcal{U}^c$, where $\mathcal{U}^c := \mathbb{R}^n \setminus \mathcal{U}$. Moreover, it is shown in Case (a) that any relative interior point of a q-dimensional face of each $\mathcal{C}_i$ is an interior point of $\mathcal{U}$, where $q \in \{1, \ldots, n-1\}$. Hence, x^* is the only boundary point of $\mathcal{U}$ and $\mathcal{U}^c$. Choose $z^* := z_k \in \mathcal{U}^c$ for some k and a nonzero vector $\widehat{z} \in \mathcal{U}$. Clearly, $z^* \neq 0$ since $z_k \neq 0$. Let $[z^*, \widehat{z}]$ denote the line segment joining z^* and $\widehat{z}$, i.e., $[z^*, \widehat{z}] := \{z : \lambda z^* + (1-\lambda)\widehat{z},\ \lambda \in [0,1]\}$. Note that $[z^*, \widehat{z}]$ contains at most one zero vector if there is any. Consider two subcases:

(b.1) $0 \notin [z^*, \widehat{z}]$. Clearly, $z^* \neq 0$ is not a boundary point of $\mathcal{U}^c$, and thus is in the interior of $\mathcal{U}^c$. Similarly, $\widehat{z}$ is an interior point of $\mathcal{U}$. Hence, there exist scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the line segments $\{z : \lambda z^* + (1-\lambda)\widehat{z},\ \lambda \in [0, \varepsilon_1]\} \subset \mathcal{U}^c$ and $\{z : \lambda z^* + (1-\lambda)\widehat{z},\ \lambda \in [1-\varepsilon_2, 1]\} \subset \mathcal{U}$. This shows that the line segment $[z^*, \widehat{z}]$ contains a boundary point of $\mathcal{U}$. In fact, let $\lambda^* := \sup\{\bar{\lambda} \in [0,1] : \lambda z^* + (1-\lambda)\widehat{z} \in \mathcal{U}^c, \forall \lambda \in [0, \bar{\lambda})\}$. In view of $\varepsilon_1 \leq \lambda^* \leq 1 - \varepsilon_2$, $\lambda z^* + (1-\lambda^*)\widehat{z}$ is a desired boundary point. Since $0 \notin [z^*, \widehat{z}]$, this contradicts the fact that the zero vector is the only boundary point of $\mathcal{U}$.

(b.2) $0 \in [z^*, \widehat{z}]$. As shown in (b.1), $z^* \neq 0$ is in the interior of $\mathcal{U}^c$ and $\widehat{z}$ is in the interior of $\mathcal{U}$. Hence, there exist a unit vector v orthogonal to $(z^* - \widehat{z})$ and a constant $\varepsilon_3 > 0$ such that $z^* + \varepsilon_3 v$ is in the interior of $\mathcal{U}^c$. It is easy to show that the line segment $[z^* + \varepsilon_3 v, \widehat{z}]$ does not contain the zero vector. A similar argument as in (b.1) leads to a contradiction.

Consequently, statement (1) follows from the induction principle. Since $x = 0$ is an extremal point of some $\mathcal{C}_i$, it follows from statement (1) that there exists a neighborhood $\mathcal{N}$ of $x = 0$ such that $\mathcal{N} \subseteq \mathcal{U}$. Furthermore, noting that the union $\mathcal{U}$ is a closed cone, statement (2) follows readily. $\square$

The next lemma treats a special case of Theorem 2.1, and it establishes an algebraic relation between different A_i's and d_i's when two polyhedra in Ξ share a common facet. Its proof is straightforward and is omitted.

Lemma 2.3. *Let f be a Lipschitz PA function such that $f(x) = A_i x + d_i, \forall x \in \mathcal{X}_i$ and $f(x) = A_j x + d_j, \forall x \in \mathcal{X}_j$. Suppose that $\mathcal{F} := \mathcal{X}_i \cap \mathcal{X}_j$ is a common facet contained in the hyperplane $\{x \in \mathbb{R}^n : c^T x = \gamma\}$ for a unit vector $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. Then there exists a vector $b \in \mathbb{R}^n$ such that $A_j = A_i + bc^T$ and $d_j = d_i - \gamma\, b$.*

Lemma 2.4. *Let $\mu > 0$ be such that $\|A_i\|_2 \leq \mu$ for all i. Suppose that $A_j = A_k + bc^T$ with $\|c\|_2 = 1$. Then $\|b\|_2 \leq 2\mu$.*

Proof. Since $A_j = A_k + bc^T$ and $\|c\|_2 = 1$, $A_j c = A_k c + bc^T c$ such that $b = A_j c - A_k c$. Hence, $\|b\|_2 \leq \|A_j\|_2 + \|A_k\|_2 = 2\mu$. $\square$

Theorem 2.1. *Consider a polyhedral subdivision $\{\mathcal{X}_i\}_{i=1}^m$. For a given $x^* \in \mathbb{R}^n$ and a fixed index $i_* \in \mathcal{I}(x^*)$, there exist vectors $b_{\ell,i,j} \in \mathbb{R}^n$ such that for any $\ell \in \mathcal{I}(x^*)$,*

$$A_\ell = A_{i_*} + \sum_{j\in\mathcal{L}(x^*,i), i\in\mathcal{I}(x^*)} b_{\ell,i,j} (C_i)_{j\bullet}, \qquad d_\ell = d_{i_*} - \sum_{j\in\mathcal{L}(x^*,i), i\in\mathcal{I}(x^*)} (h_i)_j\, b_{\ell,i,j}. \tag{2}$$

Moreover, if there exists a constant $\mu_A > 0$ such that $\|A_i\|_2 \leq \mu_A$ for any i, then $\|b_{\ell,i,j}\|_2 \leq 2\mu_A$ for all i, j, ℓ.

Proof. To avoid triviality, consider the non-singleton $\mathcal{I}(x^*)$. For any $i, j \in \mathcal{I}(x^*)$, we denote $\mathcal{X}_i \sim \mathcal{X}_j$ if there is a finite sequence $\{\mathcal{X}_{s_1}, \mathcal{X}_{s_2}, \dots, \mathcal{X}_{s_p}\}$ with $s_k \in \mathcal{I}(x^*)$ such that (i) $s_1 = i$; (ii) $s_p = j$; and (iii) for each $k = 1, \dots, s_p - 1$, $\mathcal{X}_{s_k} \cap \mathcal{X}_{s_{k+1}}$ is a facet $\mathcal{F}_{s_k,w}$ of $\mathcal{X}_{s_k}$ for some $w \in \mathcal{L}(x^*, s_k)$. By Lemma 2.1, condition (iii) says that two consecutive polyhedra $\mathcal{X}_{s_k}$ and $\mathcal{X}_{s_{k+1}}$ in the sequence share a (unique) common facet containing x^*. Geometrically, $\mathcal{X}_i \sim \mathcal{X}_j$ means that there is a path crossing several facets containing x^* that connects $\mathcal{X}_i$ and $\mathcal{X}_j$. It is easy to verify that the binary relation $\sim$ is reflexive, symmetric, and transitive, and thus it defines an equivalent relation on $\{\mathcal{X}_i,\ i \in \mathcal{I}(x^*)\}$.

For a fixed $i_* \in \mathcal{I}(x^*)$, we claim that for any $j \in \mathcal{I}(x^*)$, $\mathcal{X}_j \sim \mathcal{X}_{i_*}$. Suppose not. Let $\mathcal{E} := \{j \in \mathcal{I}(x^*) : \mathcal{X}_j \sim \mathcal{X}_{i_*}\}$. Thus there exists $k \in \mathcal{I}(x^*)$ such that $k \notin \mathcal{E}$. We prove as follows that there exists an open neighborhood $\mathcal{N}$ of x^* such that $\mathcal{N} \subseteq \cup_{j\in\mathcal{E}}\mathcal{X}_j$. To show this, recall that in view of Lemma 2.2, $\{x \in \mathcal{X}_{i_*} : (C_{i_*}x - h_{i_*})_s = 0, \forall\, s \in \mathcal{L}(x^*, i_*)\} = \{x \in \mathcal{X}_j : (C_jx - h_j)_\ell = 0, \forall\, \ell \in \mathcal{L}(x^*, j)\}$ for each $j \in \mathcal{E}$. For each $i \in \mathcal{E}$, define the polyhedral cone $\mathcal{S}_i := \{x \in \mathbb{R}^n : (C_ix - h_i)_s \geq 0, \forall\, s \in \mathcal{L}(x^*, i)\} - x^*$. Clearly, $\mathcal{S}_i = \{v \in \mathbb{R}^n : (C_iv)_s \geq 0, \forall\, s \in \mathcal{L}(x^*, i)\}$. Define the subspace $\mathcal{V} := \{v \in \mathbb{R}^n : (C_{i_*}v)_s = 0, \forall\, s \in \mathcal{L}(x^*, i_*)\}$, and let $\mathcal{C}'_i$ be the projection of $\mathcal{S}_i$ onto $\mathcal{V}^\perp$ for each $i \in \mathcal{E}$. Since $\mathcal{I}(x_*)$ is non-singleton, $\mathcal{V}$ is a proper subspace. By a same argument as in Proposition 2.1, we see that $\mathcal{C}'_i = \mathcal{S}_i \cap \mathcal{V}^\perp$ and $\{\mathcal{C}'_i, i \in \mathcal{E}\}$ satisfies the three claims shown in the proof of Proposition 2.1. Since $\mathcal{V}^\perp$ is isomorphic to $\mathbb{R}^r$ for some $r \in \mathbb{N}$, we obtain the isomorphic of $\mathcal{C}'_i$, denoted by $\mathcal{C}_i$, in $\mathbb{R}^r$. Hence, the family of polyhedral cones $\{\mathcal{C}_i, i \in \mathcal{E}\}$ in $\mathbb{R}^r$ satisfies conditions (i)–(iii) in Proposition 2.1. Consequently, $\cup_{i\in\mathcal{E}}\mathcal{C}_i = \mathbb{R}^r$. Along with the observation that for all x sufficiently close to x^*, $(C_ix - h_i)_s > 0$ for each $i \in \mathcal{E}$ and any $s \notin \mathcal{L}(x^*, i)$, it suffices to show the existence of a desired neighborhood $\mathcal{N}$ of x^* such that $\mathcal{N} \subseteq \cup_{j\in\mathcal{E}}\mathcal{X}_j$. However, since $x^* \in \mathcal{X}_k$ and $\mathcal{X}_k$ has nonempty interior, there exists a vector x' in the interior of $\mathcal{X}_k$ such that the sequence (z_s) with $z_s := (1 - 1/s)x^* + x'/s \in \mathcal{X}_k$ converges to x^* as $s \to \infty$. This yields a contradiction as $k \notin \mathcal{E}$.

Consider an arbitrary index $\ell \in \mathcal{I}(x^*)$. Since $\mathcal{X}_{i_*} \sim \mathcal{X}_\ell$, there exists a sequence $\{\mathcal{X}_{s_1}, \mathcal{X}_{s_2}, \dots, \mathcal{X}_{s_p}\}$ with $s_k \in \mathcal{I}(x^*)$ such that (i) $s_1 = i_*$; (ii) $s_p = \ell$; and (iii) for each $k = 1, \dots, s_p - 1$, $\mathcal{X}_{s_k} \cap \mathcal{X}_{s_{k+1}}$ is a facet $\mathcal{F}_{s_k,w}$ of $\mathcal{X}_{s_k}$ for some $w \in \mathcal{L}(x^*, s_k)$. This yields a sequence of common facets $\{\mathcal{F}_{s_k,w}\}$. For each s_k, it follows from Lemma 2.3 that $A_{s_{k+1}} = A_{s_k} + b_w(C_{s_k\bullet})_w$ and $d_{s_{k+1}} = d_{s_k} - (h_{s_k})_w b_w$ for some vector b_w. Repeating this process and letting $b_{\ell,i,j} := 0$ for $(C_i)_{j\bullet}$ that corresponds to a facet not in $\{\mathcal{F}_{s_k,w}\}$, we obtain the identities in (2). Finally, under the boundedness assumption on A_i, it follows from Lemma 2.4 that each nonzero $b_{\ell,i,j}$ satisfies $\|b_{\ell,i,j}\|_2 \leq 2\mu_A$. □

The following corollary presents a global extension of Theorem 2.1.

Corollary 2.1. *Given a Lipschitz PA function $f : \mathbb{R}^n \to \mathbb{R}^n$ and its polyhedral subdivision Ξ with corresponding PA pieces $\{A_ix + d_i\}_{i=1}^m$. Fix an $i_* \in \{1, \dots, m\}$. Let $\{(c_k, \gamma_k)\}$ define distinct facets of polyhedra in Ξ. Then for any $\ell \in \{1, \dots, m\}$, there exist vectors $b_{\ell,k} \in \mathbb{R}^n$ such that*

$$A_\ell = A_{i_*} + \sum_k b_{\ell,k}c_k^T, \qquad d_\ell = d_{i_*} - \sum_k \gamma_k b_{\ell,k}. \tag{3}$$

Proof. For a given index i_*, define the index set $\mathcal{H}_1 := \{j \in \{1, \dots, m\} : \mathcal{X}_j \cap \mathcal{X}_{i_*} \neq \emptyset\}$. It follows from Theorem 2.1 that for each $\ell \in \mathcal{H}_1$, there exist vectors $b_{\ell,k} \in \mathbb{R}^n$ such that (3) holds, by possibly setting some $b_{\ell,k} = 0$ if a facet defined by (c_k, γ_k) is not a common facet of $\mathcal{X}_j, j \in \mathcal{H}_1$. Then define $\mathcal{H}_2 := \{j \in \{1, \dots, m\} : \mathcal{X}_j \cap \mathcal{X}_i \neq \emptyset$ for some $i \in \mathcal{H}_1\}$. It is easy to see that (3) holds for each $\ell \in \mathcal{H}_2$ for suitable vectors $b_{\ell,k}$. Repeating this process in a similar manner, we obtain $p \in \mathbb{N}$ with $\mathcal{H}_p = \mathcal{H}_{p+1}$. Let $\mathcal{U} := \cup_{j\in\mathcal{H}_p}\mathcal{X}_j$. Since $\mathcal{U}$ is a finite union of closed sets, it is closed. On the other hand, as shown in the proof of Theorem 2.1, each point in $\mathcal{U}$ is an interior point of $\mathcal{U}$ such that $\mathcal{U}$ is open. This shows that $\mathcal{U} = \mathbb{R}^n$ and $\mathcal{H}_p = \{1, \dots, m\}$. Hence (3) holds for each $\ell \in \mathcal{H}_p$. □

2.2 Lipschitz Piecewise Affine Systems: Mode Switching and Critical Time

Consider the ODE system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz PA function. We call such a system the *Lipschitz piecewise affine system* or simply Lipschitz PAS [6]. Similarly, if f is Lipschitz piecewise linear, then we call the system *conewise linear system* (CLS). Based on a polyhedral subdivision $\{\mathcal{X}_i\}_{i=1}^m$ of f, we can write the PAS in the following equivalent form

$$\dot{x} = A_i\, x + d_i, \quad \forall\, x \in \mathcal{X}_i, \tag{4}$$

where $[\, x \in \mathcal{X}_i \cap \mathcal{X}_j \,] \Rightarrow [\, A_i x + d_i = A_j x + d_j \,]$ holds due to the continuity of f. The PAS (4) has a unique continuously differentiable solution for each initial state. In what follows, we call each affine dynamics $\dot{x} = A_i x + d_i$ and its associated polyhedron $\mathcal{X}_i$ a *mode* of the PAS. The Lipschitz PASs form a class of affine hybrid systems, for which the vector fields are affine, the invariant sets are the polyhedra $\mathcal{X}_i$ of dimension n, the guard sets are the boundaries of these polyhedra, and the reset maps are all identities. Associated with the PAS (4), we define a reverse-time system as follows: for a given terminal time $T > 0$, let $x^r(t) := x(T - t)$ and $x^r(0) = x(T)$. Then we have

$$\dot{x}^r = -A_i\, x^r - d_i, \quad \forall\, x^r \in \mathcal{X}_i. \tag{5}$$

This system remains a Lipschitz PAS. We introduce the definition of mode switching as follows.

