Observability Analysis of Conewise Linear Systems via Directional Derivative and Positive Invariance

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Abstract

Belonging to the broad framework of hybrid systems, conewise linear systems (CLSs) form a class of Lipschitz piecewise linear systems subject to state-triggered mode switchings. Motivated by state estimation of nonsmooth switched systems in applications, we exploit directional derivative and positive invariance techniques to characterize finite-time and long-time local observability of a general CLS. For the former observability notion, directional derivative results are developed via the simple switching property, and these results yield new or improved observability conditions. For the latter notion, we focus on the case where a nominal trajectory has finitely many switchings. In order to characterize long-time behaviors of the CLS, necessary and sufficient conditions are obtained for the interior of a positively invariant cone. By employing these conditions, we establish connections between finite-time and long-time local observability; underlying positive invariance properties are unveiled.

1 Introduction

Introduced in [7] for modeling Lipschitz piecewise linear systems, the conewise linear system (CLS) constitutes an important class of linear hybrid systems. Such a system consists of a finite collection of linear dynamical systems which are active on polyhedral cones that partition the entire state space. Each of these linear systems, together with its associated polyhedral cone, is called a mode of the system; transitions occur between these modes along a state trajectory. The CLS represents a large number of piecewise linear systems, e.g., Lipschitz linear complementarity systems [14], and has found broad applications in nonsmooth mechanics, switched electrical networks and control systems, and dynamical optimization in operations research and economics. See the recent papers [6, 7, 24, 27, 28] as well as the references therein for various results. An important feature of the CLS is that it is subject to state-dependent mode switchings with implicit transition times and implicit mode selection at switching times. The state-dependent switchings render many dynamical and control issues rather complicated, albeit critical in applications.

The notion of observability is fundamental and profound in systems and control theory and has been treated in great depth for smooth systems [16], which has led to important applications such as observer design [18] and asymptotic stability analysis [15, 20]. Observability of hybrid

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and switched systems, particularly linear hybrid systems and piecewise affine systems (PASs), has received growing interest in the past few years. State and mode observability of discrete-time switched linear systems is addressed in [2]; extension to continuous-time dynamics is made in [3]. Observability test and observer design of PASs are discussed in [9]. Bemporad *et al.* study discrete-time PASs with control inputs and logic-based mode switchings in [5]; computational issues are addressed. The paper [31] establishes necessary and sufficient conditions for observability of jump linear systems, under the assumption that mode switchings are arbitrary. Other related literature includes observability and detectability of jump Markov linear systems [10] and observability of discrete-event states of hybrid systems [12]. However, most of these papers assume state-irrelevant arbitrary mode switchings, and much less attention has been paid to state-dependent switchings. An exception is the recent paper [7] which initiates an extensive study of observability of the CLS with a linear output. Also see [21] for observability of nonlinear complementarity systems.

Inspired by state estimation of switched and hybrid systems in emerging applications such as genetic regulatory networks and biochemical systems [11, 29], the present paper carries out further observability analysis, especially for finite-time and long-time local observability (cf. Section 4). For the former observability notion, we develop directional derivative results and exploit them to obtain new or improved observability conditions. For the latter, we focus on the case where a nominal trajectory eventually remains in a polyhedral cone of the CLS, and we address the question of whether the two observability notions are equivalent, particularly whether long-time observability implies T-time observability for some large $T > 0$, since long-time observability is usually more difficult to check. We show that the answer is negative in general unless certain positive invariance conditions are imposed. The main contributions of the paper are threefold: (1) new directional derivative results are established based on the simple switching property (cf. Section 2); (2) necessary and sufficient conditions for the interior of positively invariant cones of the CLS are obtained; (3) finite-time and long-time observability conditions are developed for a general CLS, with the aid of directional derivative and positive invariance results developed above. It should be pointed out that partial results are obtained for finite-time and long-time observability in [7], especially for the bimodal CLS (cf. Example 1). However, further investigation for a general CLS is stalled in [7] due to the lack of the simple switching property and tools to handle long-time dynamics. Recently established in [26], the simple switching property turns out to a cornerstone for rigorous study of switchings of a general CLS. It leads to new directional derivative conditions for finite-time observability analysis. In addition, positive invariance analysis provides a major technique to deal with long-time dynamics of the CLS. By extending the lately developed positive invariance results in [25, 26], the current paper not only generalizes certain observability results reported in [7] and reveals the underlying positive invariance properties previously ignored, but it also attains new observability results, either related to or described by positive invariance.

The remaining of the paper is organized as follows. In Section 2, we introduce the CLS and establish new mode switching and directional derivative results that are crucial for latter observability analysis. Section 3 treats the positively invariant cone associated with each mode; necessary and sufficient conditions are developed to characterize the interior of the positively invariant cone. We then address finite-time and long-time local observability of the CLS in Section 4 via the directional derivative and positive invariance results. The paper concludes with final remarks and discussions for future research in Section 5.

2 Conewise Linear Systems: Mode Switching and Directional Derivative

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called piecewise linear if there exists a finite family of linear functions $\{f_i\}_{i=1}^{\ell}$ such that $f(x) \in \{f_i(x)\}_{i=1}^{\ell}$ for each $x \in \mathbb{R}^n$ [13, 23]. A continuous and piecewise linear function possesses an appealing geometric structure for its domain, which provides an alternative representation of the function. To be more specific, recall that a family of polyhedral cones $\{\mathcal{X}_i\}_{i=1}^m$ with $\mathcal{X}_i \equiv \{x \in \mathbb{R}^n \mid C_i\, x \geq 0\}$ forms a *conic subdivision* of $\mathbb{R}^n$ [13] if

(a) the union of all the polyhedral cones is equal to $\mathbb{R}^n$, i.e., $\bigcup_{i=1}^m \mathcal{X}_i = \mathbb{R}^n$,

(b) each cone is solid, i.e., it has a nonempty interior (thus is of dimension n), and

(c) the intersection of any two cones is a common proper face of both cones, i.e., there exist nonempty index sets α and β such that $\mathcal{X}_i \cap \mathcal{X}_j = \mathcal{X}_i \cap \{x \mid (C_i\, x)_\alpha = 0\} = \mathcal{X}_j \cap \{x \mid (C_j\, x)_\beta = 0\}$, where $(C_i x)_\alpha \equiv \{(C_i x)_\ell \mid \ell \in \alpha\}$ and $(C_j x)_\beta \equiv \{(C_j x)_\ell \mid \ell \in \beta\}$.

For a continuous and piecewise linear function f, one can always find a conic subdivision and finitely many linear functions $g_i(x) \equiv A_i x$ such that f coincides with one of g_i's on each polyhedral cone [13, Proposition 4.2.1]. Hence a time-invariant ODE system with a continuous and piecewise linear right-hand side can be put in the following equivalent form

$$\dot{x} \;=\; A_i\, x, \qquad \forall\; x \;\in\; \mathcal{X}_i \equiv \{x \mid C_i\, x \geq 0\}, \quad i = 1, \cdots, m, \tag{1}$$

where $A_i \in \mathbb{R}^{n \times n}$, $C_i \in \mathbb{R}^{m_i \times n}$, and the continuity condition holds:

$$x \in \mathcal{X}_i \cap \mathcal{X}_j \implies A_i\, x = A_j\, x. \tag{2}$$

We call the system (1) a *conewise linear system* (CLS) on $\mathbb{R}^n$ [7, 26], and each of the linear systems along with its associated polyhedral cone a *mode* of the CLS. The right-hand side of (1) is globally Lipschitz, albeit non-differentiable, in x and hence the CLS has a unique C^1 state trajectory, denoted by $x(t, x^0)$, for an initial state x^0 and all t. Note that $x(t, x^0)$ is generally only once time differentiable and not differentiable in x^0. We assume, without losing generality, that each C_i has no zero rows. The interior of each polyhedral cone $\mathcal{X}_i$ is thus given by $\mathrm{int}\mathcal{X}_i = \{x \mid C_i x > 0\}$.

Associated with the "forward-time" system (1) is a *backward-time* (or *reverse-time*) system that allows us to obtain reverse-time results easily from forward-time analysis. Specifically, for any terminal time $T > 0$, define $x^r(t) \equiv x(T - t)$. Hence $x^r(0) = x(T)$ and $\dot{x}^r = \widetilde{A}_i\, x^r$, if $x^r \in \mathcal{X}_i$, where $\widetilde{A}_i \equiv -A_i$. The latter system remains a CLS. In particular, the reverse-time CLS has a unique state trajectory for any initial condition.

It should be pointed out that a CLS may admit multiple conic subdivisions. Moreover, the converse implication of (2) may fail for a general conic subdivision; see Example 2 below. If the converse implication holds for a conic subdivision, or equivalently $x \in \mathcal{X}_i \cap \mathcal{X}_j \Leftrightarrow A_i\, x = A_j\, x$, we call this conic subdivision *simple*. If a CLS is described by a simple conic subdivision, then the CLS is called *simple*. We illustrate these notions in the following examples.

Example 1. A bimodal CLS is referred to as the CLS with two modes only. Each polyhedral cone of the bimodal CLS is a half-space of $\mathbb{R}^n$ [7, Example 2.1] such that the CLS takes the form

$$\dot{x} \;=\; \begin{cases} A_1 x & \text{if} \quad x \in \mathcal{X}_1 \equiv \{\, x \mid c^T x \;\geq\; 0\} \\ A_2 x & \text{if} \quad x \in \mathcal{X}_2 \equiv \{\, x \mid c^T x \;\leq\; 0\} \end{cases} \tag{3}$$

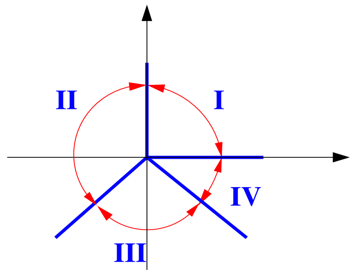

Figure 1: The conic subdivision of $\mathbb{R}^2$ in Example 2.

where $c \in \mathbb{R}^n$, and A_1 and A_2 satisfy $A_1 - A_2 = bc^T$ for $b \in \mathbb{R}^n$. Letting $A_2 \equiv A$, then $A_1 = A + bc^T$ and the bimodal CLS can be put in the compact form: $\dot{x} = Ax + b\max(0, c^T x)$. To avoid triviality, we assume that the vectors b and c are nonzero. In this case, we see that $A_1 x = A_2 x$ if and only if $c^T x = 0$; the latter is equivalent to $x \in \mathcal{X}_1 \cap \mathcal{X}_2$. This gives rise to a simple CLS.

Example 2. Consider the CLS on $\mathbb{R}^2$ with four modes, where the defining matrices of the linear dynamics are:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

and the corresponding matrices for the polyhedral cones are:

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

See Figure 1 for the associated conic subdivision of $\mathbb{R}^2$. It is easy to verify that the continuity condition (2) holds. As $A_2 = A_3$, the CLS possesses multiple conic subdivisions of $\mathbb{R}^2$ with different partitions of the union $\mathcal{X}_2 \cup \mathcal{X}_3$. However, since the cone $\mathcal{X}_2 \cup \mathcal{X}_3$ is non-convex, any conic subdivision must have at least two solid (convex) polyhedral cones, on which the linear functions coincide with $A_2 x$, to cover $\mathcal{X}_2 \cup \mathcal{X}_3$. Obviously, the converse implication (2) fails for any of such conic subdivisions. As a result, this CLS is not simple.

The significance of a simple CLS is demonstrated in the following lemma, which yields simplified conditions for the CLS; see, for instance, Proposition 19 and Corollary 20 in Section 4 for details.

Lemma 3. Consider a simple CLS on $\mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$. Then $x(t, x^0) \in \mathcal{X}_i$ on $[0, T]$ with $T > 0$ if and only if $x(t, x^0) = e^{A_i t} x^0, \forall\, t \in [0, T]$.

Proof. The "only if" part is trivial. To show the "if" part, note that $\dot{x}(t, x^0) = A_i x(t, x^0), \forall\, t \in [0, T)$. Moreover, it follows from [7, Lemma 3.3 and Theorem 3.5] that for any $t_* \in [0, T)$, there exist $\varepsilon_{t_*} > 0$ and an index j (possibly different from i) such that $t_* + \varepsilon_{t_*} \le T$ and that $x(t, x^0) = e^{A_j (t - t_*)} x(t_*, x^0), \forall\, t \in [t_*, t_* + \varepsilon_{t_*}]$; the latter shows that $\dot{x}(t, x^0) = A_j e^{A_j (t - t_*)} x(t_*, x^0), \forall\, t \in [t_*, t_* + \varepsilon_{t_*})$. By uniqueness of the CLS trajectories and their time derivatives, we have $x(t, x^0) = e^{A_j (t - t_*)} x(t_*, x^0) = e^{A_i (t - t_*)} x(t_*, x^0)$ so that $A_i x(t, x^0) = A_j x(t, x^0), \forall\, t \in [t_*, t_* + \varepsilon_{t_*})$. Since the conic subdivision is simple, we must have $x(t, x^0) \in \mathcal{X}_i \cap \mathcal{X}_j, \forall\, t \in [t_*, t_* + \varepsilon_{t_*})$. Hence, $x(t, x^0) \in \mathcal{X}_i, \forall\, t \in [0, T)$. Finally, it follows from the continuity of $x(t, x^0)$ and the closedness of $\mathcal{X}_i$ that $x(T, x^0) \in \mathcal{X}_i$. This completes the proof. $\square$

If a CLS is not simple, then the "if" part of the above lemma may fail. Consider Example 2 with $x^0 = (0, -1)^T$. It is easy to show that $x(t, x^0) = e^{A_3 t} x^0 = (-\sin t, -\cos t)^T, \forall\ t \in [0, \pi]$. Hence the trajectory $x(t, x^0)$ is in the cone $\mathcal{X}_3$ for all $t \in [0, \pi/4]$ but leaves $\mathcal{X}_3$ afterwards.