Definition 2.1. Let $x(t, x^0)$ be a state trajectory of the PAS (4) from the initial state x^0. A time instant $t_* > 0$ is *not* a switching time along $x(t, x^0)$ if there exist $\mathcal{X}_i$ and a constant $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, t_* is a *switching time* along $x(t, x^0)$, and we say that the PAS has a *mode switching* at t_* along $x(t, x^0)$.

Let $\succcurlyeq$ denote the lexicographical nonnegative order. For the ith mode of the PAS, define

$$\mathcal{Y}_i := \Big\{ x \in \mathbb{R}^n \ : \ \big(C_i x - h_i,\ C_i(A_i x + d_i),\ C_i A_i(A_i x + d_i), \cdots,\ C_i A_i^{n-1}(A_i x + d_i) \big) \succcurlyeq 0 \Big\}.$$

For any x^0, $x(t, x^0) \in \mathcal{X}_i$ for all $t \geq 0$ sufficiently small if and only if $x^0 \in \mathcal{Y}_i$. Given $\xi \in \mathbb{R}^n$, recall $\mathcal{I}(\xi) := \{ i \in \{1, \ldots, m\} : \xi \in \mathcal{X}_i \}$, and define $\mathcal{J}(\xi) := \{ i \in \{1, \ldots, m\} : \xi \in \mathcal{Y}_i \}$. It is clear that $\mathcal{J}(\xi) \subseteq \mathcal{I}(\xi)$. Similarly, we can define $\mathcal{J}^r(\xi)$ for the associated reverse-time system. Furthermore, it is easy to show that $i \in \mathcal{J}(\xi)$ if and only if there exists $\varepsilon > 0$ such that $\dot{x}(t, \xi) = A_i x(t, \xi) + d_i$ (or equivalently $x(t, \xi) \in \mathcal{X}_i$) for all $t \in (0, \varepsilon)$.

The following proposition shows that the Lipschitz PAS enjoys the *simple switching property* [31], which gives a neat algebraic characterization of a non-switching time. The proof is similar to that of [31, Proposition 2.2]; we present the proof for self-containment.

Proposition 2.2. *For any state trajectory $x(t, x^0)$ of the PAS (4), a time $t_* > 0$ is a non-switching time along $x(t, x^0)$ if and only if $\mathcal{J}(x(t_*, x^0)) = \mathcal{J}^r(x(t_*, x^0))$.*

Proof. Let $x^* := x(t_*, x^0)$. First of all, it follows from [39] that $\mathcal{J}(x^*)$ and $\mathcal{J}^r(x^*)$ are nonempty. Moreover, it can be shown using a similar argument in [6, Propositon 3.11] that t_* is a non-switching time if and only if $\mathcal{J}(x^*) \cap \mathcal{J}^r(x^*)$ is nonempty. Hence, the "if" part follows readily. To show the "only if" part, it is observed that since t_* is a non-switching time, there exists one $j \in \mathcal{J}(x^*) \cap \mathcal{J}^r(x^*)$ and there exists $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_j$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$. It suffices to show $\mathcal{J}^r(x^*) \subseteq \mathcal{J}(x^*)$ because $\mathcal{J}^r(x^*) \supseteq \mathcal{J}(x^*)$ can be proved in a similar way via the reverse-time system. Suppose not. Then there exists $i \in \mathcal{J}^r(x^*)$ but $i \notin \mathcal{J}(x^*)$. Note that this implies $i \neq j$. Since $i, j \in \mathcal{J}^r(x^*)$, $x(t, x^0) \in \mathcal{X}_i \cap \mathcal{X}_j$ on $[t_* - \varepsilon', t_*]$ for some $\varepsilon' > 0$. By the common proper face property of polyhedral subdivision, $x(t, x^0) \in \mathcal{X}_i \cap \{x : (C_i x - h_i)_\alpha = 0\} = \mathcal{X}_j \cap \{x : (C_j x - h_j)_\beta = 0\}$

on $[t_*-\varepsilon', t_*]$ for nonempty index sets α and β. This implies that $\big(C_j\, x(t,x^0)-h_j\big)_\beta=0$ for all $t\in[t_*-\varepsilon', t_*]$. Hence, letting $x^*:=x(t_*,x^0)$, we have

$$\Big(C_i x^*-h_i,\ C_i(A_ix^*+d_i),\ C_iA_i(A_ix^*+d_i),\cdots,\ C_iA_i^{n-1}(A_ix^*+d_i)\Big)_\beta = 0.$$

In view of this and $x(t,x^0)\in\mathcal{X}_j$ on $[t_*-\varepsilon, t_*+\varepsilon]$, we further have $(C_j\, x(t,x^0)-h_j)_\beta=0$ for all $t\in[t_*,t_*+\varepsilon]$. Therefore, $x(t,x^0)\in\mathcal{X}_j\cap\{x : (C_jx-h_j)_\beta=0\}=\mathcal{X}_i\cap\mathcal{X}_j$ on $[t_*,t_*+\varepsilon]$. Consequently, $x(t,x^0)\in\mathcal{X}_i$ on $[t_*,t_*+\varepsilon]$ and thus $i\in\mathcal{J}(x^*)$. This yields a contradiction. $\square$

We introduce the concept of critical time originally defined in [28] as follows.

Definition 2.2. For a given state trajectory $x(t,x^0)$, if a time t' satisfies $\mathcal{J}(x(t',x^0))\neq\mathcal{I}(x(t',x^0))$, then we call t' a *critical time* along $x(t,x^0)$ and its corresponding state $x(t',x^0)$ a *critical state*. In other words, $\mathcal{J}(x(t',x^0))$ is a proper subset of $\mathcal{I}(x(t',x^0))$ at a critical time t'.

Let $x':=x(t',x^0)$ be a critical state, and let $i\in\mathcal{I}(x')\setminus\mathcal{J}(x')$. It is easy to see that x' must be on the boundary of $\mathcal{X}_i$, i.e., $(C_ix'-h_i)_s=0$ for some s. Furthermore, we must have $x'\notin\mathcal{Y}_i$, i.e.,

$$\big(C_ix'-h_i,\ C_i(A_ix'+d_i),\ C_iA_i(A_ix'+d_i),\cdots,\ C_iA_i^{n-1}(A_ix'+d_i)\big)\not\preccurlyeq 0.$$

This implies that there exist an index s and a constant $\varepsilon'>0$ such that $(C_ix(t',x^0)-h_i)_s=0$ and $(C_ix(t,x^0)-h_i)_s<0$ for all $t\in(t',t'+\varepsilon')$. It is easy to show via a similar argument as in [28] that a switching time must be a critical time, albeit the converse may not hold. By extending the argument in [28, Proposition 7], we also obtain the following result whose proof is similar to that of [28, Proposition 7] and is omitted.

Proposition 2.3. *Given a time interval $[0,T]$ with $T>0$, there are finitely many critical times on $[0,T]$ along a state trajectory $x(t,x_0)$. Specifically, there exists a partition $0=t_0<t_1<\cdots<t_{M-1}<t_M=T$ such that for each $i=0,\dots,M-1$, there exists an index set $\mathcal{I}^*_i\subseteq\{1,\dots,m\}$ with $\mathcal{I}(x(t,x_0))=\mathcal{J}(x(t,x_0))=\mathcal{I}^*_i$ for all $t\in(t_i,t_{i+1})$.*

Consider a state trajectory $x(t,x^0)$ on the time interval $[0,T]$. Define the following sets:

$$\mathcal{T}_0 := \{t\in[0,T] : (C_ix(t,x^0)-h_i)_s=0 \text{ for some } i \text{ and } s\ \}, \tag{6}$$
$$\mathcal{I}_A := \cup_{t\in[0,T]}\,\mathcal{I}(x(t,x^0)), \tag{7}$$
$$\mathcal{I}_c := \{(C_i)_{s\bullet} : (C_ix(t,x^0)-h_i)_s=0 \text{ for some } t\in\mathcal{T}_0\ \}, \tag{8}$$
$$\mathcal{I}_h := \{(h_i)_s : (C_ix(t,x^0)-h_i)_s=0 \text{ for some } t\in\mathcal{T}_0\ \}. \tag{9}$$

Note that $\mathcal{T}_0\subseteq[0,T]$ may be empty. For a subset $\mathcal{T}\subseteq[0,T]$, we also define

$$\begin{aligned}\mathcal{I}_A(\mathcal{T}) &:= \cup_{t\in\mathcal{T}}\,\mathcal{I}(x(t,x^0)),\\ \mathcal{I}_c(\mathcal{T}) &:= \{(C_i)_{s\bullet} : (C_ix(t,x^0)-h_i)_s=0 \text{ for some } t\in\mathcal{T}_0\cap\mathcal{T}\ \},\\ \mathcal{I}_h(\mathcal{T}) &:= \{(h_i)_s : (C_ix(t,x^0)-h_i)_s=0 \text{ for some } t\in\mathcal{T}_0\cap\mathcal{T}\ \}.\end{aligned}$$

Proposition 2.4. *Given a state trajectory $x(t,x^0)$ on the time interval $[0,T]$ for some $T>0$. Fix an index $i_*\in\mathcal{I}(x^0)$. Then for each $\ell\in\mathcal{I}_A$, there exist vectors $b_{\ell,j}\in\mathbb{R}^n$ such that*

$$A_\ell = A_{i_*}+\sum_{c_j^T\in\mathcal{I}_c} b_{\ell,j}\,c_j^T, \qquad d_\ell = d_{i_*}-\sum_{\gamma_j\in\mathcal{I}_h}\gamma_j\,b_{\ell,j}.$$

Moreover, if a constant $\mu_A>0$ is such that $\|A_i\|_2\le\mu_A$ for any i, then $\|b_{\ell,j}\|_2\le 2\mu_A$ for all ℓ,j.

Proof. Since Proposition 2.3 assures finitely many critical times on $[0,T]$, we shall prove the proposition by induction on the number M of the subintervals in the partition defined by critical times.

Consider $M=1$ first, i.e., there exists an index set $\mathcal{I}^*$ such that $\mathcal{I}(x(t,x^0)) = \mathcal{J}(x(t,x^0)) = \mathcal{I}^*$ for each $t \in (0,T)$. For the initial state x^0 and the fixed $i_* \in \mathcal{I}(x^0)$, it follows from Theorem 2.1 that there exist vectors $b_{\ell,i,j} \in \mathbb{R}^n$ such that for any $\ell \in \mathcal{I}(x^0)$, $A_\ell = A_{i_*} + \sum_{j\in\mathcal{L}(x^0,i), i\in\mathcal{I}(x^0)} b_{\ell,i,j}(C_i)_{j\bullet}$ and $d_\ell = d_{i_*} - \sum_{j\in\mathcal{L}(x^0,i),i\in\mathcal{I}(x^0)}(h_i)_j\, b_{\ell,i,j}$. It is noted that $(C_i)_{j\bullet} \in \mathcal{I}_c$ and $(h_i)_j \in \mathcal{I}_h$ for each $j \in \mathcal{L}(x^0,i)$ and each $i \in \mathcal{I}(x^0)$. Hence, $\mathcal{I}_c(\{0\}) = \{(C_i)_{j\bullet} : j \in \mathcal{L}(x^0,i), i \in \mathcal{I}(x^0)\}$ and $\mathcal{I}_h(\{0\}) = \{(h_i)_j : j \in \mathcal{L}(x^0,i), i \in \mathcal{I}(x^0)\}$.

Next consider $t \in (0,T)$. By a similar argument as in [6, Proposition 3.9], we deduce that there exists a constant $\varepsilon_+ > 0$ such that $\mathcal{J}(x^0) = \mathcal{J}(x(t,x^0))$ for all $t \in [0,\varepsilon_+]$. This, along with $\mathcal{I}(x(t,x^0)) = \mathcal{J}(x(t,x^0)) = \mathcal{I}^*$ for all $t\in(0,T)$, implies that $\mathcal{I}^* = \mathcal{I}(x(t,x^0)) = \mathcal{J}(x^0) \subseteq \mathcal{I}(x^0)$ for any $t \in (0,T)$. In view of statement (3) of Lemma 2.1, we have $\mathcal{L}(x(t,x^0),i) \subseteq \mathcal{L}(x^0,i)$ for any $i\in\mathcal{I}^*$ and each $t \in (0,T)$. Therefore, $\mathcal{I}_c([0,T)) \subseteq \mathcal{I}_c(\{0\})$, and $\mathcal{I}_h([0,T)) \subseteq \mathcal{I}_h(\{0\})$. Furthermore, for each $\ell \in \mathcal{I}_A([0,T))$, $A_\ell = A_{i_*} + \sum_{c_j^T\in\mathcal{I}_c([0,T))} b_{\ell,j}\, c_j^T$ and $d_\ell = d_{i_*} - \sum_{\gamma_j\in\mathcal{I}_h([0,T))}\gamma_j\, b_{\ell,j}$.

Now consider $x(T) := x(T,x^0)$. By using [6, Proposition 3.9] again, we obtain a constant $\varepsilon_- > 0$ such that $\mathcal{J}^r(x(T)) = \mathcal{J}(x(t,x^0))$ for all $t \in [T-\varepsilon_-, T)$. It follows from the similar argument as before that $\mathcal{I}^* = \mathcal{J}^r(x(T)) \subseteq \mathcal{I}(x(T))$. Choose an index $\widetilde{i}_* \in \mathcal{I}^*$. Applying Theorem 2.1 to $x(T)$, we deduce that for each $\widetilde{\ell} \in \mathcal{I}(x(T))$, there exist vectors $b_{\widetilde{\ell},\widetilde{j}} \in \mathbb{R}^n$ such that

$$A_{\widetilde{\ell}} = A_{\widetilde{i}_*} + \sum_{c^T_{\widetilde{j}}\in\mathcal{I}_c(\{T\})} b_{\widetilde{\ell},\widetilde{j}}\, c^T_{\widetilde{j}}, \qquad d_{\widetilde{\ell}} = d_{\widetilde{i}_*} - \sum_{\gamma_{\widetilde{j}}\in\mathcal{I}_h(\{T\})} \gamma_{\widetilde{j}}\, b_{\widetilde{\ell},\widetilde{j}},$$

where $\mathcal{I}_c(\{T\}) = \{(C_i)_{j\bullet} : j \in \mathcal{L}(x(T),i), i \in \mathcal{I}(x(T))\}$ and $\mathcal{I}_h(\{T\}) = \{(h_i)_{j\bullet} : j \in \mathcal{L}(x(T),i), i \in \mathcal{I}(x(T))\}$. Since $A_{\widetilde{i}_*} = A_{i_*} + \sum_{c_j^T\in\mathcal{I}_c([0,T))} b_{i_*,j} c_j^T$, it is easy to see that for each $\widetilde{\ell}\in\mathcal{I}(x(T))$,

$$A_{\widetilde{\ell}} = A_{i_*} + \sum_{c_k^T\in\mathcal{I}_c([0,T])} b_{\widetilde{\ell},k}\, c_k^T, \qquad d_{\widetilde{\ell}} = d_{i_*} - \sum_{\gamma_k\in\mathcal{I}_h([0,T])} \gamma_k\, b_{\widetilde{\ell},k}.$$

It is obvious that for each $\ell \in \mathcal{I}_A([0,T))$, $A_\ell = A_{i_*} + \sum_{c_k^T\in\mathcal{I}_c([0,T])} b_{\ell,k}\, c_k^T$, $d_\ell = d_{i_*} - \sum_{\gamma_k\in\mathcal{I}_h([0,T])}\gamma_k\, b_{\ell,k}$ by setting $b_{\ell,k} = 0$ once $c_k^T \in \mathcal{I}_c(\{T\})$. This establishes the proposition for $M=1$.