2.1 Mode Switching Properties of the CLS

In this section, we develop mode switching properties of the CLS. Particularly, we show the finite occurrence of critical times on a compact time interval (cf. Proposition 7). This result is important to finite-time observability analysis performed in Section 4.

We introduce more notation and relevant results for the following development. An ordered real ℓ-tuple $a = \big(a_1, \cdots, a_\ell\big)$ is called lexicographically nonnegative if either $a = 0$ or its first nonzero element (from the left) is positive and we write $a \succcurlyeq 0$. If a is not only lexicographically nonnegative but also nonzero, then a is called lexicographically positive and we write $a \succ 0$. Given two ℓ-tuples a and b, we write $a \succcurlyeq (\succ) b$ if $a - b \succcurlyeq (\succ) 0$. An n-dimensional vector tuple $\big(x^1, \cdots, x^\ell\big)$ is called lexicographically nonnegative (resp. positive) if each real tuple $\big(x_i^1, \cdots, x_i^\ell\big)$ is lexicographically nonnegative (resp. positive) for all $i = 1, \cdots, n$, and we write $\big(x^1, \cdots, x^\ell\big) \succcurlyeq (\succ) 0$. For each $i = 1, \ldots, m$, let $\mathcal{Y}_i \equiv \big\{\, x \in \mathbb{R}^n \,\big|\, \big(\, C_i x, C_i A_i x, \ldots, C_i A_i^{n-1} x \,\big) \succcurlyeq 0 \,\big\}$ be the semiobservable cone associated with the pair (C_i, A_i). Given any x^0, $x(t, x^0) \in \mathcal{X}_i$ for all $t \geq 0$ sufficiently small if and only if $x^0 \in \mathcal{Y}_i$ [7]. For a pair (C_i, A_i), let $\overline{O}(C_i, A_i)$ be its unobservable subspace. Given $\xi \in \mathbb{R}^n$, define two index sets $\mathcal{I}(\xi) \equiv \{ i \,|\, \xi \in \mathcal{X}_i \}$ and $\mathcal{J}(\xi) \equiv \{ i \,|\, \xi \in \mathcal{Y}_i \}$. It is obvious that $\mathcal{J}(\xi) \subseteq \mathcal{I}(\xi), \forall\ \xi \in \mathbb{R}^n$. Similarly, we can define $\mathcal{J}^{\,r}(\xi)$ for the associated reverse-time system.

Definition 4. We say that a time instant $t_* \geq 0$ is *not* a switching time along a state trajectory $x(t, x^0)$ if there exist $i \in \{1, \cdots, m\}$ and $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i, \forall\ t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, we say that t_* is a *switching time* along $x(t, x^0)$, and that the CLS has a *mode transition* or *mode switching* along $x(t, x^0)$ at t_*.

Switching and non-switching times along $x(t, x^0)$ can be characterized by the index sets $\mathcal{J}$ and $\mathcal{J}^{\,r}$. In fact, a time $t_* > 0$ is a switching time along $x(t, x^0)$ if and only if $\mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,r}(x(t_*, x^0)) = \emptyset$ [7, Proposition 3.11]. Furthermore, it is recently shown in [26, Proposition 2] that a time $t_* > 0$ is a non-switching time along $x(t, x^0)$ if and only if $\mathcal{J}(x(t_*, x^0)) = \mathcal{J}^{\,r}(x(t_*, x^0))$. This result is the so-called *simple switching property.* Roughly speaking, this property implies that at a non-switching time t_*, if the forward-time trajectory starting from $x(t_*, x^0)$ stays in some cone $\mathcal{X}_i$ for a while, then the reverse-time trajectory starting from the same state must also remain in $\mathcal{X}_i$ for some time. While seemingly intuitive and straightforward, the simple switching property leads to various important consequences. For example, in light of this property, we have the following lemma for a non-switching trajectory.

Lemma 5. If there is no switching along $x(t, x^0)$ for all $t \in [0, T]$ with $0 < T \leq \infty$, then $\mathcal{J}(x(t, x^0)) = \mathcal{J}(x^0), \forall\ t \in [0, T]$.

Proof. Define $t_* \equiv \sup\{\, \bar{t} \geq 0 \,|\, \mathcal{J}(x(t, x^0)) = \mathcal{J}(x^0), \forall\ t \in [0, \bar{t}\,] \,\}$. It follows from Statement (a) of [7, Proposition 3.9] that $t_* > 0$. Now suppose $t_* < T$ (if $T = \infty$, this implies that t_* is finite). Then we have $\mathcal{J}(x(t, x^0)) = \mathcal{J}(x^0), \forall\ t \in [0, t_*)$ and $\mathcal{J}(x(t_*, x^0)) \neq \mathcal{J}(x^0)$. On the other hand, we deduce from Statement (b) of [7, Proposition 3.9] that there exists $\varepsilon > 0$ such that $\mathcal{J}^{\,r}(x(t_*, x^0)) = \mathcal{J}(x(t, x^0))$ for all $t \in [t_* - \varepsilon, t_*)$. Since $\mathcal{J}(x(t_* - \varepsilon, x^0)) = \mathcal{J}(x^0)$, we obtain

$\mathcal{J}^r(x(t_*, x^0)) = \mathcal{J}(x^0)$. Therefore, $\mathcal{J}(x(t_*, x^0)) \neq \mathcal{J}^r(x(t_*, x^0))$. In view of the simple switching property, we conclude that t_* is a switching time. This yields a contradiction as $t_* \in (0, T)$. $\square$

If $\mathcal{J}(x(t', x^0)) \neq \mathcal{I}(x(t', x^0))$ at some t', then we call t' a *critical time* along $x(t, x^0)$ (and its corresponding state $x(t', x^0)$ is a critical state). It can be shown that a switching time t_* must be a critical time. In fact, suppose this is not the case, i.e., $\mathcal{J}(x^*) = \mathcal{I}(x^*)$ where $x^* \equiv x(t_*, x^0)$. Since t_* is a switching time, $\mathcal{J}^r(x^*) \cap \mathcal{J}(x^*) = \emptyset$. Hence, $\mathcal{J}^r(x^*) \cap \mathcal{I}(x^*) = \emptyset$. However, there exists $\varepsilon > 0$ such that $x(t, x^0) \in \cup_{i \in \mathcal{I}(x^*)} \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$. Thus we have $\mathcal{J}^r(x^*) \subseteq \mathcal{I}(x^*)$ so that $\mathcal{J}^r(x^*) \cap \mathcal{I}(x^*) = \mathcal{J}^r(x^*)$. This leads to a contradiction as $\mathcal{J}^r(x^*) \neq \emptyset$.

For a non-switching trajectory $x(t, x^0)$ on $[0, T]$ with $T > 0$, notice that $\mathcal{J}(x(t, x^0))$ may not equal to $\mathcal{I}(x(t, x^0))$ at each $t \in [0, T]$. The following lemma asserts that there are only finitely many critical times along $x(t, x^0)$.

Lemma 6. Consider a non-switching trajectory $x(t, x^0)$ on $[0, T]$ for some $T > 0$. Then there exist finitely many times t_i satisfying: $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ that form a partition of $[0, T]$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)), \forall\, t \in (t_i, t_{i+1}),\ i = 0, \cdots, N-1$.

Proof. For each $t' \in [0, T]$, let $\widehat{x} \equiv x(t', x^0)$ and consider either of the following two cases:

(i) $\mathcal{J}(\widehat{x}) = \mathcal{I}(\widehat{x})$. In this case, there exists a neighborhood $\mathcal{N}$ of $\widehat{x}$ such that $\mathcal{N} \subseteq \cup_{j \in \mathcal{I}(\widehat{x})} \mathcal{X}_j$ [7, Lemma 2.5]. Hence by the continuity of $x(\cdot, x^0)$, we have $x(t, x^0) \in \mathcal{N}, \forall\, t \in [t' - \varepsilon, t' + \varepsilon]$ for some $\varepsilon > 0$. Therefore, $\mathcal{I}(x(t, x^0)) \subseteq \mathcal{I}(\widehat{x}), \forall\, t \in [t' - \varepsilon, t' + \varepsilon]$. Moreover, it follows from [7, Proposition 3.9] that $\mathcal{J}(x(t, x^0)) = \mathcal{J}(\widehat{x}), \forall\, t \in [t' - \varepsilon, t' + \varepsilon]$ by appropriately restricting $\varepsilon > 0$. Since $\mathcal{J}(\widehat{x}) = \mathcal{I}(\widehat{x})$ and $\mathcal{J}(x(t, x^0)) \subseteq \mathcal{I}(x(t, x^0))$, $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0))$ for all $t \in [t' - \varepsilon, t' + \varepsilon]$.

(ii) $\mathcal{J}(\widehat{x}) \neq \mathcal{I}(\widehat{x})$. Let $j \in \mathcal{I}(\widehat{x}) \setminus \mathcal{J}(\widehat{x})$. Thus for an $i \in \mathcal{J}(\widehat{x})$, $\big(C_j \widehat{x}, C_j A_i \widehat{x}, \ldots, C_j A_i^{n-1} \widehat{x}\big) \not\succcurlyeq 0$. This shows that some row of C_j, say the ℓth row denoted by $(C_j)_{\ell\bullet}$, satisfies $(C_j)_{\ell\bullet}\, x(t, x^0) = (C_j)_{\ell\bullet}\, e^{A_i(t-t')} \widehat{x} < 0, \forall\, t \in (t', t' + \varepsilon]$ for some $\varepsilon > 0$. Hence $x(t, x^0) \notin \mathcal{X}_j$, or equivalently $j \notin \mathcal{I}(x(t, x^0))$, for all $t \in (t', t' + \varepsilon]$. This shows that $\mathcal{I}(x(t, x^0)) \subseteq \mathcal{J}(\widehat{x}), \forall\, t \in (t', t' + \varepsilon]$. Since $\mathcal{J}(x(t, x^0)) = \mathcal{J}(\widehat{x})$ for all $t > t_*$ sufficiently close to t_*, we have, by appropriately refining $\varepsilon > 0$, $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}(\widehat{x}), \forall\, t \in (t', t' + \varepsilon]$. Similarly, using the reverse-time argument and [7, Proposition 3.9], we deduce that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}^r(\widehat{x}), \forall\, t \in [t' - \varepsilon, t')$ for some $\varepsilon > 0$. Hence, via the simple switching property, we obtain $\varepsilon > 0$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)), \forall\, t \in [t' - \varepsilon, t') \cup (t', t' + \varepsilon]$.

Consequently, for each $t \in [0, T]$, there exists $\varepsilon_t > 0$ such that $\mathcal{I}(x(\tau, x^0)) = \mathcal{J}(x(\tau, x^0)), \forall\, \tau \in [t - \varepsilon_t, t) \cup (t, t + \varepsilon_t]$. Since the family $\{(t - \varepsilon_t, t + \varepsilon_t) : t \in [0, T]\}$ constitutes an open cover of the compact interval $[0, T]$, it thus follows from Heine-Borel Theorem that there is a finite subcover of $[0, T]$. Hence there exist finitely many time instants $\{t'_0, t'_1, \cdots, t'_\ell\} \subset [0, T]$ such that $[0, T] \subset \bigcup_{j=0}^{\ell} \big[\, t'_j,\, t'_j + \varepsilon_{t'_j} \,\big]$. By appropriately refining the partition on the right, we obtain finitely many times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)), \forall\, t \in (t_i, t_{i+1})$ for each $i = 0, \cdots, N-1$. $\square$

Combining the above results and non-Zenoness of the CLS [7], the following proposition establishes the finite occurrence of critical times on a compact interval along any trajectory.

Proposition 7. Consider a trajectory $x(t, x^0)$ on $[0, T]$ with $T > 0$. Then there are finitely many critical times on $[0, T]$. Specifically, there exists a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$ such that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}(x(t', x^0))$ for all $t \in (t_i, t_{i+1})$ and any $t' \in (t_i, t_{i+1})$ for each $i = 0, \cdots, M-1$.

Proof. Since the CLS is non-Zeno, there are finitely many switching times along $x(t, x^0)$ [7, Theorem 3.7]. Specifically, there exists a partition $0 = \widetilde{t}_0 < \widetilde{t}_1 < \cdots < \widetilde{t}_{N-1} < \widetilde{t}_N = T$ of $[0, T]$ such that there is no switching on a subinterval $(\widetilde{t}_i, \widetilde{t}_{i+1})$ for every $i = 0, \cdots, N-1$. It follows from Lemma 6 that there are finitely many critical times on each subinterval. Hence, there are a finite number of critical times on $[0, T]$. Finally, since there is no switching on a subinterval (t_i, t_{i+1}) where t_i and t_{i+1} are two neighboring critical times on $[0, T]$, we conclude via Lemmas 5 and 6 that $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \mathcal{J}(x(t', x^0))$ for all $t \in (t_i, t_{i+1})$ and any $t' \in (t_i, t_{i+1})$. □

Proposition 7 is instrumental to finite-time sensitivity and observability analysis. In fact, given a nominal trajectory and a finite time interval $[0, T]$, one can divide $[0, T]$ into finitely many subintervals defined by consecutive critical times of the nominal trajectory on $[0, T]$. This enables one to focus on each subinterval, which is relatively easier to handle in sensitivity and observability analysis; see Theorem 9 and Proposition 19. Then one can combine obtained sensitivity or observability conditions for these subintervals, together with those at finitely many critical times, to establish desired conditions for the entire interval $[0, T]$; see Corollary 20 in Section 4.

2.2 Directional Derivative of the CLS

Sensitivity analysis of the CLS with respect to its initial conditions is essential to study various dynamic and numerical properties of the CLS, e.g., stability, robustness, observability and numerical resolution of the systems [7, 8, 17, 21, 22]. Particularly, the first-order variation of a system trajectory with respect to its initial condition is perhaps the most important and the best studied. However, unlike an ODE with a C^1 right-hand side, a trajectory of the CLS is not differentiable in its initial condition. On the other hand, it is shown in [22] that $x(t, x^0)$ is B(ouligand)-differentiable in x^0 at each t, namely, $x(t, \cdot)$ is locally Lipschitz continuous and directional differentiable at each t [13]. For a given t, the directional derivative of $x(t, x^0)$ along a direction vector $v \in \mathbb{R}^n$ is defined by

$$x'(t, x^0; v) \equiv \lim_{\tau \downarrow 0} \frac{x(t, x^0 + \tau v) - x(t, x^0)}{\tau}$$

Obviously $x(t, x^0; \cdot)$ is positively homogenous. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote the right-hand side of the CLS (1). It is known that $f(x)$ is globally Lipschitz and directionally differentiable (thus it is B-differentiable). Indeed, it can be shown via [7, Lemma 3.4] that for any x and v,

$$f'(x; v) = A_i v \tag{4}$$

for some $i \in \mathcal{I}(x)$ (such an i depends on v). Hence, it follows from [22, Theorem 7] that for any given initial condition x^0 and direction vector v, the directional derivative $x'(t, x^0; v)$ is the unique solution of the following time-varying differential system:

$$\dot{z} = f'(x(t, x^0); z), \qquad z(0) = v, \tag{5}$$

where $f'(x(t, x^0); z)$ denotes the directional derivative of f at $x(t, x^0)$ along the direction z at each t. Using (4), we can further write the right-hand side of (5) as $f'(x(t, x^0); z) = B(x(t, x^0), z)\, z$, where $B : \mathbb{R}^n \times \mathbb{R}^n \to \{A_i \mid i = 1, \cdots, m\}$. Hence, the system (5) becomes a time-varying piecewise linear system whose right-hand side $f'(x(t, x^0); z(t))$ is generally discontinuous in t. However, it shall be shown as follows that on a compact time interval, $f'(x(t, x^0); z)$ is discontinuous only at finitely many time instants. To establish this result, we need a technical lemma that asserts persistence of the duration of trajectories in a mode under small perturbations on initial conditions.

Lemma 8. Given $x^* \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ such that $x^* \in \mathcal{Y}_i$ and $(x^*+u) \in \mathcal{Y}_i$ for some $i \in \{1, \cdots, m\}$. Then there exist $\varepsilon_0 > 0$ and $\tau_0 > 0$ such that $x(t, x^* + \tau u) = e^{A_i t}(x^* + \tau u) \in \mathcal{X}_i$ for all $(t, \tau) \in [0, \varepsilon_0] \times [0, \tau_0]$.

Proof. Let $C_i \in \mathbb{R}^{m_i \times n}$. For each $\ell \in \{1, \cdots, m_i\}$, if $\big((C_i x^*)_\ell, (C_i A_i x^*)_\ell, \cdots, (C_i A_i^{n-1} x^*)_\ell \big) \succ 0$, then let μ_ℓ be the first nonnegative integer k such that $(C_i A_i^k x^*)_\ell > 0$; otherwise, let $\mu_\ell \equiv n$. In the first case, $0 \leq \mu_\ell < n$, and in the latter, we must have $x^* \in \overline{O}((C_i)_{\ell\bullet}, A_i)$, where $(C_i)_{\ell\bullet}$ denotes the ℓth row of C_i. It is easy to see that for each ℓ, $\big((C_i u)_\ell, (C_i A_i u)_\ell, \cdots, (C_i A_i^{\mu_\ell - 1} u)_\ell \big) \succcurlyeq 0$. Moreover, for every ℓ,

$$\big(C_i e^{A_i t}[x^* + \tau u]\big)_\ell = \begin{cases} \tau \displaystyle\sum_{s=0}^{\mu_\ell - 1} \frac{(C_i A_i^s u)_\ell}{s!} t^s + \sum_{j=\mu_\ell}^{\infty} \frac{(C_i A_i^j [x^* + \tau u])_\ell}{j!} t^j, & \text{if } \mu_\ell < n \\ \tau \displaystyle\sum_{s=0}^{n-1} \frac{(C_i A_i^s u)_\ell}{s!} t^s, & \text{if } \mu_\ell = n \end{cases}$$

In the case where $\mu_\ell < n$, there exists $\varepsilon'_\ell > 0$, depending on C_i, A_i and u only, such that the first summation on the right is nonnegative for all $(t, \tau) \in [0, \varepsilon'_\ell] \times [0, \infty)$. Since $(C_i A_i^{\mu_\ell} x^*)_\ell > 0$, there exist positive numbers ε''_ℓ and τ''_ℓ, depending on C_i, A_i, x^* and u only, such that the second summation

$$\sum_{j=\mu_\ell}^{\infty} \frac{(C_i A_i^j [x^* + \tau u])_\ell}{j\,!} t^j = t^{\mu_\ell} \left\{ \frac{(C_i A_i^{\mu_\ell} x^*)_\ell}{j!} + O(t) + \tau \left[\frac{(C_i A_i^{\mu_\ell} u)_\ell}{j!} + O(t) \right] \right\} \geq 0$$

for all $(t, \tau) \in [0, \varepsilon''_\ell] \times [0, \tau''_\ell]$. Letting $\varepsilon_\ell = \min(\varepsilon'_\ell, \varepsilon''_\ell)$ and $\tau_\ell = \min(\tau'_\ell, \tau''_\ell)$, we have $\big(C_i e^{A_i t}[x^* + \tau u]\big)_\ell \geq 0$ for all $(t, \tau) \in [0, \varepsilon_\ell] \times [0, \tau_\ell]$. The case where $\mu_\ell = n$ also holds true by the similar argument. Finally, letting $\varepsilon_0 = \min_\ell \varepsilon_\ell$ and $\tau_0 = \min_\ell \tau_\ell$, we obtain the desired result. □

Theorem 9. Let t_i and t_{i+1} be two consecutive critical times along $x(t, x^0)$ and $v \in \mathbb{R}^n$ be a direction vector. Then the following hold:

(a) For any interval $[T_1, T_2] \subset (t_i, t_{i+1})$, there exists a partition $T_1 = \widehat{t}_1 < \widehat{t}_2 < \cdots < \widehat{t}_{p-1} < \widehat{t}_p = T_2 < \widehat{t}_{p+1}$ of $[T_1, T_2]$ such that the directional derivative $x'(t, x^0; v)$ satisfies the linear system $\dot{z}(t) = A_k z(t)$ for some $k \in \mathcal{J}(x(T_1, x^0))$ on $(\widehat{t}_i, \widehat{t}_{i+1})$ for each $i = 1, \cdots, p$;

(b) For any interval $[T_1, T_2] \subset (t_i, t_{i+1})$, let $\xi \equiv x(T_1, x^0)$ and $\eta \equiv x'(T_1, x^0; v)$. Then for all $\tau > 0$ sufficiently small and all $t \in [T_1, T_2]$,

$$x(t - T_1, \xi + \tau\eta) = x(t, x^0) + \tau x'(t, x^0; v); \tag{6}$$

(c) $f'(x(t, x^0); x'(t, x^0; v))$ is continuous (with respect to t) on (t_i, t_{i+1}).

Proof. (a). Consider an arbitrary $t_* \in (t_i, t_{i+1})$. For notational convenience, we let $x^* \equiv x(t_*, x^0)$ and $\eta \equiv x'(t_*, x^0; v)$. Since $x'(t, x^0; v)$ is the solution of the differential system (5), it follows from the semi-group property that $x'(t, x^0; v) = x'(t - t_*, x^*; \eta)$ for all $t \geq t_*$ (this can also be shown via the B-differentiability of $x(t, x^0)$), namely,

$$x'(t, x^0; v) = \lim_{\tau \downarrow 0} \frac{x(t - t_*, x^* + \tau\,\eta) - x(t - t_*, x^*)}{\tau}, \qquad \forall \;\; t \geq t_* \tag{7}$$

Since there exists a neighborhood $\mathcal{N}$ of x^* such that $\mathcal{N} \subseteq \cup_{i \in \mathcal{I}(x^*)} \mathcal{X}_i$, we obtain a positive number $\widetilde{\tau}$ such that $(x^* + \widetilde{\tau}\eta) \in \mathcal{N}$. This further implies that $\mathcal{I}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{I}(x^*)$. Since t_* is not a critical time, $\mathcal{I}(x^*) = \mathcal{J}(x^*)$. We thus have $\mathcal{J}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{I}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{J}(x^*) = \mathcal{J}(x(T_1, x^0))$ by Proposition 7. Since $\mathcal{J}(x^* + \widetilde{\tau}\eta)$ is nonempty [7, Lemma 3.3], there exists an index $k \in \mathcal{J}(x^* + \widetilde{\tau}\eta) \subseteq \mathcal{J}(x(T_1, x^0))$ such that $(x^* + \widetilde{\tau}\eta) \in \mathcal{Y}_k$ and $x^* \in \mathcal{Y}_k$. It hence follows from Lemma 8 that $\varepsilon_0 > 0$ and $\tau_0 > 0$ exist such that

$$x(t - t_*, x^* + \tau\eta) = e^{A_k(t - t_*)}[x^* + \tau\eta] \tag{8}$$

for each pair $(t - t_*, \tau) \in [0, \varepsilon_0] \times [0, \tau_0]$. Therefore, we deduce via (7) that for all $t \in [t_*, t_* + \varepsilon_0]$, $x'(t, x^0; v) = e^{A_k(t - t_*)}\eta$, which is the unique solution of the linear system $\dot{z} = A_k z$. Hence, $x'(t, x^0; v)$ satisfies $\dot{z} = A_k z$ for all $t \in (t_*, t_* + \varepsilon_0)$. This further implies that for each $t \in (t_i, t_{i+1})$, there exist $\varepsilon_t > 0$ and A_{k_t} with $k_t \in \mathcal{J}(x(T_1, x^0))$ such that $x'(t, x^0; v)$ satisfies the linear ODE $\dot{z} = A_{k_t} z$ on the open interval $(t, t + \varepsilon_t)$. Since the collection $\{(t, t + \varepsilon_t) : t \in (t_i, t_{i+1})\}$ forms an open cover of the compact interval $[T_1, T_2]$, there exist finitely many times $\{t'_0, t'_1, \cdots, t'_\ell\} \subset (t_i, t_{i+1})$ with $t'_0 < T_1$ such that $[T_1, T_2] \subset \bigcup_{j=0}^{\ell} \left[\, t'_j, \, t'_j + \varepsilon_{t'_j} \,\right]$. By refining these intervals, we obtain finitely many times $T_1 = \widehat{t}_1 < \widehat{t}_2 < \cdots < \widehat{t}_{p-1} < \widehat{t}_p = T_2 < \widehat{t}_{p+1}$ such that $x'(t, x^0; v)$ satisfies a linear system on each $(\widehat{t}_i, \widehat{t}_{i+1})$. Hence, the desired partition holds.