Given $r\in\mathbb{N}$, and assume that the proposition is valid for the partition of $[0,T]$ into $M = 1,\ldots,r$ subintervals defined in Proposition 2.3. Consider $M = r+1$. By the induction hypothesis, the proposition holds on the interval $[0,t_r]$, namely, given some $i_* \in \mathcal{I}(x^0)$, for each $\ell \in \mathcal{I}_A([0,t_r])$, there exist vectors $b_{\ell,j}\in\mathbb{R}^n$ such that

$$A_\ell = A_{i_*} + \sum_{c_j^T\in\mathcal{I}_c([0,t_r])} b_{\ell,j}\, c_j^T, \qquad d_\ell = d_{i_*} - \sum_{\gamma_j\in\mathcal{I}_h([0,t_r])}\gamma_j\, b_{\ell,j}.$$

Now consider $t \in (t_r, t_M)$. Using the similar preceding argument, we have $\mathcal{I}_c([t_r,t_M)) \subseteq \mathcal{I}_c(\{t_r\}) \subseteq \mathcal{I}_c([0,t_r])$ and $\mathcal{I}_h([t_r,t_M)) \subseteq \mathcal{I}_h(\{t_r\}) \subseteq \mathcal{I}_h([0,t_r])$. Therefore, the proposition can be extended to the interval $[0,t_M)$. Finally, by exploiting [6, Proposition 3.9] and Theorem 2.1 to $x(T) := x(T,x^0)$, where $T = t_{r+1} = t_M$, we have, for a fixed index $\widetilde{i}_* \in \mathcal{I}(x(t,x^0))$ for some $t\in(t_r,t_{r+1})$ and each $\widetilde{\ell}\in\mathcal{I}(x(T))$, there exist vectors $b_{\widetilde{\ell},\widetilde{j}}\in\mathbb{R}^n$ such that

$$A_{\widetilde{\ell}} = A_{\widetilde{i}_*} + \sum_{c^T_{\widetilde{j}}\in\mathcal{I}_c(\{T\})} b_{\widetilde{\ell},\widetilde{j}}\, c^T_{\widetilde{j}}, \qquad d_{\widetilde{\ell}} = d_{\widetilde{i}_*} - \sum_{\gamma_{\widetilde{j}}\in\mathcal{I}_h(\{T\})} \gamma_{\widetilde{j}}\, b_{\widetilde{\ell},\widetilde{j}}.$$

The rest of the proof follows from the essentially same argument as above and is thus omitted. Hence, the proposition holds on $[0,t_M] = [0,T]$ by the induction principle. $\square$

Consider the state trajectory $x(t, x^0)$ of a Lipschitz PAS on $[0, T]$, and the index sets $\mathcal{I}_c$ and $\mathcal{I}_h$ defined in (8)–(9). Suppose that $\{A_s : s \in \mathcal{I}_A\} = \{A_1, \ldots, A_r\}$, $\mathcal{I}_c = \{c_1^T, \ldots, c_p^T\}$, and $\mathcal{I}_h = \{\gamma_1, \ldots, \gamma_p\}$, where $p \le m(m-1)/2$ (cf. Remark 2.1), $A_i \in \mathbb{R}^{n\times n}$, $c_i \in \mathbb{R}^n$ and $\gamma_i \in \mathbb{R}$. Here $\|c_i\|_2 = 1$ for all i. Without loss of generality, we assume that x^0 is contained in the polyhedron $\mathcal{X}_1$, whose affine piece is defined by (A_1, d_1). Define

$$\mathbf{C}_p := \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_p^T \end{bmatrix} \in \mathbb{R}^{p\times n}, \quad \mathbf{T} := \begin{bmatrix} \mathbf{C}_p \\ \mathbf{C}_p A_1 \\ \vdots \\ \mathbf{C}_p (A_1)^n \end{bmatrix} \in \mathbb{R}^{(n+1)p\times n}, \quad \mathbf{h}_p := \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix} \in \mathbb{R}^p.$$

Moreover, define $q^j : [0, T] \to \mathbb{R}^p$ with $j = 1, \ldots, n+1$:

$$q^1(t) := \mathbf{C}_p\, x(t, x^0) - \mathbf{h}_p, \quad q^j(t) := \mathbf{C}_p \left[A_1^{j-1} x(t, x^0) + A_1^{j-2} d_1 \right], \quad j = 2, \ldots, n+1, \tag{10}$$

and define $\mathbf{q} : [0, T] \to \mathbb{R}^{(n+1)p}$ as

$$\mathbf{q}(t) := \begin{bmatrix} q^1(t) \\ \vdots \\ q^{n+1}(t) \end{bmatrix} = \mathbf{T}\, x(t, x^0) + \mathbf{d},$$

where $\mathbf{T} \in \mathbb{R}^{(n+1)p\times n}$, and $\mathbf{d} \in \mathbb{R}^{(n+1)p}$ is a constant vector.

Proposition 2.5. *Let $\mathbf{q}(t)$ be defined above for the Lipschitz PAS $(A_i, d_i, \mathcal{X}_i)_{i=1}^m$ with $\|A_i\|_2 \le \mu_A, \forall\, i = 1, \ldots, m$ for a constant $\mu_A > 0$. Then there exists a matrix-valued measurable function $\mathbf{G} : [0, T] \to \mathbb{R}^{(n+1)p\times(n+1)p}$ such that*

$$\dot{\mathbf{q}}(t) = \mathbf{G}(t)\, \mathbf{q}(t), \quad a.e. \quad [0, T].$$

Furthermore, there exists a constant $\mu_G > 0$ (depending on μ_A and m only) such that $\|\mathbf{G}(t)\|_2 \le \mu_G$ for all $t \in [0, T]$.

Proof. It follows from the (regular) non-Zenoness of the Lipschitz PAS [6, Theorem 3.5] or [39] that for any $t' \in [0, T]$, there exist $\varepsilon_{t'} > 0$ and some index $\ell \in \mathcal{I}_A$ such that $x(t, x^0) \in \mathcal{X}_\ell$ for all $t \in (t', t' + \varepsilon_{t'})$. Hence, on the open sub-interval $(t', t' + \varepsilon_{t'})$, we have

$$\dot{\mathbf{q}}(t) = \mathbf{T}\, \dot{x}(t, x^0) = \mathbf{T}\left[A_\ell\, x(t, x^0) + d_\ell \right], \quad \forall\, t \in (t', t' + \varepsilon_{t'}).$$

In light of Proposition 2.4 and letting $i_* = 1$, we have the vectors $b_{\ell,j} \in \mathbb{R}^n$ such that

$$A_\ell = A_1 + \sum_{j=1}^p b_{\ell,j}\, c_j^T, \qquad d_\ell = d_1 - \sum_{j=1}^p \gamma_j\, b_{\ell,j}.$$

Therefore, for any $k \in \mathbb{N}$,

$$(A_1)^k \left[A_\ell\, x(t, x^0) + d_\ell \right] = (A_1)^k \left[\Big(A_1 x(t, x^0) + d_1 \Big) + \sum_{s=1}^p b_{\ell,s} \Big(c_s^T x(t, x^0) - \gamma_s \Big) \right].$$

Consequently, for each $k = 0, 1, \ldots, n-1$,

$$\begin{aligned} \mathbf{C}_p (A_1)^k \left[A_\ell\, x(t, x^0) + d_\ell \right] &= \begin{bmatrix} c_1^T \Big(A_1^{k+1} x(t, x^0) + A_1^k d_1 \Big) + \sum_{s=1}^p c_1^T A_1^k b_{\ell,s} \Big(c_s^T x(t, x^0) - \gamma_s \Big) \\ \vdots \\ c_p^T \Big(A_1^{k+1} x(t, x^0) + A_1^k d_1 \Big) + \sum_{s=1}^p c_p^T A_1^k b_{\ell,s} \Big(c_s^T x(t, x^0) - \gamma_s \Big) \end{bmatrix} \\ &= q^{k+2}(t) + H_k\, q^1(t), \end{aligned}$$

where H_k is a $p \times p$ matrix whose elements are given by $c_j^T A_1^k b_{\ell,s}$.

Let the coefficients $a_{s,j}$ be such that $A_1^n = \sum_{j=0}^{n-1} a_{s,j} A_1^j$. Since $a_{s,j}$'s are continuous in A_1 and $\|A_1\|_2 \le \mu_A$ for all s, there exists a constant $\mu_a > 0$ (dependent on μ_A only) such that $|a_{s,j}| \le \mu_a$ for all s and j. Noting that $A_1^{n+1} x(t,x^0) + A_1^n d_1 = \sum_{j=0}^{n-1} a_{s,j}(A_1^{j+1} x(t,x^0) + A_1^j d_1)$, we have

$$\begin{aligned}
\mathbf{C}_p(A_1)^n \Big[A_\ell\, x(t,x^0) + d_\ell \Big] &= \begin{bmatrix} c_1^T \Big(A_1^{n+1} x(t,x^0) + A_1^n d_1 \Big) + \sum_{s=1}^p c_1^T A_1^n b_{\ell,s} \big(c_s^T x(t,x^0) - \gamma_s \big) \\ \vdots \\ c_p^T \Big(A_1^{n+1} x(t,x^0) + A_1^n d_1 \Big) + \sum_{s=1}^p c_p^T A_1^n b_{\ell,s} \Big(c_s^T x(t,x^0) - \gamma_s \Big) \end{bmatrix} \\
&= \sum_{j=0}^{n-1} a_{s,j}\, q^{j+2}(t) + H_n\, q^1(t),
\end{aligned}$$

where H_n is a $p \times p$ matrix whose elements are given by $c_j^T A_1^n b_{\ell,s}$. Since $\|A_i\|_2 \le \mu_A$, $\|c_i^T\|_2 = 1$ and $\|b_{\ell,j}\|_2 \le 2\mu_A$ for all i, ℓ, j, there exists a constant $\mu_H > 0$ such that $\max_{k=0,1,\ldots,n} \|H_k\|_2 \le \mu_H$.

By virtue of the above results, we obtain a matrix $\mathbf{G}_\ell \in \mathbb{R}^{(n+1)p \times (n+1)p}$ such that

$$\dot{\mathbf{q}}(t) \;=\; \mathbf{G}_\ell\, \mathbf{q}(t), \quad \forall\, t \in (t', t' + \varepsilon_{t'}).$$

Note that there are at most $|\mathcal{I}_A|$ copies of such the matrices $\mathbf{G}_i$, where $|\mathcal{I}_A| \le m$. Furthermore, in view of the uniform bounds on H_k and $a_{s,j}$, there is a uniform bound $\mu_G > 0$ (dependent on μ_A and m only) such that $\|\mathbf{G}_i\|_2 \le \mu_G$ for all $i = 1, \ldots, m$.

Let $\mathbf{1}_S$ denote the indicator function of a set S, and define

$$\mathbf{G}(t) \;:=\; \sum_{s=1}^m \mathbf{G}_s \cdot \mathbf{1}_{\{t \in [0,T] \,:\, x(t,x^0) \in \mathcal{X}_s\}}.$$

Consequently, if $t \in [0,T]$ is not a critical time, then

$$\dot{\mathbf{q}}(t) \;=\; \mathbf{G}(t)\, \mathbf{q}(t). \tag{11}$$

Due to the (regular) non-Zenoness of the Lipschitz PAS [6, Theorem 3.5] or [39], we see that for each s, the set $\big\{ t \in [0,T] : x(t,x^0) \in \mathcal{X}_s$ such that $\dot{x}(t,x^0) = A_s x(t,x^0) + d_s \big\}$ is a countable (in fact, finite) union of open intervals and thus is Borel measurable [12]. This implies that $\mathbf{G}(t)$ is a measurable (piecewise constant) function on $[0,T]$. Furthermore, since (11) holds except countably (in fact, finitely) many $t \in [0,T]$, it holds almost everywhere on $[0,T]$. Moreover, $\|\mathbf{G}(t)\|_2 \le \mu_G$ for all $t \in [0,T]$, where $\mu_G > 0$ depends on μ_A and m only. □

The next result states a critical property pertaining to linear transformations of $\mathbf{q}(t)$. Given the Lipschitz PAS $(A_i, d_i, \mathcal{X}_i)_{i=1}^m$ with $\|A_i\|_2 \le \mu_A, \forall\, i = 1, \ldots, m$ for some constant $\mu_A > 0$, define the set of matrix-vector pairs $\mathcal{S} := \{(A_i, d_i)\}_{i=1}^m$. Let $c_i^T \in \mathcal{I}_c, i = 1, \ldots, p$. Define the tuple

$$\psi_i := \left((c_i^T, \gamma_i),\ (S_{i;1,1}, \widetilde{d}_1),\ S_{i;2,1}(S_{i;2,2}, \widetilde{d}_2),\ S_{i;3,1} S_{i;3,2}(S_{i;3,3}, \widetilde{d}_3),\ \ldots,\ \left(\prod_{j=1}^{n-1} S_{i;n,j} \right) (S_{i;n,n}, \widetilde{d}_n) \right),$$

where each $S_{i;k,j} \in \{A_s : s \in \mathcal{I}_A\} \subseteq \{A_1, \ldots, A_m\}$ for $k = 1, \ldots, n$ and $j = 1, \ldots, k-1$, and each $(S_{i;k,k}, \widetilde{d}_k) \in \{(A_s, d_s) : s \in \mathcal{I}_A\} \subseteq \mathcal{S}$. Note that there are at most $(m^{n+1} - 1)/(m-1)$ such tuples.

Furthermore, define $q^{\psi_i}: [0,T] \to \mathbb{R}^{n+1}$ corresponding to ψ_i as

$$
\begin{aligned}
q^{\psi_i}(t) \;&:=\; \Big(q_1^{\psi_i}(t),\, q_2^{\psi_i}(t),\, \ldots,\, q_{n+1}^{\psi_i}(t)\Big)^T \qquad (12)\\
&=\; \bigg(c_i^T x(t,x_0) - \gamma_i,\; c_i^T\Big[S_{i;1,1}x(t,x^0) + \widetilde{d}_1\Big],\; c_i^T S_{i;2,1}\Big[S_{i;2,2}x(t,x^0) + \widetilde{d}_2\Big],\;\ldots,\\
&\qquad\qquad \ldots\ldots,\qquad \ldots\ldots,\qquad c_i^T\bigg(\prod_{j=1}^{n-1} S_{i;n,j}\bigg)\Big[S_{i;n,n}x(t,x^0) + \widetilde{d}_n\Big]\bigg)^T.
\end{aligned}
$$

Define the string of the tuples $\psi := (\psi_1, \ldots, \psi_p)$, and $\mathbf{q}^\psi : [0,T] \to \mathbb{R}^{(n+1)p}$ as

$$
\mathbf{q}^\psi(t) := \Bigg(\underbrace{q_1^{\psi_1}(t), q_1^{\psi_2}(t), \ldots, q_1^{\psi_p}(t)}_{\text{with subscript } 1},\; \underbrace{q_2^{\psi_1}(t), q_2^{\psi_2}(t), \ldots, q_2^{\psi_p}(t)}_{\text{with subscript } 2},\; \ldots\ldots,\; \underbrace{q_{n+1}^{\psi_1}(t), q_{n+1}^{\psi_2}(t), \ldots, q_{n+1}^{\psi_p}(t)}_{\text{with subscript } (n+1)}\Bigg)^T. \tag{13}
$$

Proposition 2.6. *Given the Lipschitz PAS $(A_i, d_i, \mathcal{X}_i)_{i=1}^m$ with $\|A_i\|_2 \le \mu_A, \forall\, i = 1, \ldots, m$ for some constant $\mu_A > 0$, consider $\mathbf{q}^\psi(t)$ defined above for some tuple ψ. Then there exists a matrix-valued measurable function $\mathbf{G}^\psi : [0,T] \to \mathbb{R}^{(n+1)p\times(n+1)p}$ such that*

$$
\dot{\mathbf{q}}^\psi(t) = \mathbf{G}^\psi(t)\,\mathbf{q}^\psi(t), \quad a.e. \quad [0,T].
$$

Furthermore, there exists a constant $\kappa > 0$ (depending on μ_A and m only) such that

$$
\sup_{\psi,\, t\in[0,T]} \|\mathbf{G}^\psi(t)\|_2 \le \kappa.
$$

Proof. Consider a fixed i and its corresponding tuple ψ_i. Since each $S_{i;k,j} \in \{A_s : s \in \mathcal{I}_A\}$, it follows from Proposition 2.4 that $S_{i;k,j} = A_1 + \sum_{r=1}^p b_{i;k,j,r}c_r^T$ for suitable vectors $b_{i;k,j,r}$ with $\|b_{i;k,j,r}\|_2 \le 2\mu_A$. Therefore, for each $k = 1, \ldots, n$,

$$
c_i^T \prod_{j=1}^{k-1} S_{i;k,j} = c_i^T(A_1)^{k-1} + \sum_{\ell=0}^{k-2}\sum_{s=1}^{p} \eta_{i,s,\ell}\, c_s^T(A_1)^\ell,
$$

where the coefficient $\eta_{i,s,\ell} = c_i^T\big(\prod_{j=1}^{k-2-\ell} S_{i;k,j}\big)\cdot b_{i;k,\,k-1-\ell,\,s}$. Due to the uniform bounds on $\|A_1\|_2, \|b_{i;k,j,s}\|_2$, and $\|c_i\|_2$, we deduce that there exists a uniform bound $\mu_\eta > 0$ (dependent on μ_A and m only but independent of ψ and (A_i, d_i)) such that $|\eta_{i,s,\ell}| \le \mu_\eta$ for all i, s, ℓ. Hence, given $k \in \{1, \ldots, n\}$ and letting $(S_{i;k,k}, \widetilde{d}_k) = (A_w, d_w) \in \{(A_s, d_s) : s \in \mathcal{I}_A\}$, we have

$$
\begin{aligned}
& c_i^T\bigg(\prod_{j=1}^{k-1} S_{i;k,j}\bigg)\Big[S_{i;k,k}x(t,x^0) + \widetilde{d}_k\Big]\\
&= \bigg(c_i^T(A_1)^{k-1} + \sum_{\ell=0}^{k-2}\sum_{s=1}^{p}\eta_{i,s,\ell}\,c_s^T(A_1)^\ell\bigg)\bigg[\big(A_1x(t,x^0) + d_1\big) + \sum_{s=1}^{p} b_{w,s}\big(c_s^Tx(t,x^0) - \gamma_s\big)\bigg]\\
&= q_i^{k+1}(t) + \sum_{\ell=1}^{k}\sum_{s=1}^{p}\phi_{i,\ell,s}\,q_s^\ell(t), \qquad (14)
\end{aligned}
$$

where $q_s^\ell(t)$ denotes the sth component of $q^\ell(t)$ defined in (10), and the coefficients $\phi_{i,\ell,s}$ are uniformly bounded, i.e., there exists a constant $\mu_\phi > 0$ (dependent on μ_A and m only but independent of ψ and (A_i, d_i)) such that $|\phi_{i,s,\ell}| \le \mu_\phi$ for all i, s, and ℓ.

Based on (14) and the definition of $\mathbf{q}^\psi(t)$, we obtain, for a given tuple ψ,

$$\mathbf{q}^\psi(t) \;=\; \underbrace{\begin{bmatrix} I_p & & & \\ \star & I_p & & \\ \vdots & \ddots & \ddots & \\ \star & \cdots & \star & I_p \end{bmatrix}}_{\mathbf{P}^\psi} \mathbf{q}(t),$$

where I_p is the identity matrix of order p, $\mathbf{P}^\psi \in \mathbb{R}^{(n+1)p\times(n+1)p}$, and $\star$ denotes the matrix blocks whose elements are such that their absolute values are uniformly bounded by μ_ϕ, regardless of ψ and (A_i, d_i). In view of this, we conclude that there exists a uniform bound on $\|\mathbf{P}^\psi\|_2$, regardless of ψ and (A_i, d_i). Let $\text{adj}(\cdot)$ denote the adjugate of a square matrix. In light of $\det(\mathbf{P}^\psi) = 1$ and $\big(\mathbf{P}^\psi\big)^{-1} = \text{adj}(\mathbf{P}^\psi)/\det(\mathbf{P}^\psi) = \text{adj}(\mathbf{P}^\psi)$, there exists a uniform bound on $\|\big(\mathbf{P}^\psi\big)^{-1}\|_2$, regardless of ψ and (A_i, d_i). In summary, there exists a constant $\mu_P > 0$, which depends on μ_A and m only, such that $\sup_\psi(\|\mathbf{P}^\psi\|_2, \|\big(\mathbf{P}^\psi\big)^{-1}\|_2) \le \mu_P$. Let $\mathbf{G}^\psi(t) := \mathbf{P}^\psi\, \mathbf{G}(t)\, \big(\mathbf{P}^\psi\big)^{-1}$. Hence, in view of (11),

$$\dot{\mathbf{q}}^\psi(t) \;=\; \mathbf{G}^\psi(t)\, \mathbf{q}^\psi(t), \quad \text{a.e.} \quad [0, T].$$

Due to the uniform bounds on $\|\mathbf{P}^\psi\|_2, \|\big(\mathbf{P}^\psi\big)^{-1}\|_2$ and $\|\mathbf{G}(t)\|_2$, we have

$$\kappa := \sup_{\psi, t\in[0,T]} \|\mathbf{G}^\psi(t)\|_2 < \infty.$$

where κ depends on μ_A and m only but is independent of ψ, t, and (A_i, d_i). $\square$

The above proposition can be extended to a family of Lipschitz PASs with uniformly bounded $\|A_i\|_2$ and m via an almost same argument. Precisely, we have the following corollary.

Corollary 2.2. *Given a family of Lipschitz PASs $\{(A_{\sigma,i}, d_{\sigma,i}, \mathcal{X}_{\sigma,i})_{i=1}^{m_\sigma}\}$. Suppose that there exist a constant $\mu_A > 0$ and $m_* \in \mathbb{N}$ such that for each σ, $\|A_{\sigma,i}\|_2 \le \mu_A, \forall\, i = 1, \ldots, m_\sigma$, and $m_\sigma \le m_*$. For a PAS indexed by σ, let $\mathbf{q}^\psi_\sigma(t)$ denote $\mathbf{q}^\psi(t)$ on $[0, T]$ defined in (13) for some tuple ψ. Then there exists a matrix-valued measurable function $\mathbf{G}^\psi_\sigma(t)$ such that*

$$\dot{\mathbf{q}}^\psi_\sigma(t) = \mathbf{G}^\psi_\sigma(t)\, \mathbf{q}^\psi_\sigma(t), \quad a.e. \quad [0, T].$$

Furthermore, there exists a constant $\kappa > 0$ (dependent on μ_A and m_* only) such that

$$\sup_{\psi, \sigma, t\in[0,T]} \|\mathbf{G}^\psi_\sigma(t)\|_2 \le \kappa. \tag{15}$$

2.3 Robust Non-Zenoness of Piecewise Affine Systems: Main Results

In this section, we establish a main robust non-Zeno result for a family of Lipschitz PASs. This result states that under suitable uniform bounds on the number of modes and subsystem matrices, there is a uniform bound on the number of mode switchings of each PAS on a given time interval.

Specifically, consider a family of Lipschitz PASs $\{(A_{\sigma,i}, d_{\sigma,i}, \mathcal{X}_{\sigma,i})_{i=1}^{m_\sigma}\}$ on $\mathbb{R}^n$, where σ is the index and $\Xi_\sigma := \{X_{\sigma,i}\}_{i=1}^{m_\sigma}$ is a polyhedral subdivision of $\mathbb{R}^n$ corresponding to PA pieces $\{A_{\sigma,i}x + d_{\sigma,i}\}_{i=1}^{m_\sigma}$. Let $x_\sigma(t, x^0)$ denote the state trajectory of the PAS indexed by σ from the initial state x^0. We introduce the following standing assumptions for robust non-Zenoness:

(H1) There exists an $m_* \in \mathbb{N}$ such that each PAS has at most m_* modes, i.e., $m_\sigma \le m_*, \forall\, \sigma$;

(H2) There exists a constant $\mu_A > 0$ such that $\|A_{\sigma,i}\|_2 \le \mu_A$ for each $i = 1, \ldots, m_\sigma$ and each σ.

Some equivalent assumptions are presented as follows. Since each polyhedron has at least one facet, it follows from Remark 2.1 that an equivalent condition for (**H1**) is:

(H1') There exists $p_* \in \mathbb{N}$ such that the total number of facets of all the polyhedra of each PAS indexed by σ is not greater than p_*.

Moreover, let $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be the right-hand side function of the PAS indexed by σ. It is easy to show, via [11, Proposition 4.2.2(c)] for example, that condition (**H2**) is equivalent to

(H2') The family of functions $\{f_\sigma\}$ is uniformly Lipschitz, i.e., there exists a real number $\mu > 0$ (independent of σ) such that for each σ,

$$\|f_\sigma(x) - f_\sigma(y)\|_2 \leq \mu\|x - y\|_2, \quad \forall\, x, y \in \mathbb{R}^n.$$

Given the fact that a mode switching occurs on a facet of a polyhedron, it is clear that condition (**H1**) is necessary for robust non-Zenoness. We show shortly that condition (**H2**) is not restrictive either. In fact, the violation of (**H2**) may yield the failure of robust non-Zenoness (see Example 2.1). Under these assumptions, we present a main result at follows; its proof is given in Section 2.3.1.

Theorem 2.2. *Consider a family of Lipschitz PASs* $\{(A_{\sigma,i}, d_{\sigma,i}, \mathcal{X}_{\sigma,i})_{i=1}^{m_\sigma}\}$ *on* $\mathbb{R}^n$*, each of which has a polyhedral subdivision* $\{\mathcal{X}_{\sigma,i}\}_{i=1}^{m_\sigma}$ *of* $\mathbb{R}^n$*. Suppose that conditions* (**H1**) $-$ (**H2**) *hold. Then for a time interval* $[0, T]$ *with* $T > 0$*, there exists* $N(T, \mu_A) \in \mathbb{N}$ *such that for any* σ *and any initial state* $x^0 \in \mathbb{R}^n$*, there are at most* $N(T, \mu_A)$ *critical times on* $[0, T]$ *along the state trajectory* $x_\sigma(t, x^0)$*.*

Since a switching time is a critical time, we obtain a direct consequence of Theorem 2.2.

Theorem 2.3. *Under the same conditions stated in Theorem 2.2, for a time interval* $[0, T]$ *with* $T > 0$*, there exists* $N'(T, \mu_A) \in \mathbb{N}$ *such that for any* σ *and any initial state* $x^0 \in \mathbb{R}^n$*, there are at most* $N'(T, \mu_A)$ *mode switchings on* $[0, T]$ *along the state trajectory* $x_\sigma(t, x^0)$*.*

Remark 2.3. It is noted that the robust non-Zenoness of PASs does not impose any assumption on the constant drift vector d_i in each subsystem. In other words, the robust non-Zenoness remains valid even if the constant drift vectors $d_{\sigma,i}$ are unbounded.

The following example shows that if the uniform bound on $\|A_{\sigma,i}\|_2$ is dropped, then robust non-Zenoness may fail.

Example 2.1. Consider a planar bimodal conewise linear system (CLS), i.e., $n = 2$. This system can be compactly written as: $\dot{x} = Ax + b\max(0, -c^T x)$, where $A \in \mathbb{R}^{2\times 2}$ and $b, c \in \mathbb{R}^2$. Suppose that A and $A - bc^T$ have complex eigenvalues $\mu_1 \pm \imath\omega_1$ and $\mu_2 \pm \imath\omega_2$ respectively, where ω_1, ω_2 are positive real numbers. It is shown in [5] via the Poincaré map that if $\frac{\mu_1}{\omega_1} + \frac{\mu_2}{\omega_2} = 0$, then the CLS has a periodic solution from any nonzero initial state with the constant period $\frac{\pi}{\omega_1} + \frac{\pi}{\omega_2}$. Let

$$A = \begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalues of A and $A - bc^T$ are $\pm\imath\omega_1$ and $\pm\imath\sqrt{\omega_1(\omega_1 + 1)}$, respectively. Clearly for a fixed $T > 0$, the number of mode switchings along a trajectory from a nonzero initial state is roughly proportional to ω_1 for all $\omega_1 > 0$ sufficiently large. Consequently, a uniform bound on the number of switchings of a family of the CLSs does not exist if $\|A_\sigma\|_2 \to \infty$ for an index sequence of σ.

2.3.1 Proof of Theorem 2.2

A state trajectory $x(t, x^0)$ is generally known to be only once time differentiable with respect to t, due to possible switching at t. In spite of this, the next lemma states a combinatorial property of zeros on a hyperplane. A similar idea is given in [4, 29].