(b). Recall that $\xi \equiv x(T_1, x^0)$ and $\eta \equiv x'(T_1, x^0; v)$. Since there is no critical time on $[T_1, T_2]$, $\mathcal{I}(x(t, \xi)) = \mathcal{J}(x(t, \xi)) = \mathcal{J}(\xi) = \mathcal{I}(\xi)$ for all $t \in [T_1, T_2]$. Hence, it follows from the Lipschitz property that there exists $\tau_0 > 0$ such that $\mathcal{I}(x(t, \xi + \tau\eta)) \subseteq \mathcal{I}(\xi)$ for all $t \in [T_1, T_2]$ and all $\tau \in (0, \tau_0]$, which yields $\mathcal{J}(x(t, \xi + \tau\eta)) \subseteq \mathcal{J}(\xi)$, $\forall\, (t, \tau) \in [T_1, T_2] \times (0, \tau_0]$. Using the partition obtained in (a) and letting $\widehat{x}^i \equiv x(\widehat{t}_i, x^0)$ and $\widehat{\eta}^i = x'(\widehat{t}_i, x^0; v)$, we deduce, in light of (8), that for each subinterval $[\widehat{t}_i, \widehat{t}_{i+1}]$, there exists $\tau_i > 0$ such that $x(t - \widehat{t}_i, \widehat{x}^i + \tau\widehat{\eta}^i) = e^{A_k(t - \widehat{t}_i)}\widehat{x}^i + \tau x'(t, x^0; v)$ for some A_k with $k \in \mathcal{J}(\xi)$ for all $t \in [\widehat{t}_i, \widehat{t}_{i+1}]$ and all $\tau \in [0, \tau_i]$, $i = 1, \cdots, p-1$. Let $\widehat{\tau} \equiv \min_{i \in \{0, \cdots, p-1\}} \tau_i$. In what follows, we prove (6) by induction. Consider the first subinterval $[\widehat{t}_1, \widehat{t}_2]$. Since $\widehat{t}_1 = T_1$, $\widehat{x}^1 = \xi$ and $\widehat{\eta}^1 = \eta$, we have $x(t - T_1, \xi + \tau\eta) = e^{A_k(t - T_1)}\xi + \tau x'(t, x^0; v)$ for all $t \in [\widehat{t}_1, \widehat{t}_2]$ and all $\tau \in (0, \widehat{\tau}]$. Hence, (6) holds on $[\widehat{t}_1, \widehat{t}_2]$ because $e^{A_k(t - T_1)}\xi = x(t, x^0)$ on $[T_1, T_2]$. Now assume that (6) holds on the subintervals $[\widehat{t}_i, \widehat{t}_{i+1}]$ for all $i = 1, \cdots, \ell$, where $1 \leq \ell \leq p - 2$. Consider the subinterval $[\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$. Thus $x(t - \widehat{t}_{\ell+1}, \widehat{x}^{\ell+1} + \tau\widehat{\eta}^{\ell+1}) = e^{A_k(t - \widehat{t}_{\ell+1})}\widehat{x}^{\ell+1} + \tau x'(t, x^0; v)$ for some A_k with $k \in \mathcal{J}(\xi)$ for all $t \in [\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$ and all $\tau \in (0, \widehat{\tau}]$. Note that $\widehat{x}^{\ell+1} + \tau\widehat{\eta}^{\ell+1} = x(\widehat{t}_{\ell+1} - T_1, \xi) + \tau x'(\widehat{t}_{\ell+1}, x^0; v)$. It follows from the induction hypothesis that the latter equals $x(\widehat{t}_{\ell+1} - T_1, \xi + \tau\eta)$. Therefore $x(t - \widehat{t}_{\ell+1}, \widehat{x}^{\ell+1} + \tau\widehat{\eta}^{\ell+1}) = x\big(t - \widehat{t}_{\ell+1}, x(\widehat{t}_{\ell+1} - T_1, \xi + \tau\eta)\big) = x(t - T_1, \xi + \tau\eta)$ for all $t \in [\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$. Consequently, (6) holds on $[\widehat{t}_{\ell+1}, \widehat{t}_{\ell+2}]$ since $e^{A_k(t - \widehat{t}_{\ell+1})}\widehat{x}^{\ell+1} = x(t, x^0)$ on the subinterval in consideration. By the induction principle, we see that (6) holds on $[T_1, T_2]$.

(c). Given any $t' \in (t_i, t_{i+1})$, it belongs to the interior of some compact interval $[T_1, T_2]$ contained in (t_i, t_{i+1}). Hence (6) holds true on a small open interval $I \subseteq [T_1, T_2]$ containing t'. Therefore

$$x'(t, x^0; v) = \frac{x(t - T_1, \xi + \tau\eta) - x(t, x^0)}{\tau}, \;\; \forall \;\; t \in I$$

for a fixed small $\tau > 0$. Since both $x(t - T_1, \xi + \tau\eta)$ and $x(t, x^0)$ have continuous time derivatives on I, so does $x'(t, x^0; v)$. This shows that $f'(x(t, x^0); x'(t, x^0; v))$ is continuous at t'. Since t' is arbitrary in (t_i, t_{i+1}), we obtain (c). □

The results in this section will be employed to characterize finite-time observability of the CLS in Section 4.

3 Positive Invariance of Conewise Linear Systems

The concept of positive invariance plays a crucial role in asymptotic analysis of dynamical systems and Lyapunov stability theory [19]. Roughly speaking, a set is positively invariant if each trajectory starting from the given set remains in that set for all positive times. In the realm of switched and hybrid systems, positive invariance also sheds light on reachability analysis and control of hybrid systems with applications in engineering [4, 30] and systems biology [1, 29]. Pertaining to the CLS, it is shown lately in [26] that global and long-time dynamics are closely related to the positively invariant cone of each mode. These cones are used to investigate global switching behaviors and stability properties of the CLS. In this section, we establish necessary and sufficient conditions for the interior of a positively invariant cone. These results will be exploited to characterize long-time observability in Section 4.

3.1 Preliminary Discussions

The positively invariant cone associated with the ith mode of the CLS (1) is defined by:

$$\mathcal{A}_i \;\equiv\; \{\, x \in \mathbb{R}^n \mid C_i\, e^{A_i t} x \geq 0, \;\forall\, t \geq 0 \,\} \tag{9}$$

It is clear that $\mathcal{A}_i$ is closed and convex and is the largest positively invariant set contained in $\mathcal{X}_i$. While the formulation (9) is simple and neat, an explicit characterization of $\mathcal{A}$ in term of x only is highly challenging due to the difficulty of removing the quantifier t. This is a major obstacle in understanding various analytical properties of such a cone. In the following, we focus on each of these cones and drop the subscripts in $\mathcal{A}_i$, C_i and A_i for notational simplicity. For the given matrix C, we also let $\mathcal{X} \equiv \{x \in \mathbb{R}^n \,|\, Cx \geq 0\,\}$.

Since $\mathcal{X}$ is polyhedral, it is natural to ask whether its positively invariant cone $\mathcal{A}$ is also polyhedral, in that a polyhedral cone is both analytically and numerically more tractable than a non-polyhedral one. However, the following example shows that the answer is negative in general; this demonstrates another difficulty in analyzing a positively invariant cone.

Example 10. Let $A \in \mathbb{R}^{n\times n}$ be nilpotent and C be an n-row, i.e., $C = c^T \in \mathbb{R}^{1\times n}$. In this case, $\mathcal{X} = \{x \,|\, c^T x \geq 0\}$ becomes a half-space of $\mathbb{R}^n$. Without loss of generality, we assume that A is in the Jordan canonical form, i.e., $A = \text{diag}(J_1, \cdots, J_\ell)$, where J_i is a Jordan block associated with the zero eigenvalue, and we partition c accordingly as $c^T = \big(c_1^T, \cdots, c_\ell^T\big)$. Moreover, we may assume, without losing generality, that each pair (c_i, J_i) is of the observable canonical form, i.e.,

$$c_i^T = \big(\,0,\, \widetilde{c}_i^{\,T}\,\big), \qquad J_i = \begin{bmatrix} J_{iu} & J_{i2} \\ 0 & \widetilde{J}_i \end{bmatrix},$$

such that $(\widetilde{c}_i^{\,T}, \widetilde{J}_i)$ is an observable pair. Note that each $\widetilde{J}_i$ remains a Jordan block. Let v be an eigenvector of A. Thus $\text{sgn}(c^T v)\, v$ is always in the half-space $\mathcal{X}$. As a result, $\mathcal{A}$ contains nonzero vectors. We claim that $\mathcal{A}$ is polyhedral if and only if each Jordan block $\widetilde{J}_i$ is at most of order 2. To show this, we partition $x \in \mathbb{R}^n$ as $x^T = \big((x^1)^T, \cdots, (x^\ell)^T\big)$, where each x^i corresponds to the block J_i. Therefore $c^T e^{At} x = \sum_{i=1}^{\ell} c_i^T e^{J_i t} x^i = \sum_{i=1}^{\ell} \widetilde{c}_i^T e^{\widetilde{J}_i t} \widetilde{x}^i$, where $\widetilde{x}^i$ is a sub-vector of x^i corresponding to the observable pair $(\widetilde{c}_i^T, \widetilde{J}_i)$ for each i. Since the sub-vector $x^i \backslash \widetilde{x}^i$ plays no role in determining the polyhedrality of $\mathcal{A}$, we thus assume that each pair (c_i^T, J_i) is observable. We first show sufficiency. If each J_i is at most of order 2, then $c^T e^{At} x = a_0(x) + a_1(x) t$, where a_0 and a_1 are real-valued linear functions of x. Hence, $\mathcal{A} = \{x \mid a_0(x) \ge 0, \, a_1(x) \ge 0\}$ is polyhedral. We prove necessity by contradiction. Suppose $\mathcal{A}$ is polyhedral but one of the Jordan blocks, say J_1, is of order greater than 2. Let x_j denote the j-th entry of x and define the polyhedral cone $P_1 \equiv \{\, x \in \mathbb{R}^n \mid x_j = 0, \, j \ge 4\}$. Furthermore, let $P_2 \equiv \mathcal{A} \cap P_1 = \{x \in \mathbb{R}^n \mid c^T e^{At} x \ge 0, \forall\, t \ge 0, \, x_j = 0, \, j \ge 4\}$ which is polyhedral. Therefore $c^T e^{At} x = a_0(x) + a_1(x) t + a_2(x)\, t^2/2$ for $x \in P_1$, where $a_k(x) = c_1^T (J_1)^k x$ with $k = 0, 1, 2$. Since the pair (c_1^T, J_1) is observable, $\{c_1, J_1 c_1, (J_1)^2 c_1\}$ is linearly independent. This leads to a linear transformation $z = Tx$ with an invertible $T \in \mathbb{R}^{n \times n}$ such that $\widetilde{P}_2 = T P_2 = \{z \in \mathbb{R}^n \mid z_1 + z_2 t + z_3 t^2 \ge 0, \forall\, t \ge 0, \; z_j = 0, \, j \ge 4\}$. Clearly, $\widetilde{P}_2$ is also polyhedral. Let $\gamma < 0$ be a given scalar. Therefore $\widetilde{P}_3 \equiv \widetilde{P}_2 \cap \{z \mid z_2 = \gamma\}$ is a polyhedron (provided that it is nonempty). Since a quadratic polynomial $a_0 + a_1 t + a_2 t^2 \ge 0, \forall\, t \ge 0$ if and only if $a_0, a_2 > 0$ and $a_1 + 2\sqrt{a_0 a_2} \ge 0$, we deduce that $\widetilde{P}_3 = \{z \in \mathbb{R}^n \mid z_1 \ge 0, \, z_3 \ge 0, \, z_1 z_3 \ge \gamma^2/4, \, z_2 = \gamma, \; z_j = 0, \, j \ge 4\}$, which is clearly nonempty. This further implies that the convex set $\{(z_1, z_3)^T \in \mathbb{R}^2 \mid z_1 \ge 0, \, z_3 \ge 0, \, z_1 z_3 \ge \gamma^2/4\}$ is polyhedral, a contradiction. Consequently, the claim holds.

3.2 Interior of Positively Invariant Cone

For a given x^* to be in the interior of $\mathcal{A}$, it is easy to expect that $C e^{At} x^* > 0, \forall\, t \ge 0$. However, the following example shows that this condition alone is not enough; some additional condition related to the "largest mode" defined by the pair (C, A) is needed.

Example 11. Let $C = (1, 1)$, $A = \text{diag}(1, \, 2) \in \mathbb{R}^{2 \times 2}$, and $x^* = (1, 0)^T$. Hence, $C e^{At} x^* = e^t > 0, \forall\, t \ge 0$ and $x^* \in \mathcal{A}$. Consider $\widehat{x} = (1, \, -\varepsilon)^T$, where $\varepsilon > 0$. Clearly, $\widehat{x} \to x^*$ as $\varepsilon \downarrow 0$. However, for any $\varepsilon > 0$, $C e^{At} \widehat{x} = e^t - \varepsilon e^{2t} < 0$ for all $t \ge 0$ sufficiently large. Therefore, $x^* \notin \text{int}\mathcal{A}$.

To characterize the interior of a positively invariant cone, we need certain technical results. The next two lemmas provide major tools to treat oscillatory modes corresponding to complex eigenvalues of the defining matrix in positive invariance analysis of linear dynamics. The first lemma states that a nontrivial linear combination of sinusoidal functions has persistent sign alternating and its positive/negative variations are not diminishing as time goes.