Lemma 2.5. *Let $x(t, x^0)$ be a trajectory of a Lipschitz PAS whose subsystems are defined by the set of matrix-vector pairs $\mathcal{S} := \{(A_i, d_i)\}_{i=1}^m$, and $[t_1, t_2]$ be a time interval. Given $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. If $c^T x(t, x^0) - \gamma$ has $\prod_{j=1}^k (m^{j-1} + 1)$ zeros on $[t_1, t_2]$ for some $k \in \mathbb{N}$, then there exist $t_* \in (t_1, t_2)$, $S_j \in \{A_i\}_{i=1}^m$ with $j = 1, \ldots, k-1$, and $(S_k, d_k) \in \mathcal{S}$ such that $c^T \Big(\prod_{j=1}^{k-1} S_j \Big) \big[S_k\, x(t_*, x^0) + d_k \big] = 0$.*

Proof. We prove the lemma by induction on k. Consider $k = 1$. Since $c^T x(t, x^0) - \gamma$ has two zeros on $[t_1, t_2]$ and $c^T x(t, x^0) - \gamma$ is (time-)differentiable, there exists $t_* \in (t_1, t_2)$ such that $c^T \dot{x}(t_*, x^0) = 0$ by the mean-value theorem. Thus we have $c^T [A_i x(t_*, x^0) + d_i] = 0$ for some $(A_i, d_i) \in \mathcal{S}$. Hence the lemma holds. Now suppose that the lemma holds true for all $k = 1, \ldots, \ell$, where $\ell \in \mathbb{N}$. Consider $k = \ell + 1$. Without loss of generality, we assume that the zeros $\tau_i \in [t_1, t_2]$, $i = 1, \ldots, \prod_{i=1}^{\ell+1}(m^{i-1}+1)$, of $c^T x(t, x^0) - \gamma$ are in the strictly increasing order. For notational simplicity, let $p := \prod_{i=1}^{\ell}(m^{i-1}+1)$. Define the time interval $I_j := [\tau_{1+(j-1)p}, \tau_{j\,p}] \subset [t_1, t_2]$, where $j = 1, \ldots, m^\ell + 1$. Hence $c^T x(t, x^0) - \gamma$ has p zeros on each I_j. It follows from the induction hypothesis that for each j, there exist $\widetilde{\tau}_j \in (\tau_{1+(j-1)p}, \tau_{j\,p})$ and $S_{j,i} \in \{A_i\}_{i=1}^m$ such that $c^T \Big(\prod_{i=1}^{\ell-1} S_{j,i} \Big) \big[S_{j,\ell}\, x(\widetilde{\tau}_j, x^0) + d_\ell \big] = 0$, where $(S_{j,\ell}, d_\ell) \in \mathcal{S}$ and all the $\widetilde{\tau}_j$'s are distinct. Since the tuple $\{\, S_{j,1}, S_{j,2}, \ldots, S_{j,\ell-1}, (S_{j,\ell}, d_\ell)\}$ has m^ℓ combinations but $j = 1, \ldots, m^\ell + 1$, there must be two identical tuples, denoted by $\{S_1^\circ, S_2^\circ, \ldots, S_{\ell-1}^\circ, (S_\ell^\circ, d_\ell^\circ)\}$. Therefore, $c^T \Big(\prod_{i=1}^{\ell-1} S_i^\circ \Big) \big[S_\ell^\circ\, x(t, x^0) + d_\ell^\circ \big]$ has two distinct zeros on $[t_1, t_2]$. Using the mean-value theorem again, we obtain $t_* \in (t_1, t_2)$ and $(S_{\ell+1}^\circ, d_{\ell+1}) \in \mathcal{S}$ such that $c^T \Big(\prod_{i=1}^{\ell} S_i^\circ \Big) \big[S_{\ell+1}^\circ\, x(t_*, x^0) + d_{\ell+1} \big] = 0$. This show that the lemma holds for $k = \ell + 1$, and thus for all $k \in \mathbb{N}$. $\square$

The following lemma is due to Sussmann [38, Lemma A1]:

Lemma 2.6. *Let $\kappa > 0$ and $n \in \mathbb{N}$. Let $\Delta T > 0$ be such that $\Delta T < \min\Big(1, \dfrac{e^{-\kappa n}}{n^{\frac{3}{2}}\kappa}\Big)$. If $\phi_1(t), \ldots, \phi_n(t)$ are absolutely continuous functions on a time interval I of length ΔT that satisfy a linear system of differential equations:*

$$\dot{\phi}_i(t) = \sum_{j=1}^{n} \alpha_{ij}(t)\, \phi_j(t), \quad i = 1, \ldots, n, \quad a.e.\ \ I,$$

where the coefficients $\alpha_{ij}(t)$ are measurable real-valued functions on I such that $|\alpha_{ij}(t)| \le \kappa$ for all $1 \le i, j \le n$ and all $t \in I$, then either (i) all the $\phi_i(t)$ vanish identically on I, or (ii) at least one of $\phi_i(t)$ has no zeros on I.

Equipped with the above results, we are ready to prove Theorem 2.2 as follows.

Proof of Theorem 2.2. We prove the theorem by contradiction. Suppose not. Then for a fixed time interval $[0, T]$ with $T > 0$, there exists a sequence of the pairs $\{(\sigma_k, x^{0,k})\}$ such that the number of critical times on $[0, T]$ along the state trajectory $x_{\sigma_k}(t, x^{0,k})$, denoted by N_k, satisfies $N_k \to \infty$ as $k \to \infty$, where $x_{\sigma_k}(t, x^{0,k})$ represents the trajectory of the Lipschitz PAS indexed by σ_k in the given family from the initial state $x^{0,k}$.

For each Lipschitz PAS indexed by σ_k, let $(c_{\sigma_k,s}^T, \gamma_{\sigma_k,s})_{s=1}^{p_k}$ be a collection of all the facets of the associated polyhedral subdivision, where each $p_k \le m_*(m_* - 1)/2$ and each $\|c_{\sigma_k,s}\|_2 = 1$. At a critical time $t' \in [0,T]$ of $x_{\sigma_k}(t, x^{0,k})$, it follows from the discussion before Proposition 2.3 that $c^T_{\sigma_k,s}\, x_{\sigma_k}(t', x^{0,k}) - \gamma_{\sigma_k,s} = 0$ for some $s \in \{1,\ldots,p_k\}$. Since the number of critical times tends to infinity as $k \to \infty$, we deduce via [4, Lemma 8] that for any $M \in \mathbb{N}$, there exist a sub-interval $\mathcal{T} \subseteq [0,T]$ and $K \in \mathbb{N}$ such that for each $s \in \{1,\ldots,p_K\}$, $c^T_{\sigma_K,s}\, x_{\sigma_K}(t, x^{0,K}) - \gamma_{\sigma_K,s}$ either has no zeros on $\mathcal{T}$ or has more than M zeros on $\mathcal{T}$.

Let $\kappa > 0$ be defined in (15) for the family of Lipschitz PASs, where κ depends on μ_A and m_* only. Furthermore, define $p_* := m_*(m_*-1)/2$, $N_* := \prod_{j=1}^n \big(m_*^{j-1}+1\big)$, $M_* := \lceil T/\varepsilon_T \rceil \cdot N_*$, where $\lceil \cdot \rceil$ denotes the ceiling of a real number, and

$$\varepsilon_T := \frac{1}{2} \min\left(1, \; \frac{e^{-\kappa p_*(n+1)}}{\big[p_*(n+1)\big]^{3/2}\kappa}\right).$$

Now choose $M \geq M_* \in \mathbb{N}$. This implies that there exist a Lipschitz PAS indexed by σ_* from the given family, a time sub-interval $\mathcal{T}_* \subseteq [0,T]$, and an initial state $x^{0,*}$ such that $c^T_{\sigma_*,s}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s}$ has more than M_* zeros on $\mathcal{T}_*$ for $s = 1,\ldots,p$ with $p \leq p_*$, and each of the remaining $c^T_{\sigma_*,s}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s}$ has no zero on $\mathcal{T}_*$.

Since the length of $\mathcal{T}_*$ is no more than T, we obtain, by the definition of M_*, a time interval $\mathcal{T}_{\varepsilon_T} \subseteq \mathcal{T}_*$ of length $\varepsilon_T > 0$ such that (i) for each $s = 1,\ldots,p$, $c^T_{\sigma_*,s}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s}$ has more than N_* zeros on $\mathcal{T}_{\varepsilon_T}$; and (ii) each of the rest of $c^T_{\sigma_*,s}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s}$ has no zeros on $\mathcal{T}_{\varepsilon_T}$. Therefore, $\mathcal{I}_c(\mathcal{T}_{\varepsilon_T}) = \{c^T_{\sigma_*,s} : s = 1,\ldots,p\}$ and $\mathcal{I}_h(\mathcal{T}_{\varepsilon_T}) = \{\gamma_{\sigma_*,s} : s = 1,\ldots,p\}$. Since $m_{\sigma_*} \leq m_*$ and $N_* := \prod_{j=1}^n \big(m_*^{j-1}+1\big)$, it follows from Lemma 2.5 that for each $i = 1,\ldots,p$, there is a tuple

$$\psi^*_i := \left((c^T_i, \gamma_i),\; (S^*_{i;1,1}, d^*_1),\; S^*_{i;2,1}(S^*_{i;2,2}, d^*_2),\; S^*_{i;3,1}S^*_{i;3,2}(S^*_{i;3,3}, d^*_3),\; \ldots,\; \Big(\prod_{j=1}^{n-1} S^*_{i;n,j}\Big)(S^*_{i;n,n}, d^*_n) \right)$$

such that each component of the vector-valued function $q^{\psi^*_i}(t)$ defined in (12) based on ψ^*_i has at least one zero on $\mathcal{T}_{\varepsilon_T}$, where each $S^*_{i;k,j} \in \{A_{\sigma_*,s} : s \in \mathcal{I}_A(\mathcal{T}_{\varepsilon_T})\} \subseteq \{A_1,\ldots,A_{m_{\sigma_*}}\}$ for $k = 1,\ldots,n$ and $j = 1,\ldots,k-1$, and each $(S^*_{i;k,k}, d^*_k) \in \{(A_{\sigma_*,s}, d_{\sigma_*,s}) : s \in \mathcal{I}_A(\mathcal{T}_{\varepsilon_T})\} \subseteq \mathcal{S}$. Let $\psi^* := (\psi^*_1,\ldots,\psi^*_p)$, where $p \leq p_*$. This shows that each component of $\mathbf{q}^{\psi^*}(t)$ (cf. (13)) has at least one zero on $\mathcal{T}_{\varepsilon_T}$. For the given tuple ψ^*, it follows from Proposition 2.4 and Corollary 2.2 that there exists a measurable function $\mathbf{G}^{\psi^*}_{\sigma_*}(t)$ with $\sup_{t \in \mathcal{T}_{\varepsilon_T}} \|\mathbf{G}^{\psi^*}_{\sigma_*}(t)\|_2 \leq \kappa$ such that

$$\dot{\mathbf{q}}^{\psi^*}(t) \;=\; \mathbf{G}^{\psi^*}_{\sigma_*}(t)\, \mathbf{q}^{\psi^*}(t), \quad \text{a.e. } \mathcal{T}_{\varepsilon_T}.$$

Note that $\left| \big(\mathbf{G}^{\psi^*}_{\sigma_*}(t)\big)_{i,j} \right| \leq \big\|\mathbf{G}^{\psi^*}_{\sigma_*}(t)\big\|_2 \leq \kappa$ for any i, j and all $t \in [0,T]$. In view of Lemma 2.6, we deduced that each component of $\mathbf{q}^{\psi^*}(t)$ is identically zero on $\mathcal{T}_{\varepsilon_T}$. This thus shows that for each $s = 1,\ldots,p$, $c^T_{\sigma_*,s}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s} = 0$ for all $t \in \mathcal{T}_{\varepsilon_T}$. For the rest of s', either $c^T_{\sigma_*,s'}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s'} > 0$ or $c^T_{\sigma_*,s'}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,s'} < 0$ for all $t \in \mathcal{T}_{\varepsilon_T}$. On the other hand, since there are multiple critical times on $\mathcal{T}_{\varepsilon_T}$ along $x_{\sigma_*}(t, x^{0,*})$, there exists at least a critical time t' in the interior of $\mathcal{T}_{\varepsilon_T}$. By observing the discussions before Proposition 2.3, we conclude that at the critical time t', there exist an index $\widehat{s}$ and a constant $\varepsilon' > 0$ such that $(c^T_{\sigma_*,\widehat{s}}\, x_{\sigma_*}(t', x^{0,*}) - \gamma_{\sigma_*,\widehat{s}}) = 0$, and $(c^T_{\sigma_*,\widehat{s}}\, x_{\sigma_*}(t, x^{0,*}) - \gamma_{\sigma_*,\widehat{s}}) < 0$ for all $t \in (t', t'+\varepsilon')$. This yields a contradiction. $\square$

3 Robust Non-Zenoness of Non-Lipschitz Piecewise Affine Systems

In this section, we extend robust non-Zeno analysis to a family PASs with discontinuous (or equivalently non-Lipschitz) right-hand sides. The discontinuous property of this class of PASs renders many difficulties not only in switching dynamics analysis but also in fundamental issues, such as solution well-posedness. Partial robust non-Zeno results are obtained for well-posed biomdal PASs via the recent work on well-posed PASs and a technical result of Sussmann.

3.1 Non-Lipschitz PASs: Solution Concepts and Mode Switching

We introduce a non-Lipschitz PAS on a polyhedral subdivision first. Let $\Xi = \{\mathcal{X}_i\}_{i=1}^m$ be a polyhedral subdivision of $\mathbb{R}^n$, and let $\{(A_i, d_i)\}_{i=1}^m$ be a tuple of matrix-vector pairs, where the affine functions $A_i x + d_i$ need not be continuous on the boundary of polyhedral $\mathcal{X}_i$ in Ξ. This gives rise to the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$:

$$F(x) := \begin{cases} A_i x + d_i, & \text{if } x \in \operatorname{int} \mathcal{X}_i \\ \big\{A_i x + d_i : i \in \mathcal{I}(x)\big\}, & \text{if } x \in \cap_{i\in\mathcal{I}(x)} \mathcal{X}_i \end{cases},$$

where int denotes the interior of a set. A non-Lipschitz PAS is thus defined by the following differential inclusion (DI):

$$\dot{x} \in F(x). \tag{16}$$

Note that if the right-hand side of a PAS is continuous, then $F(x)$ is singleton for any x so that (16) becomes a Lipschitz PAS. Moreover, if each $d_i = 0$ and each polyhedron in Ξ is a cone, then (16) becomes a non-Lipschitz piecewise linear system (PLS).

In what follows, we introduce several widely studied solution concepts [8].

Definition 3.1. Given an initial state $x^0 \in \mathbb{R}^n$. A $\mathbb{R}^n$-valued function $x(t, x^0)$ with $x(0, x^0) = x^0$ and $t \in [0, \infty)$ is:

(1) a *weak solution* in the sense of Carathéodory (or simply a Carathéodory solution) if $x(t, x^0)$ is absolutely continuous in t and satisfies the DI (16) for almost all $t \in [0, \infty)$;

(2) a *forward Carathéodory solution* if it is a weak solution in the sense of Carathéodory and for any $t_0 \geq 0$, there exist the ith mode and a real $\varepsilon_{t_0} > 0$ such that $\dot{x}(t, x^0) = A_i x(t, x^0) + d_i$ for all $t \in (t_0, t_0 + \varepsilon_{t_0})$ (a backward Carathéodory solution can be defined in a similar way);

(3) a *Filippov solution* if $x(t, x^0)$ is absolutely continuous in t and satisfies the DI: $\dot{x} \in \operatorname{conv}(F(x))$ for almost all $t \in [0, \infty)$, where $\operatorname{conv}(\cdot)$ denotes the convex hull of a set.

Clearly, the concept of forward Carathéodory solutions is the most restrictive one among the three, and it rules out the existence of a left Zeno time but allows a right Zeno time [8, 17]. Furthermore, a Carathéodory solution is a Filippov solution but not vice versa in general. Extensive research has been carried out to characterize well-posedness of non-Lipschitz PASs under the above solution concepts. For instance, necessary and sufficient algebraic conditions are derived in [17] for well-posedness of forward Carathéodory solutions of bimodal PLSs and their extensions, and this issue is addressed in [8, 40] for Filippov solutions of bimodal PLSs and PASs, which has led to the non-Zeno property of well-posed bimodal PLSs and PASs. It should be pointed out that in spite of the above-mentioned progress, verifiable algebraic conditions for general well-posed non-Lipschitz PASs remain unavailable, even for forward Carathéodory solutions.

Given a solution $x(t, x^0)$ of a non-Lipschitz PAS (16) on a polyhedral subdivision, mode switching can be defined in the same way as in Definition 2.1. Moreover, for a forward Carathéodory solution $x(t, x^0)$, the index set $\mathcal{J}(x(t, x^0))$ is nonempty for any $t \geq 0$. Therefore, critical times along $x(t, x^0)$ can be defined, namely, a time t' is a critical time along $x(t, x^0)$ if $\mathcal{J}(x(t', x^0)) \neq \mathcal{I}(x(t', x^0))$.

3.2 Robust Non-Zenoness of Well-posed Bimodal Non-Lipschitz PASs

Robust non-Zeno analysis of a general non-Lipschitz PAS is much more challenging than that of its Lipschitz counterpart due to the discontinuity induced analytic difficulties. To elaborate on this, recall that one of important steps in robust non-Zeno analysis of Lipschitz PASs in Section 2.3 is Lemma 2.5, where one obtains zero derivatives using time differentiability of a state trajectory. However, a trajectory of a non-Lipschitz PAS (in any sense of Definition 3.1) is absolutely continuous and need not be time differentiable everywhere. This invalidates Lemma 2.5 and its subsequent argument. Another analytic difficulty pertains to solution well-posedness. It is known that if solution uniqueness of a switching system fails, then its trajectory may be Zeno [2, 34, 36]. On the other hand, there are few characterization results of general well-posed non-Lipschitz PASs except bimodal PASs, even when a relatively simple solution concept (e.g., forward Carathéodory solutions) is considered. For these reasons, we focus on bimodal non-Lipschitz PASs, whose well-posedness conditions are recently established by Çamlibel [8, 40].

A bimodal PAS consists of two modes characterized by affine functions $A_i x + d_i,\ i = 1, 2$ and a polyhedral subdivision $\Xi = \{\mathcal{X}_1, \mathcal{X}_2\}$ with $\mathcal{X}_i := \{x \in \mathbb{R}^n : (-1)^{i-1}(c^T x - \gamma) \geq 0\}$ for $i = 1, 2$, where $A_i \in \mathbb{R}^{n\times n}$, $d_i \in \mathbb{R}^n$, $0 \neq c \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$. In this case, the two polyhedra are halfspaces of $\mathbb{R}^n$ defined by $c^T x \geq \gamma$ and $c^T x \leq \gamma$, respectively. To describe the well-posedness conditions of the bimodal PASs, let h_i be the observability index of the pair $(c^T, A_i), i = 1, 2$, i.e., the largest integer such that $(c, A_i^T c, \ldots, (A^{h_i})^T c)^T$ has full row rank, and define the following matrices and vectors similar to those before Proposition 2.5:

$$T_i := \begin{bmatrix} c^T \\ c^T A_i \\ c^T A_i^2 \\ \vdots \\ c^T A_i^{h_i+1} \end{bmatrix} \in \mathbb{R}^{(h_i+2)\times n}, \quad p_i := \begin{bmatrix} -\gamma \\ c^T d_i \\ c^T A_i d_i \\ \vdots \\ c^T A_i^{h_i} d_i \end{bmatrix} \in \mathbb{R}^{(h_i+2)}, \qquad i = 1, 2.$$

It is shown in [40, Theorem 3.1] that the bimodal PAS has a unique Filippov solution for any initial state if the following conditions hold:

(H3) (i) $h_1 = h_2$; (ii) there exists a lower triangular matrix M of order $(h_1 + 2)$ with positive diagonal elements such that $T_1 = MT_2$ and $p_1 = Mp_2$; and (iii) the implication $[T_1 x + p_1 = 0] \Longrightarrow [A_1 x + d_1 = A_2 x + d_2]$ holds.

Furthermore, under (**H3**), any Filippov solution is both a forward and backward Carathéodory solution and thus is non-Zeno [40].

We introduce more notation. Let $h := h_1 = h_2$ under (**H3**). Since h is the observability index of both $(c^T, A_i), i = 1, 2$, we obtain real numbers $\alpha_{i,j}$ such that $c^T A_i^{h+1} = \sum_{j=0}^{h} \alpha_{i,j}\, c^T A_i^j$. Moreover, for any $x \in \mathbb{R}^n$,

$$T_i\Big(A_i x + d_i\Big) = \begin{bmatrix} c^T(A_i x + d_i) \\ \vdots \\ c^T A_i^{h+1} x + c^T A_i^h d_i \\ c^T A_i^{h+1}(A_i x + d_i) \end{bmatrix} = \begin{bmatrix} c^T A_i x + c^T d_i \\ \vdots \\ c^T A_i^{h+1} x + c^T A_i^h d_i \\ \sum_{j=0}^{h} \alpha_{i,j}\, c^T(A_i^{j+1} x + A_i^j d_i) \end{bmatrix} = W_i \begin{bmatrix} c^T x - \gamma \\ c^T A_i x + c^T d_i \\ \vdots \\ c^T(A_i^{h+1} x + A_i^h d_i) \end{bmatrix},$$

where the matrices

$$W_i := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ 0 & \alpha_{i,0} & \cdots & \cdots & \alpha_{i,h-1} & \alpha_{i,h} \end{bmatrix} \in \mathbb{R}^{(h+2)\times(h+2)}, \qquad i = 1, 2. \tag{17}$$

We present a technical lemma due to Sussmann that gives a bound on the number of zeros of absolutely continuous functions. This result plays a key role in robust non-Zeno analysis of bimodal non-Lipschitz PASs.

Lemma 3.1. *[37, Lemma 3] Let $\varphi_1, \ldots, \varphi_N$ be real-valued absolutely continuous functions that satisfy the system of equations for almost all $t \in [0, T]$:*

$$\begin{aligned} \dot\varphi_1(t) &= \alpha_{11}(t)\varphi_1(t) + \beta_1(t)\varphi_2(t) \\ \dot\phi_2(t) &= \alpha_{21}(t)\varphi_1(t) + \alpha_{22}(t)\varphi_2(t) + \beta_2(t)\varphi_3(t) \\ \vdots &= \quad \vdots \\ \dot\varphi_i(t) &= \alpha_{i1}(t)\varphi_1(t) + \alpha_{i2}(t)\varphi_2(t) + \cdots + \alpha_{ii}(t)\varphi_i(t) + \beta_i(t)\varphi_{i+1}(t) \\ \vdots &= \quad \vdots \\ \dot\varphi_N(t) &= \alpha_{N1}(t)\varphi_1(t) + \alpha_{N2}(t)\varphi_2(t) + \cdots + \alpha_{NN}(t)\varphi_N(t), \end{aligned}$$

where α_{ij}, $i = 1, \ldots, N, j = 1, \ldots, i$ and $\beta_i, i = 1, \ldots, N-1$ are real-valued measurable functions. Suppose that there exist positive constants $\nu < \mu$ such that for all suitable i, j, $|\alpha_{ij}(t)| \le \mu$ and $\nu \le |\beta_j(t)| \le \mu$ for all $t \in [0, T]$. Then there exists a positive real number $\delta(N, \mu, \nu)$ such that for any time sub-interval $\mathcal{T} \subseteq [0, T]$ of length not greater than $\delta(N, \mu, \nu)$, either $\varphi_1(t)$ is identically zero on $\mathcal{T}$ or $\varphi_1(t)$ has at most $N-1$ zeros on $\mathcal{T}$.

For a matrix $S = (s_{ij})$, let $\|S\|_{\max}$ denote the max norm of S, i.e., $\|S\|_{\max} := \max_{i,j} |s_{ij}|$.

Proposition 3.1. *Consider a family of bimodal PASs defined by $\{(A_{\sigma,i}, d_{\sigma,i}), c_\sigma, \gamma_\sigma\}$ satisfying (**H3**) with the observability index h_σ, the lower triangular matrix $M_\sigma = (m_{\sigma,ij})$, and the matrices $W_{\sigma,i}$ defined in (17), where σ is the index and h_σ may be different. If there exist two positive constants ν and μ with $\nu < \mu$ such that for each σ,*

(1) $\nu \le \frac{m_{\sigma,ii}}{m_{\sigma,(i+1)(i+1)}} \le \mu$ for all $i = 1, \ldots, h_\sigma + 1$, and

(2) $\max\big(\|W_{\sigma,1}\|_{\max}, \|M_\sigma W_{\sigma,2} M_\sigma^{-1}\|_{\max}\big) \le \mu$,

then the family of bimodal PASs is robust non-Zeno, namely, for a time interval $[0, T]$ with $T > 0$, there exists $N(T, \mu, \nu) \in \mathbb{N}$ such that for any σ and any initial state x^0, there are at most $N(T, \mu, \nu)$ switchings (resp. critical times) on $[0, T]$ along any Filippov solution $x_\sigma(t, x^0)$.

Proof. Consider a Filippov trajectory $x(t)$ of a bimodal PAS satisfying the condition (**H3**), where we drop the dependence on an initial state x^0 for notational simplicity when the setting is clear. Given the (common) observability index h, define the $\mathbb{R}^{(h+2)}$-valued functions q and $\widetilde{q}$ as

$$q(t) := \begin{bmatrix} c^T x(t) - \gamma \\ c^T A_1 x(t) + c^T d_1 \\ \vdots \\ c^T (A_1^{h+1} x(t) + A_1^h d_1) \end{bmatrix} = T_1\, x(t) + p_1, \quad \widetilde{q}(t) := \begin{bmatrix} c^T x(t) - \gamma \\ c^T A_2 x(t) + c^T d_2 \\ \vdots \\ c^T (A_2^{h+1} x(t) + A_2^h d_2) \end{bmatrix} = T_2\, x(t) + p_2.$$

Note that $q(t), \widetilde{q}(t)$ are absolutely continuous, and $q(t) = M\widetilde{q}(t)$ for a lower triangular matrix M with positive diagonal entries. It follows from condition (**H3**) and [40, Theorem 3.1] that $x(t)$ is a forward Carathéodory solution. Hence, for any $t_* \geq 0$, there exists a scalar $\varepsilon_{t_*} > 0$ such that for all $t \in (t_*, t_* + \varepsilon_{t_*})$, either (i) $\dot{x}(t) = A_1 x(t) + d_1$ or (ii) $\dot{x}(t) = A_2 x(t) + d_2$. For case (i), in view of the definition of W_1, we have $\dot{q}(t) = T_1\big(A_1 x(t) + d_1\big) = W_1 q(t)$ on $(t_*, t_* + \varepsilon_{t_*})$. For case (ii), we have

$$\dot{q}(t) = T_1\big(A_2 x(t) + d_2\big) = MT_2\big(A_2 x(t) + d_2\big) = MW_2\widetilde{q}(t) = MW_2M^{-1}q(t), \quad \forall\, t \in (t_*, t_* + \varepsilon_{t_*}).$$

It follows from the lower triangular structure of $M = (m_{ij})$ and the definition of W_2 in (17) that

$$MW_2M^{-1} = \begin{bmatrix} \star & \frac{m_{11}}{m_{22}} & 0 & 0 & \cdots & 0 \\ \star & \star & \frac{m_{22}}{m_{33}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ \star & \cdots & & \cdots & \star & \frac{m_{(h+1)(h+1)}}{m_{(h+2)(h+2)}} \\ \star & \cdots & & \star & \cdots & \star \end{bmatrix} \in \mathbb{R}^{(h+2)\times(h+2)},$$

where $\star$ denotes possibly nonzero elements in the matrix. For each $i = 1, 2$, define the sets

$$\mathcal{S}_i := \Big\{ t_* \geq 0 \,:\, \text{there exists } \varepsilon_{t_*} > 0 \text{ such that } \dot{x}(t) = A_i x(t) + d_i \text{ for all } t \in (t_*, t_* + \varepsilon_{t_*}) \Big\},$$

and $\mathcal{T}_i := \cup_{t_* \in \mathcal{S}_i} (t_*, t_* + \varepsilon_{t_*})$. Clearly, $\mathcal{S}_i$ is countable, and $\mathcal{T}_i$ is a countable union of open intervals in $\mathbb{R}$ and thus is Borel measurable. This gives rise to a measurable function

$$\mathbf{W}(t) := W_1 \cdot \mathbf{1}_{\mathcal{T}_1}(t) + MW_2M^{-1} \cdot \mathbf{1}_{\mathcal{T}_2}(t),$$

where $\mathbf{1}_S(t)$ denotes the indicator function, such that the absolutely continuous $q(t)$ satisfies $\dot{q}(t) = \mathbf{W}(t)q(t)$ for almost all $t \geq 0$.

Now consider a Filippov solution $x_\sigma(t)$ and its corresponding absolutely continuous function $q_\sigma : \mathbb{R}_+ \to \mathbb{R}^{(h_\sigma+2)}$ of a bimodal PAS indexed by σ from the given family. Hence, $\dot{q}_\sigma(t) = \mathbf{W}_\sigma(t)q_\sigma(t)$ for almost all $t \geq 0$. In light of the properties of $\mathbf{W}_\sigma(t)$ established above and the stated uniform bounds in the proposition, it follows from Sussmann's Lemma 3.1 that there exists a real number $\delta(h_\sigma, \mu, \nu) > 0$ such that for any time interval whose length is not bigger than $\delta(h_\sigma, \mu, \nu)$, $\big(q_\sigma(t)\big)_1 = c^T x_\sigma(t) - \gamma$ either is identically zero or has at most $(h_\sigma + 1)$ zeros on that time interval. For the first case, it is easy to show via condition (iii) of (**H3**) and the fact that $x_\sigma(t)$ is both a forward and a backward Carathéodory solution that $c^T x_\sigma(t) - \gamma$ is identically zero for all t. For the second case, since $0 \leq h_\sigma \leq n-1$, we define $\widetilde{\delta}(\mu, \nu) := \min_{0 \leq h_\sigma \leq n-1} \delta(h_\sigma, \mu, \nu) > 0$ such that $c^T x_\sigma(t) - \gamma$ has at most n zeros on the time interval of length no greater than $\widetilde{\delta}(\mu, \nu)$, regardless of σ. Therefore, the uniform bounds on the number of switching and critical times on a time interval $[0, T]$ along any Filippov solution $x_\sigma(t)$ follow readily. $\square$

4 Robust Non-Zenoness of Linear Complementarity Systems

In this section, we apply the robust non-Zeno results to a class of Lipschitz CLSs, i.e., linear complementarity systems (LCSs) [3, 16, 32]. As a special class of differential variational inequalities (DVIs) [21, 22, 23] at the interface between differential systems and constrained optimization, complementarity systems provide a unified modeling framework for applied problems containing dynamics, inequality constraints, and mode switching. From a dynamical system perspective, complementarity systems constitute a class of nonsmooth hybrid systems subject to state-dependent switchings determined by differential dynamics and complementarity conditions.

4.1 Linear Complementarity Systems: Switching and Non-Zeno Concepts

Given the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$, the linear complementarity system, denoted by LCS(A, B, C, D), is defined as

$$\dot{x} = Ax + Bu, \qquad 0 \le u \perp Cx + Du \ge 0, \tag{18}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $u \perp v$ means that two real vectors u and v are orthogonal, i.e., $u^T v = 0$. The latter condition in (18) is known as the linear complementarity problem (LCP) denoted by LCP(Cx, D) for a given x. In what follows, let SOL(Cx, D) denote the solution set of LCP(Cx, D).

The non-Zeno property of the LCS critically depends on the definitions of "modes" and "mode switching" of the system. This leads to several different non-Zeno notions, such as strong non-Zenoness and weak non-Zenoness [15, 22, 32, 33]. This fine classification of a seemingly intuitive concept turns out to be necessary and crucial due to mathematical subtleties associated with the complementarity condition. We introduce the notions of strong and weak non-Zenoness as follows.