Lemma 12. [26, Corollary 15] Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) \equiv \displaystyle\sum_{i=1}^{m} \big[\alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t)\big]$, where $\omega_i > 0$, $\omega_i \ne \omega_j$ for $i \ne j$, and $|\alpha_i| + |\beta_i| \ne 0$ for all i. Then there exist two scalars $\gamma_1 > 0$ and $\gamma_2 < 0$ such that for any t_*, $t_1, t_2 \in [t_*, \infty)$ exist satisfying $f(t_1) \ge \gamma_1$ and $f(t_2) \le \gamma_2$.

More properties for the above $f(t)$ are presented as follows. For notational simplicity, let $d_i(t) \equiv \alpha_i \cos(\omega_i t) + \beta_i \sin(\omega_i t)$. By considering the rationality of ratios of the frequencies, we obtain the collection of (distinct and disjoint) equivalent classes $E_{\omega_j} = \{d_i(t) \mid \omega_i/\omega_j \text{ is rational}\}$ [26, Lemma 14]. Note that each equivalent class E_{ω_j} attains a basis frequency $\widetilde{\omega}_s > 0$, namely,

$\omega_i/\widetilde{\omega}_s$ is a positive integer for any frequency ω_i associated with $d_i(t) \in E_{\omega_j}$. Let $E_{\widetilde{\omega}_s}$ denote the equivalent class, and let $q_{\widetilde{\omega}_s}(t) := \sum_{d_i \in E_{\widetilde{\omega}_s}} d_i(t)$. Then the following statements hold:

(1) $q_{\widetilde{\omega}_s}(\cdot)$ is a real-valued smooth and periodic function with the frequency $\widetilde{\omega}_s$;

(2) if $q_{\widetilde{\omega}_s}(\cdot)$ is not identically zero, then it attains the maximal and minimal values $\sigma_{\widetilde{\omega}_s} > 0$ and $\nu_{\widetilde{\omega}_s} < 0$ on $(-\infty, \infty)$ respectively;

(3) $q_{\widetilde{\omega}_s}(\cdot)$ is onto $[\,\nu_{\widetilde{\omega}_s},\ \sigma_{\widetilde{\omega}_s}\,]$;

(4) the ratio of any two basis frequencies associated with distinct equivalent classes is irrational.

Suppose there are k equivalent classes $E_{\widetilde{\omega}_s}$, i.e. $s = 1, \cdots, k$. Hence, $f(t) = \sum_{s=1}^k q_{\widetilde{\omega}_s}(t)$. We have the following result:

Lemma 13. [25, Lemma 5] Let $\sigma_{\widetilde{\omega}_s}$ and $\nu_{\widetilde{\omega}_s}$ be defined in the above setting for the function f. Then $\sum_{s=1}^{k} \sigma_{\widetilde{\omega}_s} = \sup_{[t_*,\infty)} f(t)$ and $\sum_{s=1}^{k} \nu_{\widetilde{\omega}_s} = \inf_{[t_*,\infty)} f(t)$ for any $t_* \in \mathbb{R}$.

It is worth pointing out that Lemma 13 implies that $\sum_{s=1}^k \nu_{\widetilde{\omega}_s} = \inf_{(-\infty,\infty)} f(t)$, and that $f(t) \geq \rho, \forall\, t$ for some scalar ρ if and only if $\sum_{s=1}^k \nu_{\widetilde{\omega}_s} \geq \rho$.

We introduce more notions for the following development. We assume that A has the real Jordan canonical form via a real similarity transformation $A = \text{diag}\big(\widetilde{J}_1,\, \widetilde{J}_2,\, \cdots,\, \widetilde{J}_p\big)$, where each submatrix $\widetilde{J}_i$ contains all the Jordan blocks associated with a real eigenvalue λ_i or a complex eigenvalue and its conjugate (i.e. μ_i and $\bar{\mu}_j$) whose real parts are $\widetilde{\lambda}$, i.e., $\lambda_i = \widetilde{\lambda}$ and $\mu_i = \widetilde{\lambda} + \imath\, \omega_i$. Without loss of generality, we also assume that $\widetilde{J}_i$'s are ordered in a way such that the real parts of their corresponding eigenvalues are strictly decreasing. Furthermore, each $\widetilde{J}_i$ is given by $\widetilde{J}_i = \text{diag}\big(J_{i1},\, J_{i2},\, \cdots,\, J_{ir(i)}\big)$. Here $r(i)$ is the number of Jordan blocks in $\widetilde{J}_i$ and J_{ij} is the jth real Jordan block given by

(i) $J_{ij} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}$ corresponding to the real eigenvalue λ_i, or

(ii) $J_{ij} = \begin{bmatrix} D_i & I_2 & & & \\ & D_i & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & D_i \end{bmatrix}$ corresponding to the complex eigenvalue pair $\lambda_i \pm \imath\, \omega_i$,

where $\lambda_i, \omega_i \in \mathbb{R}$ with $\omega_i > 0$, I_2 is the 2×2 identity matrix, and $D_i = \begin{bmatrix} \lambda_i & \omega_i \\ -\omega_i & \lambda_i \end{bmatrix}$.

Accordingly, the matrix C can be partitioned as

$$C = \Big[\widetilde{C}_1 \quad \widetilde{C}_2 \quad \cdots \quad \widetilde{C}_p\Big], \qquad \text{with} \quad \widetilde{C}_i = \Big[C_{i1} \quad C_{i2} \quad \cdots \quad C_{ir(i)}\Big] \tag{10}$$

Obviously, C_{ij} has at least one column (resp. two columns) if it corresponds to a real eigenvalue (resp. a complex eigenvalue pair). For each $\widetilde{J}_i$, define the index set

$$\mathcal{K}_i^r \equiv \{\, j \in \{1, \cdots, r(i)\} \,|\, J_{ij} \text{ corresponds to the real eigenvalue } \lambda_i \,\},$$

and $\mathcal{K}_i^c \equiv \{1, \cdots, r(i)\} \backslash \mathcal{K}_i^r$. Hence, for each $j \in \mathcal{K}_i^c$, the Jordan block J_{ij} corresponds to a complex eigenvalue pair with the real part λ_i.

Let $(\widetilde{C}_a)_{\ell \bullet}$ be the first nonzero block in $(C)_{\ell \bullet}$, i.e., $(\widetilde{C}_i)_{\ell \bullet} = 0$ for $i = 1, \cdots, a-1$ and $(\widetilde{C}_a)_{\ell \bullet} \neq 0$. Notice that $(\widetilde{C}_a)_{\ell \bullet}\, e^{\widetilde{J}_a t}$ is a finite sum of the terms of the form $\kappa e^{(\lambda_a \pm \imath\, \omega_s)t}\, t^{b_s}$, where b_s is a non-negative integer. Let b denote the largest such b_s. We call $e^{\lambda_a t}\, t^b$ the *principal mode* associated with the pair $\big((C)_{\ell \bullet}, A\big)$. It is noted that $(C)_{\ell \bullet} e^{At} x = \mu_\ell^0(t,x) e^{\lambda_a t}\, t^b + \sum_{k \geq 1} \mu_\ell^k(t,x)\, e^{\lambda_k t}\, t^{b_k}$, where $\big(\lambda_a, b\big) \succ (\lambda_k, b_k)$ for all k. Here for each $i \geq 0$, $\mu_\ell^i(t,x)$ takes the form $c_\ell^i(x) + \sum_s \big(g^i_{\ell,\,\omega_s}(x) \cos(\omega_s t) + h^i_{\ell,\,\omega_s}(x) \sin(\omega_s t)\big)$, where $c_\ell^i(x)$, $g^i_{\ell,\,\omega_s}(x)$ and $h^i_{\ell,\,\omega_s}(x)$ are all linear. In particular, we show as follows how to determine $c_\ell^0(x)$, $g^0_{\ell,\,\omega_s}(x)$, and $h^0_{\ell,\,\omega_s}(x)$; this eventually leads to the concept of "principal coefficient".

(1) For each $j \in \mathcal{K}_a^r$, let $J_{aj} \in \mathbb{R}^{m_j \times m_j}$ and k_j be the index corresponding to the first nonzero number in $(C_{aj})_{\ell \bullet}$ (from the left), i.e., $(C_{aj})_{\ell\, k_j} \neq 0$. It is easy to verify that the dominating mode in $(C_{aj})_{\ell \bullet} e^{J_{aj} t}$ for large $t \geq 0$ is given by $\frac{1}{(m_j - k_j)!}\, (C_{aj})_{\ell\, k_j}\, e^{\lambda_a t}\, t^{(m_j - k_j)}$. In other words, the principal mode associated with $\big((C_{aj})_{\ell \bullet}, J_{aj}\big)$ is $e^{\lambda_a t}\, t^{(m_j - k_j)}$. Define the index set $\mathcal{L}_a^r \equiv \{ j \in \mathcal{K}_a^r \,|\, m_j - k_j = b \}$ (note that by the definition of the principal mode associated with the pair $\big((C)_{\ell \bullet}, A\big)$, $m_j - k_j \leq b$ for all $j \in \mathcal{K}_a^r$). Therefore, writing $x \in \mathbb{R}^n$ as $\big((x^{11})^T, \cdots, (x^{ij})^T, \cdots\big)^T$, where x^{ij} is the sub-vector corresponding to the Jordan block J_{ij}, we see that $c_\ell^0(x) = \dfrac{1}{b!} \sum_{j \in \mathcal{L}_a^r} (C_{aj})_{\ell\, k_j}\, x^{aj}_{m_j}$, where each $x^{aj}_{m_j}$ is the last element of the sub-vector x^{aj}.

(2) For each $s \in \mathcal{K}_a^c$, let $J_{as} \in \mathbb{R}^{2m_s \times 2m_s}$, and we write the row $(C_{as})_{\ell \bullet} \in \mathbb{R}^{1 \times 2m_s}$ as $(C_{as})_{\ell \bullet} = \big((C_{as})_{\ell\, 1}, \cdots, (C_{as})_{\ell\, m_s}\big)$, where each $(C_{as})_{\ell\, j}$ is a sub-row of two entries. Let $(C_{as})_{\ell\, k_s}$ be the first nonzero sub-row (from the left) such that $m_s - k_s = b$. For each (distinct) imaginary part ω_j of the complex eigenvalues $\lambda_a \pm \omega_j$ of A, define the index set $\mathcal{L}^c_{a,\omega_j} \equiv \{\, s \in \mathcal{K}_a^c \mid m_s - k_s = b \text{ and } J_{as} \text{ corresponds to } \lambda_a \pm \imath\, \omega_j \,\}$ and $\mathcal{L}_a^c \equiv \cup_{\omega_j} \mathcal{L}^c_{a,\omega_j}$. Let $x^{as}_{2m_j - 1}$ and $x^{as}_{2m_j}$ denote the last two elements of the sub-vector x^{as} corresponding to the Jordan block J_{as}. It can be shown that $g^0_{\ell,\,\omega_j}(x) = \frac{1}{b!} \sum_{s \in \mathcal{L}^c_{a,\omega_j}} (C_{as})_{\ell\, k_s} \big(x^{as}_{2m_s - 1}, x^{as}_{2m_s}\big)^T$ and $h^0_{\ell,\,\omega_j}(x) = \frac{1}{b!} \sum_{s \in \mathcal{L}^c_{a,\omega_j}} (C_{as})_{\ell\, k_s} \big(x^{as}_{2m_s}, -x^{as}_{2m_s - 1}\big)^T$. Moreover, $\mathcal{L}_a^r \cup \mathcal{L}_a^c$ is nonempty by the definition of the principal mode.

We comment more on $\sum_s \big(g^0_{\ell,\,\omega_s}(x) \cos(\omega_s t) + h^0_{\ell,\,\omega_s}(x) \sin(\omega_s t)\big)$, following Lemmas 12 and 13. Let $p_{\ell,\,\omega_j}(t,x) \equiv g^0_{\ell,\,\omega_j}(x) \cos(\omega_j t) + h^0_{\ell,\,\omega_j}(x) \sin(\omega_j t)$. We thus obtain the collection of (disjoint) equivalent classes $E_{\omega_j} = \{\, p_{\ell,\,\omega_i}(t,x) \mid \omega_i/\omega_j \text{ is rational} \,\}$ (which is independent of x). Let $\widetilde{\omega}_s > 0$ be the basis frequency associated with each equivalent class E_{ω_j} and denote it by $E_{\widetilde{\omega}_s}$. Let $q_{\ell,\,\widetilde{\omega}_s}(t,x) := \sum_{p_{\ell,\omega_i} \in E_{\widetilde{\omega}_s}} p_{\ell,\,\omega_i}(t,x)$. Then we obtain the similar properties for $q_{\ell,\,\widetilde{\omega}_s}(\cdot,x)$ as shown before Lemma 13, namely, for a fixed $x \in \mathbb{R}^n$, (i) $q_{\ell,\,\widetilde{\omega}_s}(\cdot,x)$ is a smooth and periodic function with the frequency $\widetilde{\omega}_s$; (ii) if $q_{\ell,\,\widetilde{\omega}_s}(\cdot,x)$ is not identically zero, then it attains the maximal and minimal values $\sigma_{\ell,\,\widetilde{\omega}_s}(x) > 0$ and $\nu_{\ell,\,\widetilde{\omega}_s}(x) < 0$ on $(-\infty, \infty)$ respectively; (iii) $q_{\ell,\,\widetilde{\omega}_s}(\cdot,x)$ is onto $[\,\nu_{\ell,\,\widetilde{\omega}_s}(x),\ \sigma_{\ell,\,\widetilde{\omega}_s}(x)\,]$; and (iv) the ratio of any two basis frequencies associated with distinct equivalent classes is irrational. Suppose there are k equivalent classes $E_{\widetilde{\omega}_s}$, i.e. $s = 1, \cdots, k$. We have

$\mu_\ell^0(t,x) = c_\ell^0(x) + \sum_{s=1}^k q_{\ell,\widetilde{\omega}_s}(t,x)$. Define $\varphi_\ell(x) \equiv \big[\, c_\ell^0(x) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x) \,\big]$. For a given x, we call $\varphi_\ell(x)$ the *principal coefficient* associated with the tuple $\big((C)_{\ell\bullet}, A, x\big)$. It can be shown, via the above properties (i-iv) and Lemma 13, that for any x, $\varphi_\ell(x) = \inf_{[t_*,\infty)} \mu_\ell^0(\cdot, x)$ for any $t_* \geq 0$.