Given a solution pair $\big(x(t), u(t)\big)$ of the LCS (18) for which we drop the dependence on x^0 for notational simplicity, let the fundamental triple of index sets for $\big(x(t), u(t)\big)$ be:

$$\begin{aligned}
\alpha(t) &:= \{\, j : u_j(t) > 0 = (\, Cx(t) + Du(t)\,)_j \,\}, \\
\beta(t) &:= \{\, j : u_j(t) = 0 = (\, Cx(t) + Du(t)\,)_j \,\}, \\
\gamma(t) &:= \{\, j : u_j(t) = 0 < (\, Cx(t) + Du(t)\,)_j \,\}.
\end{aligned}$$

Definition 4.1. For a solution pair $\big(x(t), u(t)\big)$ of the LCS (18), a time $t_* \ge 0$ is a *non-switching time in the strong sense* if there exist a real number $\varepsilon_* > 0$ and a triple of index sets $(\alpha_*, \beta_*, \gamma_*)$ such that for all $t \in [t_* - \varepsilon_*, t_* + \varepsilon_*]$, $\big(\alpha(t), \beta(t), \gamma(t)\big) = \big(\alpha_*, \beta_*, \gamma_*\big)$; otherwise, we call t_* a *switching time in the strong sense* or there is a *switching in the strong sense* at t_* along $\big(x(t), u(t)\big)$.

A solution pair $\big(x(t), u(t)\big)$ is *strongly non-Zeno* on a time interval $[0, T]$ if there are finitely many switchings in the strong sense on $[0, T]$ along $\big(x(t), u(t)\big)$. Equivalently, this implies that for each $t_* \in [0, T]$, there exist two real $\varepsilon_\pm > 0$ and two triples of index sets $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ such that

$$\begin{aligned}
\big(\alpha(t), \beta(t), \gamma(t)\big) &= \big(\alpha_+, \beta_+, \gamma_+\big), \quad \forall\, t \in (t_*, t_* + \varepsilon_+], \\
\big(\alpha(t), \beta(t), \gamma(t)\big) &= \big(\alpha_-, \beta_-, \gamma_-\big), \quad \forall\, t \in [t_* - \varepsilon_-, t_*).
\end{aligned}$$

To define switching and non-Zenoness in the weak sense for a solution pair $\big(x(t), u(t)\big)$, consider a linear differential algebraic equation (LDAE) characterized by a pair of disjoint index sets $(\theta, \bar{\theta})$ whose union is $\{1, \ldots, m\}$:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (\, Cx(t) + Du(t)\,)_\theta = 0, \quad u_{\bar{\theta}}(t) = 0. \tag{19}$$

It is clear that there are finitely many LDAEs (namely, modes) and that every solution pair $\big(x(t), u(t)\big)$ of the LCS must satisfy one of such the LDAEs for some pair $\big(\theta, \bar{\theta}\big)$ at each time.

Definition 4.2. A time $t_* \ge 0$ is a *non-switching time in the weak sense* if there exist a real number $\varepsilon_* > 0$ and a pair of index sets $(\theta_*, \bar{\theta}_*)$ such that $\big(x(t), u(t)\big)$ satisfies the LDAE (19) corresponding to $\big(\theta_*, \bar{\theta}_*\big)$ for all $t \in [t_* - \varepsilon_*, t_* + \varepsilon_*]$; otherwise, we call t_* a *switching time in the weak sense* or there is a *switching in the weak sense* at t_* along $\big(x(t), u(t)\big)$.

Similarly, $\big(x(t), u(t)\big)$ is *weakly non-Zeno* on a time interval $[0, T]$ if there are finitely many switchings in the weak sense on $[0, T]$ along $\big(x(t), u(t)\big)$. This means that for each time $t_* \in [0, T]$, there exist a scalar $\varepsilon_* > 0$ and two pairs of index sets $(\theta_+, \bar{\theta}_+)$ and $(\theta_-, \bar{\theta}_-)$ such that

(i) $\big(x(t), u(t)\big)$ satisfies the LDAE (19) corresponding to $(\theta_+, \bar{\theta}_+)$ for all $t \in (t_*, t_* + \varepsilon_*]$, and

(ii) $\big(x(t), u(t)\big)$ satisfies the LDAE (19) corresponding to $(\theta_-, \bar{\theta}_-)$ for all $t \in [t_* - \varepsilon_*, t_*)$.

Moreover, for a given x-trajectory and each $t_* \in [0, T]$, if the above positive scalar ε_* and the index partition hold for any u-trajectory in both forward-time and backward-time directions, then we call the LCS *uniformly weakly non-Zeno* along $x(t)$ on $[0, T]$. If the strong (resp. weak) non-Zenoness holds for any solution pair $\big(x(t), u(t)\big)$ of the LCS on any $[0, T]$, then we simply call the LCS *strongly non-Zeno* (resp. *(uniformly) weakly non-Zeno*).

Some important classes of LCSs are shown to be non-Zeno under certain singleton properties of the associated LCP solution set. Specifically, for a given LCS(A, B, C, D) and a matrix $F \in \mathbb{R}^{\ell \times m}$, we call that FSOL(Cx, D) is singleton for any $x \in \mathbb{R}^n$, if for each $x \in \mathbb{R}^n$, SOL(Cx, D) is nonempty and FSOL(Cx, D) is a singleton set. The singleton property implies that FSOL(Cx, D) is a Lipschitz piecewise linear function on $\mathbb{R}^n$ [6, 33]. The following non-Zeno results are established for LCS(A, B, C, D) under singleton properties [33]:

(1) If SOL(Cx, D) is singleton for any $x \in \mathbb{R}^n$, then the LCS is strongly non-Zeno;

(2) If both BSOL(Cx, D) and DSOL(Cx, D) are singleton for any $x \in \mathbb{R}^n$, then the LCS is uniformly weakly non-Zeno.

For a given matrix $F \in \mathbb{R}^{\ell \times m}$, let FSOL(Cx, D) be singleton for any $x \in \mathbb{R}^n$. The piecewise linear form of FSOL(Cx, D) and its conic subdivision play an important role in the subsequent robust non-Zeno analysis; for one thing, the (uniform) bounds on the number of polyhedral cones in a subdivision and their facets critically depend on construction of a conic subdivision of FSOL(Cx, D). To obtain a piecewise linear form of FSOL(Cx, D), consider $u \in$ SOL(Cx, D) for an arbitrary $x \in \mathbb{R}^n$. Let $w := Cx + Du$. Hence, there exists an index subset $\theta \subseteq \{1, \ldots, m\}$ such that $u_\theta \geq 0$, $u_{\overline{\theta}} = 0$, $w_{\overline{\theta}} \geq 0$, and $w_\theta = 0$, where $\overline{\theta}$ denotes the complement of θ. Hence, $G_\theta(D)v = Cx$, where $G_\theta(D) = \big[I_{\bullet\overline{\theta}} \quad -D_{\bullet\theta}\big]$ is the complementarity matrix of D corresponding to θ [9, Section 1.3] and $v = \big(w_{\overline{\theta}}, u_\theta\big) \geq 0$. Let $\psi \subseteq \{1, \ldots, m\}$ be the support of v, i.e., $\psi := \{i \,:\, v_i \neq 0\}$. (If ψ is an empty set, then $v = 0$.) Without loss of generality, we assume $\big(G_\theta(D)\big)_{\bullet\psi} \neq 0$. Therefore, there exists an index subset $\phi \subseteq \psi$ such that the columns of $\big(G_\theta(D)\big)_{\bullet\phi}$ are linearly independent and $\big(G_\theta(D)\big)_{\bullet\phi} v_\phi = Cx$ with $v_\phi > 0$ and $v_{\overline{\phi}} = 0$. This shows that

$$v_\phi = \Big(\big(G_\theta(D)\big)^T_{\bullet\phi} \big(G_\theta(D)\big)_{\bullet\phi}\Big)^{-1} \big(G_\theta(D)\big)^T_{\bullet\phi} Cx.$$

Let $\phi' := \theta \cap \phi$. Hence $z := F$SOL$(Cx, D) = Fu = F_{\bullet\phi'} v_{\phi'}$. (If ϕ' is empty, then $z = 0$.) This shows that FSOL$(Cx, D) : \mathbb{R}^n \to \mathbb{R}^\ell$ is a piecewise linear function with at most 2^{2m} linear pieces. We summarize this result as follows.

Lemma 4.1. *Given an $m \in \mathbb{N}$, then for any $F \in \mathbb{R}^{\ell \times m}$ such that $FSOL(Cx, D)$ is singleton for any $x \in \mathbb{R}^n$, $FSOL(Cx, D)$ admits a conic subdivision Ξ of $\mathbb{R}^n$ with at most 2^{2m} linear pieces.*

4.2 Robust Non-Zenoness of Linear Complementarity Systems with Singleton Properties

In this section, we show robust strong and weak non-Zenoness of a family of LCSs under singleton properties and uniform Lipschitz properties.

Theorem 4.1 (Robust Strong Non-Zenoness). *Consider a family of LCSs $\{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)\}$, where σ is the index. Assume that*

(1) For each σ, SOL$(C_\sigma x, D_\sigma)$ is singleton for any $x \in \mathbb{R}^n$, and $\{SOL(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz piecewise linear functions, i.e., there exists $\kappa_1 > 0$ such that

$$\big\| SOL(C_\sigma x, D_\sigma)\big\|_2 \le \kappa_1 \|x\|_2, \quad \forall\, \sigma,\, x \in \mathbb{R}^n;$$

(2) $\{A_\sigma x + B_\sigma SOL(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz piecewise linear functions, i.e., there exists $\kappa_2 > 0$ such that

$$\big\| A_\sigma x + B_\sigma SOL(C_\sigma x, D_\sigma)\big\|_2 \le \kappa_2 \|x\|_2, \quad \forall\, \sigma,\, x \in \mathbb{R}^n;$$

(3) $\{C_\sigma x + D_\sigma SOL(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz piecewise linear functions, i.e., there exists $\kappa_3 > 0$ such that

$$\big\| C_\sigma x + D_\sigma SOL(C_\sigma x, D_\sigma)\big\|_2 \le \kappa_3 \|x\|_2, \quad \forall\, \sigma,\, x \in \mathbb{R}^n.$$

Then the family of the LCSs $\{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)\}$ is robust strongly non-Zeno, i.e., on any finite time interval $[0,T]$, there exists $N(T, \kappa_1, \kappa_2, \kappa_3) \in \mathbb{N}$ such that for any initial state $x^0 \in \mathbb{R}^n$ and any index σ, the solution pair $\big(x_\sigma(t, x^0), u_\sigma(t, x^0)\big)$ of the LCS$(A_\sigma, B_\sigma, C_\sigma, D_\sigma)$ has at most $N(T, \kappa_1, \kappa_2, \kappa_3)$ switchings in the strong sense on $[0,T]$.

Proof. It follows from Assumption (1) and Lemma 4.1 (with $F = I_m$) that for each σ, SOL$(C_\sigma x, D_\sigma)$ is a Lipschitz continuous, piecewise linear function on $\mathbb{R}^n$ and thus admits a conic subdivision Ξ_σ of $\mathbb{R}^n$ with at most m_* linear pieces, where m_* depends on m only but is independent of σ. Hence, each $A_\sigma x + B_\sigma \mathrm{SOL}(C_\sigma x, D_\sigma)$ is a Lipschitz piecewise linear function on $\mathbb{R}^n$ with the same conic subdivision Ξ_σ. Under Assumption (2), each $A_\sigma x + B_\sigma \mathrm{SOL}(C_\sigma x, D_\sigma)$ has uniformly bounded matrices associated with linear pieces. This yields a family of Lipschitz CLSs indexed by σ whose system matrices are uniformly bounded by κ_2. By Theorem 2.3, this implies that for a given time interval $[0,T]$, there are at most $\widetilde{N}(T, \kappa_2)$ mode switchings in the sense of Definition 2.1 on $[0,T]$ along each trajectory $x_\sigma(t, x^0)$ of any LCS, regardless of x^0 and σ.

For a state trajectory $x_\sigma(t, x^0)$, consider two consecutive switching times $t_i, t_{i+1} \in [0,T]$ in the sense of Definition 2.1 along $x_\sigma(t, x^0)$. Let $z := x_\sigma(t_i, x^0)$. Since there is no switching on (t_i, t_{i+1}), there exists a matrix $\widetilde{A} \in \mathbb{R}^{n\times n}$ with $\|\widetilde{A}\|_2 \le \kappa_2$ such that $x_\sigma(t, x^0) = e^{\widetilde{A}(t-t_i)} z$ for all $t \in [t_i, t_{i+1}]$. In addition, let the conic subdivision Ξ_σ of SOL$(C_\sigma x, D_\sigma)$ be $\Xi_\sigma = \{\mathcal{P}_{\sigma,s}\}_{s=1}^{m_\sigma}$ and SOL$(Cx, D) = E_{\sigma,s} x$ for all $x \in \mathcal{P}_{\sigma,s}$ for some matrix $E_{\sigma,s}$ with $\|E_{\sigma,s}\|_2 \le \kappa_1$ for each s, where $m_\sigma \le m_*$. It is easy to see that $x_\sigma(t, x^0) = e^{\widetilde{A}(t-t_i)} z$ has at most $\widehat{N}(T, \kappa_2)$ mode switchings in the sense of Definition 2.1 on $[0,T]$ in the conic subdivision Ξ_σ, thus on $[t_i, t_{i+1}]$, where $\widehat{N}(T, \kappa_2)$ depends on T and κ_2 only (but independent of σ and z). Let τ_j, τ_{j+1} be two consecutive switchings of this kind, where $t_i \le \tau_j < \tau_{j+1} \le t_{i+1}$. Hence, there exists a matrix $\widetilde{E}$ with $\|\widetilde{E}\|_2 \le \kappa_1$ such that $u_\sigma(t, x^0) = \widetilde{E} x_\sigma(t, x^0) = \widetilde{E} e^{\widetilde{A}(t-\tau_i)} \widetilde{z} \ge 0$ for all $t \in [\tau_j, \tau_{j+1}]$, where $\widetilde{z} := e^{\widetilde{A}(\tau_j - t_i)} z$. Furthermore,

$$C_\sigma x_\sigma(t, x^0) + D_\sigma u_\sigma(t, x^0) = \big(C_\sigma + D_\sigma \widetilde{E}\big) e^{\widetilde{A}(t-\tau_i)} \widetilde{z} \ge 0, \quad \forall\, t \in [\tau_j, \tau_{j+1}].$$

Fix $k \in \{1, \dots, m\}$, and define the following real-valued function on $[\tau_j, \tau_{j+1}]$:

$$h_{\sigma,k}(t) \;:=\; \big(u_\sigma(t, x^0)\big)_k \;=\; \widetilde{E}_{k\bullet} e^{\widetilde{A}(t-\tau_i)} \widetilde{z}.$$

Since $\|\widetilde{E}\|_2 \le \kappa_1$, we have $\|\widetilde{E}_{k\bullet}\|_2 \le \kappa_1$ for any k. Therefore, $h_{\sigma,k}$ and its ℓth order derivatives with $\ell = 1, \dots, n-1$ satisfy a time-invariant linear ODE whose $n \times n$ system matrix is bounded by a positive constant depending on κ_1 and κ_2 only. It follows from [38, Lemma A1] of Sussmann (cf.