It should be noted that the closed-form expression of $\nu_{\ell,\widetilde{\omega}_s}(x)$, and thus that of $\varphi_\ell(x)$, is hardly obtained. In spite of this, φ_ℓ possesses nice properties, e.g., Lipschitz continuity, as shown below.

Lemma 14. The function $\varphi_\ell(x)$ is Lipschitz continuous for each ℓ.

Proof. For each ℓ, $\varphi_\ell(x) \equiv c_\ell^0(x) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x)$. Since $c_\ell^0(x)$ is linear and thus Lipschitz continuous, it is sufficient to show that each $\nu_{\ell,\widetilde{\omega}_s}(x)$ is Lipschitz continuous. Recall that $\nu_{\ell,\widetilde{\omega}_s}(x) = \min\, q_{\ell,\widetilde{\omega}_s}(\cdot, x)$, where $q_{\ell,\widetilde{\omega}_s}(t,x)$ is smooth and periodic in t for any fixed x. In addition, since $q_{\ell,\widetilde{\omega}_s}(t,x)$ is the summation of finitely many sinusoidal functions whose coefficient is linear in x, we have (i) for any $x, y \in \mathbb{R}^n$, $q_{\ell,\widetilde{\omega}_s}(t,x) = q_{\ell,\widetilde{\omega}_s}(t,y) + q_{\ell,\widetilde{\omega}_s}(t, x-y)$, and (ii) there exists $\rho > 0$ such that $|q_{\ell,\widetilde{\omega}_s}(t,x)| \leq \rho\,\|x\|, \forall\, t$. Notice that for any $x, y \in \mathbb{R}^n$,

$$q_{\ell,\widetilde{\omega}_s}(t,x) \;\leq\; q_{\ell,\widetilde{\omega}_s}(t,y) + |q_{\ell,\widetilde{\omega}_s}(t,x-y)| \;\leq\; q_{\ell,\widetilde{\omega}_s}(t,y) + \rho\|x-y\|, \quad \forall\, t \in \mathbb{R}$$

Let t' be a minimum of $q_{\ell,\widetilde{\omega}_s}(\cdot, y)$. Therefore,

$$\min\, q_{\ell,\widetilde{\omega}_s}(\cdot, x) \;\leq\; q_{\ell,\widetilde{\omega}_s}(t', x) \leq q_{\ell,\widetilde{\omega}_s}(t', y) + \rho\|x-y\| \;=\; \min\, q_{\ell,\widetilde{\omega}_s}(\cdot, y) + \rho\|x-y\|$$

This shows that $\nu_{\ell,\widetilde{\omega}_s}(x) \leq \nu_{\ell,\widetilde{\omega}_s}(y) + \rho\|x-y\|$. Similarly, $\nu_{\ell,\widetilde{\omega}_s}(y) \leq \nu_{\ell,\widetilde{\omega}_s}(x) + \rho\|x-y\|$. Consequently, $|\nu_{\ell,\widetilde{\omega}_s}(x) - \nu_{\ell,\widetilde{\omega}_s}(y)| \leq \rho\|x-y\|$. Hence, $\nu_{\ell,\widetilde{\omega}_s}$ is Lipschitz continuous. $\square$

The following theorem provides necessary and sufficient conditions for the interior of $\mathcal{A}$. Note that the two conditions given below are independent, i.e., they do not imply each other in general. See Example 11 for illustration.

Theorem 15. Let $x^* \in \mathbb{R}^n$. Then $x^* \in \operatorname{int}\mathcal{A}$ if and only if the following conditions both hold:

(a) $Ce^{At}x^* > 0, \forall\, t \geq 0$;

(b) for each $\ell \in \{1, \cdots, m\}$, the principal coefficient associated with $\big((C)_{\ell\bullet}, A, x^*\big)$ is positive.

Proof. "Necessity". We show this by contradiction. Suppose $x^* \in \operatorname{int}\mathcal{A}$ but (a) does not hold, then there exists $t_* \geq 0$ such that $(C)_{\ell\bullet}e^{At_*}x^* = 0$ for some ℓ. Since $(C)_{\ell\bullet} \neq 0$ (recall that C has no zero rows), we have $(C)_{\ell\bullet}e^{At_*}\big[x^* - \varepsilon e^{-At_*}\big((C)_{\ell\bullet}\big)^T\big] = -\varepsilon (C)_{\ell\bullet}\big((C)_{\ell\bullet}\big)^T < 0$ for all $\varepsilon > 0$. Hence, $x^* \notin \operatorname{int}\mathcal{A}$, a contradiction. Now we assume that (b) does not hold. Then there is an ℓ such that the principal coefficient associated with $\big((C)_{\ell\bullet}, A, x^*\big)$ is non-positive, i.e., $\varphi_\ell(x^*) \equiv c_\ell^0(x^*) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x^*) \leq 0$. We consider a vector of the form $\widehat{x} = x^* + \varepsilon v$, where the scalar $\varepsilon > 0$, and the vector $v \neq 0$ is chosen in the following two cases:

(1) $\mathcal{L}_a^r$ is nonempty. Let $j \in \mathcal{L}_a^r$ and $v = \big(0, \cdots, 0, (v^{aj})^T, 0, \cdots, 0\big)^T$. Here v^{aj} is the sub-vector corresponding to J_{aj} whose last element is chosen as $-(C_{aj})_{\ell\, k_j} \neq 0$, i.e., $v^{aj}_{m_j} = -(C_{aj})_{\ell\, k_j}$. Note that this choice of $\widehat{x}$ does not affect $g^0_{\ell,\omega_j}(x)$ and $h^0_{\ell,\omega_j}(x)$ (see the expressions of these functions above), and hence does not change $\nu_{\ell,\widetilde{\omega}_s}$. Therefore, $\varphi_\ell(\widehat{x}) = c_\ell^0(\widehat{x}) + \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x^*)$. Moreover, since $c_\ell^0(x)$ is linear, we have $c_\ell^0(\widehat{x}) = c_\ell^0(x^*) - \frac{1}{b!}\,\varepsilon\,[(C_{aj})_{\ell\, k_j}]^2$. Thus the principal coefficient $\varphi_\ell(\widehat{x}) = \varphi_\ell(x^*) - \frac{1}{b!}\,\varepsilon\,[(C_{aj})_{\ell\, k_j}]^2 < 0$ for all $\varepsilon > 0$.

(2) $\mathcal{L}_a^r$ is empty but $\mathcal{L}_a^c$ is nonempty. Therefore $c_\ell^0(x) = 0$ for all x such that $\varphi_\ell(x) = \sum_{s=1}^k \nu_{\ell,\widetilde{\omega}_s}(x)$, where $\nu_{\ell,\widetilde{\omega}_s}(x) \le 0, \forall\, s$. Consider two subcases: (2.1) $\big(g^0_{\ell,\omega_j}(x^*),\, h^0_{\ell,\omega_j}(x^*)\big) \neq (0,0)$ for some $j \in \mathcal{L}_a^c$, and (2.2) $\big(\widetilde{g}^0_{\ell,\omega_j}(x^*),\, \widetilde{h}^0_{\ell,\omega_j}(x^*)\big) = (0,0)$ for all $j \in \mathcal{L}_a^c$. In subcase (2.1), let $\omega_j \in E_{\widetilde{\omega}_s}$ for some s. Then $\nu_{\ell,\widetilde{\omega}_s}(x^*) < 0$ and thus $\varphi(x^*) < 0$. Let $v \neq 0$ be arbitrary. It follows from the continuity of φ_ℓ (cf. Lemma 14) that $\varphi(\widehat{x}) < 0$ for all $\varepsilon > 0$ sufficiently small. For subcase (2.2), $\nu_{\ell,\widetilde{\omega}_s}(x^*) = 0, \forall\, s$ and thus $\varphi(x^*) = 0$. Let $s \in \mathcal{L}^c_{\omega_j}$ for some ω_j and $v = \big(0,\cdots,0,(v^{as})^T,0,\cdots,0\big)^T$, where v^{as} is the sub-vector corresponding to J_{as} whose last two elements satisfy $\big(v^{as}_{2m_s-1}, v^{as}_{2m_s}\big) = -(C_{as})_{\ell k_s} \in \mathbb{R}^{1\times 2}$. Since $(C_{as})_{\ell k_s} \neq 0$, it can be shown via the formulation of $g^0_{\ell,\omega_j}(x)$ and $h^0_{\ell,\omega_j}(x)$ that $g^0_{\ell,\omega_j}(\widehat{x}) = -\frac{1}{b!}\,\varepsilon\,\|(C_{as})_{\ell k_s}\|_2^2 < 0$ and $h^0_{\ell,\omega_j}(\widehat{x}) = 0$. This implies that for all $\varepsilon > 0$, $\nu_{\ell,\widetilde{\omega}_{s'}}(\widehat{x}) < 0$ for some s'. Therefore $\varphi_\ell(\widehat{x}) < 0$ for all $\varepsilon > 0$.

Noting $\varphi_\ell(\widehat{x}) < 0$ for all $\varepsilon > 0$ sufficiently small in both of the above cases, we see via Lemma 12 that for any small $\varepsilon > 0$ and any $t_* \ge 0$, there is $t' \in [t_*, \infty)$ such that $\mu^0_\ell(t', \widehat{x}) \le \varphi_\ell(\widehat{x})/2 < 0$. Furthermore, since $C_{\ell\bullet} e^{At}\widehat{x}$ tends to $\mu^0_\ell(t,\widehat{x}) e^{\lambda_a t} t^b$ as $t \to +\infty$ (i.e., for any $\epsilon > 0$, there is $t_\epsilon \ge 0$ such that $\big|\, C_{\ell\bullet} e^{At}\widehat{x} - \mu^0_\ell(t,\widehat{x}) e^{\lambda_a t} t^b \,\big| \le \epsilon, \forall\, t \ge t_\epsilon$), we deduce that for any $\varepsilon > 0$ sufficiently small, $C_{\ell\bullet} e^{A\widehat{t}}\widehat{x} < 0$ for some large $\widehat{t} \ge 0$. As a result, $x^* \notin \operatorname{int}\mathcal{A}$. This is a contradiction.

"Sufficiency". Consider an $\ell \in \{1,\cdots,m\}$. For each $\mu^k_\ell(t,x) \equiv c^k_\ell(x) + \sum_s g^k_{\ell,\omega_s}(x)\cos(\omega_s t) + h^k_{\ell,\omega_s}(x)\sin(\omega_s t)$ with $k \ge 1$, define $d^k_\ell(x) \equiv |c^k_\ell(x)| + \sum_s \big(|g^k_{\ell,\omega_s}(x)| + |h^k_{\ell,\omega_s}(x)|\big)$. Hence, $d^k_\ell(x)$ is continuous and provides an upper bound for $\mu^k_\ell(\cdot,x)$ on $\mathbb{R}$ for any given x, i.e., $d^k_\ell(x) \ge \max|\mu^k_\ell(\cdot,x)|$. In view of the continuity of φ_ℓ (cf. Lemma 14), we deduce that there exists a neighborhood $\mathcal{N}_1$ of x^* such that for each ℓ, $d^k_\ell(x)$ is bounded on $\mathcal{N}_1$ for all k and that $\varphi_\ell(x) \ge \frac{2\varphi_\ell(x^*)}{3} > 0$ for all $x \in \mathcal{N}_1$. Note that for each ℓ, $C_{\ell\bullet} e^{At}x \ge \varphi_\ell(x) e^{\lambda_a t} t^b - \sum_{k\ge 1} d^k_\ell(x) e^{\lambda_k t} t^{b_k}$ for all $t \ge 0$ and that $\big(\lambda_a, b\big) \succ (\lambda_k, b_k)$ for each k. Consequently, we obtain a scalar $t_* > 0$, via the bounds for $d^k_\ell(x)$ and $\varphi_\ell(x)$, such that for each ℓ and all $x \in \mathcal{N}_1$, $C_{\ell\bullet} e^{At}x \ge \frac{\varphi_\ell(x^*)}{2}\, e^{\lambda_a t} t^b > 0, \ \forall\, t \ge t_*$. Furthermore, since $Ce^{At}x^* > 0$ on the compact time interval $[0,t_*]$, we claim that there exists a neighborhood $\mathcal{N}_2$ of x^* such that $Ce^{At}x > 0, \forall\,(t,x) \in [0,t_*] \times \mathcal{N}_2$. To see this, define $r_\ell(x) \equiv \min_{[0,t_*]} C_{\ell\bullet} e^{At}x$. It follows from the compactness of $[0,t_*]$ and the similar argument of Lemma 14 that each r_ℓ is Lipschitz continuous. Since $r_\ell(x^*) > 0$ for each ℓ, we deduce, via the continuity of r_ℓ, that there exists a neighborhood $\mathcal{N}_2$ of x^* such that $r_\ell(x) > 0, \forall\, x \in \mathcal{N}_2$ for each ℓ, or equivalently, $Ce^{At}x > 0, \forall\,(t,x) \in [0,t_*] \times \mathcal{N}_2$. Finally, letting $\mathcal{N} \equiv \mathcal{N}_1 \cap \mathcal{N}_2$, we have $Ce^{At}x > 0, \forall\, t \ge 0$ for any x in the neighborhood $\mathcal{N}$ of x^*. Therefore, $x^* \in \operatorname{int}\mathcal{A}$. □

For further illustration, we revisit Example 11. For the given $C = (1,1)^T$ and $A = \operatorname{diag}(1,2) \in \mathbb{R}^{2\times 2}$, the principal mode associated with (C,A) is e^{2t} and for the given $x^* = (1,0)^T$, the principal coefficient associated with the triple (C,A,x^*) is 0. Hence, x^* is not in the interior of the positively invariant cone $\mathcal{A}$, even though $Ce^{At}x^* > 0, \forall\, t \ge 0$. This agrees with the conclusion drawn in Example 11. Furthermore, it can be shown via Theorem 15 that $x^* = (0,1)^T$ is in the interior of $\mathcal{A}$. The interior conditions established in Theorem 15 play an important role in characterization of the connection between finite-time and long-time local observability discussed in Section 4; see Theorem 21 and Example 23 for more details.

4 Finite-time and Long-time Observability Analysis of the CLS via Directional Derivative and Positive Invariance

Throughout this section, let $H \in \mathbb{R}^{r\times n}$ be a given matrix that defines the linear output Hx for the CLS (1). We recall some observability notions as follows:

Definition 16. A pair of states $(\xi, \eta) \in \mathbb{R}^{n+n}$ is called
• *short-time indistinguishable* if $\varepsilon > 0$ exists such that $Hx(t,\xi) = Hx(t,\eta)$ for all $t \in [0,\varepsilon]$;
• *T-time indistinguishable* for a given $T > 0$ if $Hx(t,\xi) = Hx(t,\eta)$ for all $t \in [0,T]$;
• *long-time indistinguishable* if $Hx(t,\xi) = Hx(t,\eta)$ for all $t \geq 0$.

Clearly, long-time indistinguishability implies T-time indistinguishability, and the latter implies short-time indistinguishability for any pair of states.

Definition 17. A state $\xi \in \mathbb{R}^n$ is called

• *short-time (*resp. *T-time/long-time) locally observable* if there exists a neighborhood $\mathcal{N}$ of ξ such that no pair (ξ,η) with $\eta \in \mathcal{N} \setminus \{\xi\}$ is short-time (resp. T-time/long-time) indistinguishable;

• *finite-time locally observable* if there exist a $T > 0$ and a neighborhood $\mathcal{N}$ of ξ such that no pair (ξ,η) with $\eta \in \mathcal{N} \setminus \{\xi\}$ is T-time indistinguishable.

To unify the notation, we allow $T = \infty$. In such a case, T-time observability means long-time observability. Global observability can be defined in the similar manner; see [7] for details.

4.1 Finite-time and Long-time Local Observability

It is clear that short-time observability implies finite-time observability, which in turn yields long-time observability, but the converse generally do not hold [7]. While short-time observability has been recently extensively studied in [7], much less is known about finite-time and long-time observability, except some limited results for the bimodal CLS. The difficulty in addressing the latter observability concepts is largely attributed to the lack of understanding of state-dependent mode switchings of the CLS, particularly finite-time and long-time switching behaviors. In this section, we exploit the results of mode switching, directional derivative and positive invariance developed in the prior sections to obtain concrete observability conditions. We begin with T-time local observability.

Given a finite $T > 0$. A neat sufficient condition for a state ξ to be T-time locally observable is given by the following implication in term of directional derivatives [21, Theorem 10]:

$$\big[\ Hx'(t,\xi;\eta) = 0,\ \forall\ t \in [0,T]\ \big] \implies \eta = 0 \tag{11}$$

It turns out that if there is no critical time on $[0,T]$ along $x(t,\xi)$, then this condition is also necessary as indicated in the following theorem. It is worth mentioning that although the nominal trajectory $x(t,\xi)$ has no critical time, a perturbed trajectory may have critical times and even mode switchings on $[0,T]$. This substantially complicates the observability analysis.

Theorem 18. Given $\xi \in \mathbb{R}^n$ and $T > 0$. Suppose there is no critical time on $[0,T]$ along $x(t,\xi)$. Then ξ is T-time locally observable if and only if the condition (11) holds.

Proof. We prove the necessity only. In light of (b) of Theorem 9, we see that for all $\tau > 0$ sufficiently small, $x(t, \xi + \tau\eta) = x(t, \xi) + \tau x'(t, \xi; \eta)$ on $[0, T]$. Suppose $\eta \neq 0$ but $Hx'(t, \xi; \eta) = 0, \ \forall\, t \in [0, T]$. Then $Hx(t, \xi + \tau\eta) = Hx(t, \xi)$ on $[0, T]$ for all $\tau > 0$ sufficiently small. Noticing $\eta \neq 0$, we deduce that ξ is not T-time locally observable. This is a contradiction. $\square$

Theorem 18 completely characterizes T-time local observability of a state whose corresponding trajectory does not have critical times via directional derivatives, which in turn rely on critical time and mode switching results from Section 2. Admittedly, the closed-form expression of directional derivatives is generally difficult to obtain. In the following, we provide more explicit sufficient and necessary conditions; the obtained necessary condition even holds for $T = \infty$ and a nominal non-switching trajectory (possibly with critical times along it).

Proposition 19. Consider a state $\xi \in \mathbb{R}^n$ such that $x(t, \xi)$ has no mode switching on $[0, T]$ with $0 < T \leq \infty$. Then ξ is T-time locally observable only if

$$\overline{O}(H, A_i) \cap \Big\{ v \mid C_i e^{A_i t} v \geq 0, \ \forall\, t \in [0, T] \Big\} = \{0\}, \quad \forall \ \ i \in \mathcal{J}(\xi) \tag{12}$$

Moreover, suppose $T > 0$ is finite and $x(t, \xi)$ has no critical time on $[0, T]$. Then ξ is T-time locally observable if the following conditions both hold:

$$\overline{O}(H, A_i) \cap \Big\{ v \mid x(t, \xi + v) = e^{A_i t}(\xi + v), \ \forall\, t \in [0, T] \Big\} = \{0\}, \ \ \forall \ \ i \in \mathcal{J}(\xi), \tag{13}$$

$$\overline{O}(H, A_i) \cap \overline{O}(H, A_j) = \{0\}, \ \ \forall\, i, j \in \mathcal{J}(\xi) \ \text{ with } \ A_i \neq A_j \tag{14}$$

Furthermore, if the CLS is simple, then the condition (13) is equivalent to

$$\overline{O}(H, A_i) \cap \Big\{ v \mid C_i e^{A_i t}(\xi + v) \geq 0, \ \forall\, t \in [0, T] \Big\} = \{0\}, \ \ \forall \ \ i \in \mathcal{J}(\xi) \tag{15}$$

Proof. Note that if the condition (12) fails, then there exist $i \in \mathcal{J}(\xi)$ and $v \neq 0$ such that $v \in \overline{O}(H, A_i)$ and $C_i e^{A_i t} v \geq 0$ on $[0, T]$. Consider the sequence $\{\eta^k\}$, where $\eta^k = \xi + v/k, \forall\, k \in \mathbb{N}$. Hence, $\{\eta^k\}$ converges to ξ with $\eta^k \neq \xi$ for all k. Moreover, it follows from Lemma 5 that $C_i e^{A_i t}\xi \geq 0, \forall\, t \in [0, T]$. This shows that $C_i e^{A_i t}\eta^k = C_i e^{A_i t}[\xi + v/k] \geq 0, \forall\, t \in [0, T]$ for each k. Hence, $x(t, \eta^k) = e^{A_i t}\eta^k$ on $[0, T]$. However, each pair (ξ, η^k) is T-time indistinguishable as $Hx(t, \xi) = Hx(t, \eta^k)$ on $[0, T]$. This is contradictory to the T-time local observability at ξ.

We prove the second statement by contradiction. Suppose ξ is not T-time locally observable. It follows from Theorem 18 that there exists a nonzero η such that $Hx'(t, \xi; \eta) = 0$ for all $t \in [0, T]$. Moreover, we deduce via Theorem 9 that there exist $0 = \widehat{t}_0 < \widehat{t}_1 < \cdots < \widehat{t}_{p-1} < \widehat{t}_p = T$ and matrices A_{k_i} with $k_i \in \mathcal{J}(\xi)$ such that for all $t \in [\widehat{t}_i, \widehat{t}_{i+1}]$, $x'(t, \xi; \eta) = e^{A_{k_i}(t - \widehat{t}_i)} x'(\widehat{t}_i, \xi; \eta)$, $i = 0, \cdots, p-1$. Without loss of generality, we assume that A_{k_i}'s associated with two neighboring intervals are distinct. We consider two case as follows: (1) $p = 1$, and (ii) $p \geq 2$. For the first case, $x'(t, \xi; \eta) = e^{A_{k_1} t}\eta, \forall\, t \in [0, T]$. Hence, $He^{A_{k_1} t}\eta = 0, \forall\, t \in [0, T]$, which implies $\eta \in \overline{O}(H, A_{k_1})$. Moreover, we have, via statement (b) of Theorem 9, $x(t, \xi + \tau\eta) = x(t, \xi) + \tau e^{A_{k_1} t}\eta = e^{A_{k_1} t}(\xi + \tau\eta)$ on $[0, T]$ for all $\tau > 0$ sufficiently small. This contradicts the condition (13). For the second case, it is known from Theorem 9 that $x'(t, \xi; \eta) = e^{A_{k_1} t}\eta, \forall\, t \in [0, \widehat{t}_1]$ and $x'(t, \xi; \eta) = e^{A_{k_2}(t - \widehat{t}_1)} x'(\widehat{t}_1, \xi; \eta), \forall\, t \in [\widehat{t}_1, \widehat{t}_2]$, where $k_1, k_2 \in \mathcal{J}(\xi)$ and $A_{k_1} \neq A_{k_2}$. Since $Hx'(t, \xi; \eta) = 0, \forall\, t \in [0, T]$, we have $x'(\widehat{t}_1, \xi; \eta) \in \overline{O}(H, A_{k_1}) \cap \overline{O}(H, A_{k_2})$. On the other hand, notice that $x'(\widehat{t}_1, \xi; \eta) \neq 0$ as $\eta \neq 0$. This is contradictory to (14). Finally, the equivalence between (13) and (15), under the assumption that the CLS is simple, follows from Lemma 3. $\square$

By virtue of this result and Proposition 7, we combine observability conditions for each subintervals defined by consecutive critical times to obtain the following corollary pertaining to an arbitrary nominal trajectory without further proof; the obtained conditions can be further simplified if the CLS is simple.

Corollary 20. Consider a state $\xi \in \mathbb{R}^n$ and $T > 0$. Let $t_i \in [0, T], i = 1, \cdots, p-1$ be the critical times such that $0 = t_0 < t_1 < \cdots < t_{p-1} < t_p = T$ and the interval (t_i, t_{i+1}) does not contain a critical time for each $i = 0, \cdots, p-1$. If, for some interval (t_j, t_{j+1}) with $j \in \{0, \cdots, p-1\}$, there exists a compact interval $[\widehat{t}_1, \widehat{t}_2] \subset (t_j, t_{j+1})$ such that $\overline{O}(H, A_i) \cap \{ v \,|\, x(t, x^* + v) = e^{A_i t}(x^* + v), \forall\, t \in [0, \widehat{t}_2 - \widehat{t}_1] \} = \{0\}, \forall\, i \in \mathcal{J}(x^*)$ and $\overline{O}(H, A_i) \cap \overline{O}(H, A_j) = \{0\}, \forall\, i, j \in \mathcal{J}(x^*)$ with $A_i \neq A_j$, where $x^* \equiv x(\widehat{t}_1, \xi)$, then ξ is T-time locally observable.