Lemma 2.6) that that either (i) $h_{\sigma,k}(t)$ is identically zero on $[\tau_j, \tau_{j+1}] \subseteq [0, T]$, or (ii) $h_{\sigma,k}(t)$ has at most $N'(T, \kappa_1, \kappa_2)$ (isolated) zeros on $[0, T]$, where $N'(T, \kappa_1, \kappa_2)$ depends on T, κ_1, κ_2 only. In case (ii), we see that between any two consecutive times where $h_{\sigma,k}$ is zero, $h_{\sigma,k}(t)$ must be strictly positive such that $\big(C_\sigma x_\sigma(t, x^0) + D_\sigma u_\sigma(t, x^0)\big)_k$ is identically zero. It follows from time continuity of $x_\sigma(t, x^0)$ and $u_\sigma(t, x^0)$ that $\big(C_\sigma x_\sigma(t, x^0) + D_\sigma u_\sigma(t, x^0)\big)_k$ is identically zero on $[\tau_j, \tau_{j+1}]$. This implies that the kth component of the solution pair $\big(x_\sigma(t, x^0), u_\sigma(t, x^0)\big)$ has at most $N'(T, \kappa_1, \kappa_2)$ switchings in the strong sense on $[\tau_j, \tau_{j+1}]$. In case (i) where $h_{\sigma,k}(t) \equiv 0$ on $[\tau_j, \tau_{j+1}]$, it follows from Assumption (3) and a similar argument for $u_\sigma(t, x^0)$ that for each k, $\big(C_\sigma x_\sigma(t, x^0) + D_\sigma u_\sigma(t, x^0)\big)_k$ is either identically zero on $[\tau_j, \tau_{j+1}]$ or has at most $N''(T, \kappa_2, \kappa_3)$ zeros on $[0, T]$, where $N''(T, \kappa_2, \kappa_3)$ depends on T, κ_2, κ_3 only. Similarly, this shows that the kth component of the pair $\big(x_\sigma(t, x^0), u_\sigma(t, x^0)\big)$ has at most $N''(T, \kappa_2, \kappa_3)$ switchings in the strong sense on $[\tau_j, \tau_{j+1}]$.

In light of the above results, there are at most $m \cdot \max(N'(T, \kappa_1, \kappa_2), N''(T, \kappa_2, \kappa_3))$ switchings in the strong sense on $[\tau_j, \tau_{j+1}]$, regardless of x^0 and σ. In total, we deduce that there are at most

$$N(T, \kappa_1, \kappa_2, \kappa_3) := \widetilde{N}(T, \kappa_2) \cdot \widehat{N}(T, \kappa_2) \cdot m \cdot \max(N'(T, \kappa_1, \kappa_2), N''(T, \kappa_2, \kappa_3))$$

mode switchings in the strong sense on $[0, T]$, where $N(T, \kappa_1, \kappa_2, \kappa_3)$ is independent of x^0 and σ. This yields robust strong non-Zenoness of the family of LCSs. $\square$

An important special case is when each D_σ is a P-matrix. Recall that a square matrix is a P-matrix if all its principal minors are positive; see [9] for other equivalent characterizations. The class of P-matrices plays a fundamental role in the LCP theory since $\mathrm{LCP}(q, M)$ has a unique solution for all q if and only if M is a P-matrix [9]. The P-class includes several important subclasses of matrices, e.g., positive definite matrices (but not necessarily symmetric). In the following, let $\varphi(D_\sigma)$ denote the smallest principal minor of D_σ.

Corollary 4.1. *Consider the family of LCSs$\{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)\}$, where each D_σ is a P-matrix. If there exist real numbers $\kappa > 0$ and $\chi > 0$ such that $\max(\|A_\sigma\|_2, \|B_\sigma\|_2, \|C_\sigma\|_2, \|D_\sigma\|_2) \leq \kappa$ and $\varphi(D_\sigma) \geq \chi$ for all σ, then the family of LCSs$\{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)\}$ is robust strongly non-Zeno.*

Proof. We shall show that under the given conditions, the LCSs satisfy the assumptions in Theorem 4.1. First of all, for each σ, due to the global solution existence and uniquness under the P-property, $\mathrm{SOL}(C_\sigma x, D_\sigma)$ is singleton for any x. Therefore, it suffices to show that $\{\mathrm{SOL}(C_\sigma x, D_\sigma)\}$ satisfies the uniform Lipschitz property; the rest of the proof follows directly from the uniform bounds on $\|A_\sigma\|_2$, $\|B_\sigma\|_2$, $\|C_\sigma\|_2$, and $\|D_\sigma\|_2$.

Toward this end, we consider complementarity cones associated with $\mathrm{LCP}(C_\sigma x, D_\sigma)$. For notational simplicity, we shall drop the subscript σ when the setting is clear. Specifically, for each index subset $\theta \subseteq \{1, \ldots, m\}$ whose complement is denoted by $\overline{\theta}$, define the polyhedral cone

$$\mathcal{C}_\theta = \left\{ q \in \mathbb{R}^m : \begin{bmatrix} -\big(D_{\theta\theta}\big)^{-1} & 0 \\ -D_{\overline{\theta}\theta}\big(D_{\theta\theta}\big)^{-1} & I_{\overline{\theta}\overline{\theta}} \end{bmatrix} q \geq 0 \right\},$$

where $D_{\theta\theta}$ denotes the principal matrix defined by θ. By the P-property of the matrix D, the union of $\mathcal{C}_\theta$'s is $\mathbb{R}^m$. Let the matrix

$$K_\theta = \begin{bmatrix} -\big(D_{\theta\theta}\big)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m \times m}.$$

It follows from the LCP theory that if $Cx \in \mathcal{C}_\theta$, then $\mathrm{SOL}(Cx, D) = K_\theta C x$. In what follows, we show that $\|\big(D_{\theta\theta}\big)^{-1}\|_2$, thus $\|K_\theta\|_2$, is uniformly bounded, regardless of θ and σ. Note that

$$\big(D_{\theta\theta}\big)^{-1} = \frac{\mathrm{adj}(D_{\theta\theta})}{\det(D_{\theta\theta})},$$

where adj denotes the adjugate of a square matrix. Without loss of generality, we assume that θ is nonempty. If $|\theta| = 1$, then $\|\big(D_{\theta\theta}\big)^{-1}\|_2 \leq 1/\chi$. If $2 \leq |\theta| \leq m$, then the absolute value of each element of $\operatorname{adj}(D_{\theta\theta})$ is bounded above by $(|\theta|-1)^2 \cdot \|D\|_1$. Hence, $\|\operatorname{adj}(D_{\theta\theta})\|_1 \leq |\theta|(|\theta|-1)^2 \cdot \|D\|_1 \leq m^3 m^{1/2}\kappa$ for any θ and σ. Furthermore, since $\det(D_{\theta\theta})$ is a principal minor of D, we have $\det(D_{\theta\theta}) \geq \chi$ for any θ and σ. This shows $\|\big(D_{\theta\theta}\big)^{-1}\|_2 \leq m^4\kappa/\chi$, regardless of θ and σ. This also leads to the uniform boundedness of $\|K_\theta\|_2$ and thus the uniform Lipschitz property of $\{\mathrm{SOL}(C_\sigma x, D_\sigma)\}$. □

The next result extends Theorem 4.1 to robust uniformly weak non-Zenoness under similar singleton and uniform Lipschitz properties.

Theorem 4.2 (Robust Uniform Weak Non-Zenoness)**.** *Consider a family of LCSs $\{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)\}$, where σ is the index. Assume that*

(1) For each σ, $B_\sigma SOL(C_\sigma x, D_\sigma)$ and $D_\sigma SOL(C_\sigma x, D_\sigma)$ are singleton for any $x \in \mathbb{R}^n$;

(2) $\{A_\sigma x + B_\sigma SOL(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz piecewise linear functions, i.e., there exists $\kappa_1 > 0$ such that

$$\big\|A_\sigma x + B_\sigma SOL(C_\sigma x, D_\sigma)\big\|_2 \leq \kappa_1 \|x\|_2, \quad \forall\, \sigma,\ x \in \mathbb{R}^n;$$

(3) $\{C_\sigma x + D_\sigma SOL(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz piecewise linear functions, i.e., there exists $\kappa_2 > 0$ such that

$$\big\|C_\sigma x + D_\sigma SOL(C_\sigma x, D_\sigma)\big\|_2 \leq \kappa_2 \|x\|_2, \quad \forall\, \sigma,\ x \in \mathbb{R}^n.$$

Then the family of the LCSs $\{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)\}$ is robust uniformly weakly non-Zeno, i.e., on any finite time interval $[0, T]$, there exists $N(T, \kappa_1, \kappa_2) \in \mathbb{N}$ such that for any initial state $x^0 \in \mathbb{R}^n$ and any index σ, the state trajectory $x_\sigma(t, x^0)$ of the LCS$(A_\sigma, B_\sigma, C_\sigma, D_\sigma)$ has at most $N(T, \kappa_1, \kappa_2)$ mode switchings in the uniform weak sense on $[0, T]$.

Proof. Since the proof is similar to that of Theorem 4.1, we will be brief on the overlapping part but focus more on the distinct part. First, it follows from the singleton property of $B_\sigma\mathrm{SOL}(C_\sigma x, D_\sigma)$, Lemma 4.1, and Assumption (2) that the family of the LCSs gives rise to a family of Lipschitz CLSs whose system matrices are uniformly bounded by κ_1 with uniformly bounded numbers of modes. This leads to a uniform bound $\widetilde{N}(T, \kappa_1)$ on the number of switchings of any state trajectory $x_\sigma(t, x^0)$ in the sense of Definition 2.1 on $[0, T]$. Hence, for two consecutive switching times t_i, t_{i+1} of this kind, we have $x_\sigma(t, x^0) = e^{\widetilde{A}(t-t_i)}z$ with $z := x_\sigma(t_i, x^0)$ and $\|\widetilde{A}\|_2 \leq \kappa_1$. Moreover, it can be shown via the singleton property of $D_\sigma\mathrm{SOL}(C_\sigma x, D_\sigma)$ and Assumption (3) that $C_\sigma x + D_\sigma\mathrm{SOL}(C_\sigma x, D_\sigma)$ is Lipschitz continuous and piecewise linear in x and admits a conic subdivision Ξ_σ of $\mathbb{R}^n$ with uniformly bounded numbers of pieces. In addition, coefficient matrices of its linear pieces are uniformly bounded by κ_2. This further shows that there exists $\widehat{N}(T, \kappa_1, \kappa_2)$ such that each component of $C_\sigma x_\sigma(t, x^0) + D_\sigma u_\sigma(t, x^0)$ is either identically zero or has at most $\widehat{N}(T, \kappa_1, \kappa_2)$ (isolated) zeros on $[t_i, t_{i+1}]$, regardless of $u_\sigma(t, x^0)$, x^0, and σ. By a similar argument as in Theorem 4.1, we conclude that there are at most

$$N(T, \kappa_1, \kappa_2) := \widetilde{N}(T, \kappa_1) \cdot \widehat{N}(T, \kappa_1, \kappa_2)$$

mode switchings in the weak sense on $[0, T]$ along $x_\sigma(t, x^0)$, regardless of $u_\sigma(t, x^0)$, x^0, and σ. □

A particular class of uniformly weak non-Zeno LCSs is when the matrix D is positive semidefinite plus (or simply PSD-plus) that appears in a wide range of applications (cf. [33]). One definition for a PSD-plus matrix D is that D is written as $D = FMF^T$ for some matrix F and a positive definite matrix M (not necessarily symmetric). This definition is matrix-theoretic analog of the"monotonicity-plus" property of a nonlinear mapping [11], which has been employed in the study of DVIs [23]. Under this condition, it is known that for any q, $\mathrm{SOL}(Fq, FMF^T)$ is nonempty and $F^T\mathrm{SOL}(Fq, FMF^T)$ is singleton. In light of this result, it is shown in [33] that the LCS (A, BF^T, FC, FMF^T) is uniformly weakly non-Zeno. In what follows, we establish a robust non-Zeno result for a family of PSD-plus LCSs with the help of Theorem 4.2.

Corollary 4.2. *Consider a family of LCSs* $\{(A_\sigma, B_\sigma F_\sigma^T, F_\sigma C_\sigma, F_\sigma M_\sigma F_\sigma^T)\}$*, where each* $M_\sigma \in \mathbb{R}^{\ell_\sigma \times \ell_\sigma}$ *is positive definite and* $F_\sigma \in \mathbb{R}^{m \times \ell_\sigma}$*,* $A_\sigma \in \mathbb{R}^{n \times n}$*,* $B_\sigma \in \mathbb{R}^{n \times \ell_\sigma}$*,* $C_\sigma \in \mathbb{R}^{\ell_\sigma \times n}$*. If there exist real numbers* $\kappa > 0$ *and* $\chi > 0$ *such that* $\max(\|A_\sigma\|_2, \|B_\sigma\|_2, \|C_\sigma\|_2, \|F_\sigma\|_2, \|M_\sigma\|_2) \leq \kappa$ *and* $\lambda_{\min}(\frac{M_\sigma + M_\sigma^T}{2}) \geq \chi$ *for all* σ*, then the family of the LCSs* $\{(A_\sigma, B_\sigma F_\sigma^T, F_\sigma C_\sigma, F_\sigma M_\sigma F_\sigma^T)\}$ *is robust uniformly weakly non-Zeno.*

Proof. We show that the family $\{F_\sigma^T \mathrm{SOL}(F_\sigma q, F_\sigma M_\sigma F_\sigma^T)\}$ is uniformly Lipschitz in q first. Given any $q \in \mathbb{R}^{\ell_\sigma}$, let $d := F_\sigma^T u$ for any $u \in \mathrm{SOL}(F_\sigma q, F_\sigma M_\sigma F_\sigma^T)$. Hence, $d^T(q + M_\sigma d) = 0$. By the Cauchy-Schwarz inequality, we have $0 = d^T(q + M_\sigma d) \geq d^T M_\sigma d - \|q\|_2 \cdot \|d\|_2$ such that $d^T M_\sigma d \leq \|q\|_2 \cdot \|d\|_2$. Since M_σ is positive definite and $\lambda_{\min}(\frac{M_\sigma + M_\sigma^T}{2}) \geq \chi$ for all σ, we see that $\|d\|_2 \leq \|q\|_2 / \chi$ for any q and σ. This establishes the uniform Lipschitz property of $\{F_\sigma^T \mathrm{SOL}(F_\sigma q, F_\sigma M_\sigma F_\sigma^T)\}$. The rest of the proof follows directly from the uniform bound on $\|A_\sigma\|_2, \|B_\sigma\|_2, \|C_\sigma\|_2, \|F_\sigma\|_2, \|M_\sigma\|_2$ and Theorem 4.2. $\square$

5 Conclusion

In this paper, we establish a uniform bound on the number of mode switchings and critical times for a family of Lipschitz PASs whose right-hands are uniformly Lipschitz. This result is applied to several classes of Lipschitz linear complementarity systems under different switching notions. Partial results are also obtained for non-Lipschitz PASs, particularly well-posed bimodal PASs. An open question is whether the robust non-Zeno property holds for general well-posed non-Lipschitz PASs. Addressing this question calls for comprehensive understanding of well-posedness, which will be a future research topic. The robust non-Zeno analysis of PASs sheds light on that of piecewise (nonlinear) smooth systems that will also be addressed. Furthermore, application of robust non-Zenoness to sensitivity and uncertainty analysis of PASs will be studied in the future.

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