In the following, we establish subtle connections between finite-time and long-time observability by making use of the positive invariance results. Especially we address the question of whether long-time observability implies finite-time observability, since the former is much more difficult to check in general. We focus on the case where a nominal trajectory eventually remains in a polyhedral cone of the CLS. It should be pointed out that a perturbed trajectory, even locally perturbed, may not stay in the same polyhedral cone in a long time unless certain positive invariance conditions are imposed for the nominal trajectory (cf. Theorem 21). This is one of major difficulties in large-time observability analysis and a motivation for employing positive invariance.

The first result affirms the equivalence of finite-time and long-time local observability under the assumption that a nominal trajectory will enter the interior of a positively invariant cone.

Theorem 21. Given a state $\xi \in \mathbb{R}^n$. Suppose there exists $t_* \geq 0$ such that $x(t_*, \xi) \in \operatorname{int} \mathcal{A}_i$, where $\mathcal{A}_i$ is the positively invariant cone of the ith mode. Then the following are equivalent:

(a) ξ is long-time locally observable;

(b) ξ is finite-time locally observable;

(c) ξ is t_*-time locally observable.

Proof. The implication $(c) \Rightarrow (b) \Rightarrow (a)$ is obvious. In order to prove the other implications, it is sufficient to consider $(a) \Rightarrow (c)$ which we shall prove by contradiction as follows. Suppose ξ is not t_*-time locally observable. Then, there exists a sequence $\{\eta^\nu\}$ with $\eta^\nu \neq \xi, \forall\, \nu \in \mathbb{N}$ such that $\{\eta^\nu\}$ converges to ξ and the pair (ξ, η^ν) is t_*-time indistinguishable for all ν, namely, $Hx(t, \xi) = H(x, \eta^\nu)$ for all $t \in [0, t_*]$. Since the CLS is globally Lipschitz, $\|x(t_*, \xi) - x(t_*, \eta^\nu)\| \leq e^{Lt_*}\|\xi - \eta^\nu\|$ for all ν, where $L > 0$ is the Lipschitz constant. This, together with the assumption that $x(t_*, \xi)$ is in the interior of $\mathcal{A}_i$, implies that there exist a neighborhood $\mathcal{U}$ of $x(t_*, \xi)$ and a subsequence $\{\widetilde{\eta}^\nu\}$ of $\{\eta^\nu\}$ such that $x(t_*, \widetilde{\eta}^\nu) \in \mathcal{U} \subseteq \operatorname{int} \mathcal{A}_i$ for all ν. Hence, $x(t, \widetilde{\eta}^\nu) \in \mathcal{X}_i$ for all $t \geq t_*$ and all ν due to positive invariance. It further follows from the continuity of the CLS trajectories that for each pair $(\xi, \widetilde{\eta}^\nu)$, there exists $\varepsilon_\nu > 0$ such that $x(t, \xi) \in \mathcal{A}_i$ and $x(t, \widetilde{\eta}^\nu) \in \mathcal{A}_i$ for all $t \in [t_* - \varepsilon_\nu, t_*]$. Since each pair $(\xi, \widetilde{\eta}^\nu)$ is t_*-time indistinguishable, we have $Hx(t, \xi) = Hx(t, \widetilde{\eta}^\nu)$ for all $t \in [t_* - \varepsilon_\nu, t_*]$. Note that $\mathcal{A}_i \subseteq \mathcal{X}_i$ and $x(t, \xi) = e^{A_i(t - t_* + \varepsilon_\nu)}x(t_* - \varepsilon_\nu, \xi)$ and $x(t, \widetilde{\eta}^\nu) = e^{A_i(t - t_* + \varepsilon_\nu)}x(t_* - \varepsilon_\nu, \widetilde{\eta}^\nu)$ for all $t \geq (t_* - \varepsilon_\nu)$. In view of $He^{A_i(t - t_* + \varepsilon_\nu)}x(t_* - \varepsilon_\nu, \xi) = He^{A_i(t - t_* + \varepsilon_\nu)}x(t_* - \varepsilon_\nu, \widetilde{\eta}^\nu)$ for all $t \in [t_* - \varepsilon_\nu, t_*]$, we deduce that $[x(t_* - \varepsilon_\nu, \widetilde{\eta}^\nu) - x(t_* - \varepsilon_\nu, \xi)] \in \overline{O}(H, A_i)$. Thus $[x(t_*, \widetilde{\eta}^\nu) - x(t_*, \xi)] \in \overline{O}(H, A_i)$ also holds such that $He^{A_i(t - t_*)}x(t_*, \xi) = He^{A_i(t - t_*)}x(t_*, \widetilde{\eta}^\nu)$ for all $t \geq t_*$, or equivalently $Hx(t, \xi) =$

$Hx(t, \widetilde{\eta}^\nu)$ for all $t \geq t_*$. This thus shows that $Hx(t, \xi) = Hx(t, \widetilde{\eta}^\nu)$ for all $t \geq 0$. Consequently, the pair $(\xi, \widetilde{\eta}^\nu)$ is long-time indistinguishable for each ν. Therefore ξ is not long-time locally observable. This is a contradiction. Hence $(a) \Rightarrow (c)$ holds, so does $(a) \Rightarrow (b)$. □

The interior condition in the above theorem can be verified using Theorem 15. While this condition seems strict, it is shown via Proposition 22 and Example 23 below that if this condition fails, then long-time local observability may not give rise to finite-time observability, even if a nominal trajectory will eventually remain in the interior of a polyhedral cone. Indeed, it is revealed in Proposition 22 and Example 23 that for a given state ξ and a time $t_* > 0$, even if $x(t_*, \xi)$ satisfies condition (a) of Theorem 15, the failure of condition (b) of Theorem 15 at $x(t_*, \xi)$ may lead to the non-equivalence between finite-time and long-time local observability of ξ. In other words, without the interior condition, a state may be long-time locally observable even though it is *not* T-time locally observable for any $T > 0$. To further elaborate on this, we need the following result:

Proposition 22. Consider a state $\xi \in \mathbb{R}^n$. Suppose there exist $t_* > 0$ and a cone $\mathcal{X}_i$ such that $x(t, \xi) \in \operatorname{int} \mathcal{X}_i$ for all $t \geq t_*$. Then ξ is finite-time locally observable if and only if either (a) ξ is t_*-time locally observable or (b) (H, A_i) is an observable pair.

Proof. The "if" part is trivial under the condition (a); we only need to consider the condition (b). Since $x(t_*, \xi)$ is in the interior of $\mathcal{X}_i$, we deduce, with the aid of the global Lipschitz property of the CLS, that there exists a neighborhood $\mathcal{N}$ of ξ such that $x(t_*, x^0)$ is in the interior of $\mathcal{X}_i$ for all $x^0 \in \mathcal{N}$. This implies that $x(t, x^0) \in \mathcal{X}_i$ for all t sufficiently close to t_*. Hence, in view of the condition (b), we conclude that $x(t_*, \xi)$ is small-time locally observable. This further shows that ξ is finite-time locally observable, following from [7, Proposition 4.11].

We prove the "only if" part by contraposition. Suppose both (a) and (b) fail. As ξ is not t_*-time locally observable, it is obvious that ξ is not T-time locally observable for any $T \in (0, t_*]$. Moreover, the former implies that there exists a sequence $\{\eta^\nu\}$ converging to ξ such that for each $\nu \in \mathbb{N}$, $\eta^\nu \neq \xi$ and $Hx(t, \xi) = Hx(t, \eta^\nu)$ for all $t \in [0, t_*]$. By appropriately restricting the sequence $\{\eta^\nu\}$, we may assume, without loss of generality, that $x(t_*, \eta^\nu) \in \mathcal{U} \subseteq \operatorname{int} \mathcal{X}_i$ for some neighborhood $\mathcal{U}$ of $x(t_*, \xi)$ and each ν. We further deduce, via an argument similar to Theorem 21, that $[x(t_*, \eta^\nu) - x(t_*, \xi)] \in \overline{O}(H, A_i)$. (Indeed, we only need to replace $\operatorname{int} \mathcal{A}_i$ by $\operatorname{int} \mathcal{X}_i$ in order to obtain this result.) Define $\tau_\nu \equiv \sup\{\, t \geq t_* \,|\, x(t, \eta^\nu) \in \mathcal{X}_i\}$. Notice that $\tau_\nu > t_*$ and $Hx(t, \xi) = Hx(t, \eta^\nu)$ on $[t_*, \tau_\nu]$. Since $x(t, \xi) \in \operatorname{int} \mathcal{X}_i$ for all $t \geq t_*$, $C_i\, x(t, \xi) > 0, \forall\, t \geq t_*$ (cf. Section 2). Moreover, for any $T > t_*$, let $\rho_T \equiv \min_{t \in [t_*, T],\, \ell \in \{1, \cdots, m_i\}} (C_i)_{\ell \bullet}\, x(t, \xi)$, where $(C_i)_{\ell \bullet}$ denotes the ℓth row of $C_i \in \mathbb{R}^{m_i \times n}$. Therefore, $\rho_T > 0$. It follows from

$$\|C_i\big(x(t, \eta^\nu) - x(t, \xi)\big)\| \,\leq\, e^{L(T - t_*)} \|C_i\| \, \|x(t_*, \eta^\nu) - x(t_*, \xi)\| \,\leq\, e^{LT} \|C_i\| \, \|\eta^\nu - \xi\|, \ \forall\, t \in [t_*, T]$$

that there exists $K_T \in \mathbb{N}$ such that for all $\nu \geq K_T$, $|(C_i)_{\ell \bullet}\, x(t, \eta^\nu) - (C_i)_{\ell \bullet}\, x(t, \xi)| \leq \rho_T / 2$, for all $t \in [t_*, T]$ and all $\ell \in \{1, \cdots, m_i\}$. Hence, $C_i\, x(t, \eta^\nu) > 0$ on $[t_*, T]$ for all $\nu \geq K_T$. This yields $x(t, \eta^\nu) \in \mathcal{X}_i$ on $[t_*, T]$ for each $\nu \geq K_T$. This thus implies that $\tau_\nu \geq T$ and $Hx(t, \eta^\nu) = Hx(t, \xi), \forall\, t \in [0, T]$ for each $\nu \geq K_T$. In other words, the sequence $\{\eta^\nu\}_{\nu = K_T}^\infty$ converges to ξ such that $\eta^\nu \neq \xi$ and the pair (η^ν, ξ) is T-time locally indistinguishable for each $\nu \geq K_T$. Hence, ξ is not T-time locally observable. Since $T \geq t_*$ is arbitrary, ξ is not finite-time locally observable. □

In what follows, we revisit the bimodal CLS that first appears in [7, Example 5.14] to illustrate the above observability results and their connection to the positive invariance conditions. We reveal and emphasize the underlying positive invariance property that has been neglected in the original example.

Example 23. Consider the bimodal CLS in $\mathbb{R}^3$ with

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \omega \\ 0 & -\omega & \alpha \end{bmatrix}, \quad b = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

where $\lambda < 0$, $\alpha > 0$, $\omega > 0$, $b_1 \neq 0$, and c_1 and c_2 are both nonzero. Let $\xi = (\xi_1, 0, 0)^T$ with $c_1 \xi_1 < 0$. It is easy to verify that $c^T x(t, \xi) < 0, \forall\, t \geq 0$ such that $x(t, \xi)$ remains in the interior of the polyhedral cone $\mathcal{X}_2 \equiv \{x \,|\, c^T x \leq 0\}$ for all $t \geq 0$. Notice that the principal mode associated with $(-c^T, A)$ is $e^{\alpha t}$ and the principal coefficient associated with $(-c^T, A, \xi)$ is zero. Hence, we deduce via Theorem 15 that ξ is not in the interior of the positively invariant cone $\mathcal{A}_2$ associated with $\mathcal{X}_2$. The same holds true for $x(t, \xi)$ for any $t > 0$. Let $t_* > 0$ be given. Since $x(t, \xi) \in \operatorname{int} \mathcal{X}_2$ on $[0, t_*]$ and (H, A) is not an observable pair, ξ is not t_*-time locally observable. It thus follows from Proposition 22 that ξ is not finite-time locally observable. While this result has been established via elementary computations in [7, Example 5.14], Proposition 22 generalizes it to a broader setting. More importantly, Proposition 22 unveils the critical positive invariance property for the failure of finite-time observability. Another interesting observation is that ξ is long-time locally observable as shown in [7, Example 5.14]. Hence, this example shows that without the interior condition requested in Theorem 21, long-time and finite-time observability are generally not equivalent.

4.2 Finite-time and Long-time Observability of the Bimodal CLS

We turn to the bimodal CLS (3) to obtain more concrete observability conditions. If there is no switching along $x(t, x^0)$ on $[0, T]$ with $0 < T \leq \infty$ for a given x^0, then $c^T x(t, x^0) \equiv 0$ and $\mathcal{I}(x(t, x^0)) = \mathcal{J}(x(t, x^0)) = \{1, 2\}$ on $[0, T]$. This further shows that there is no critical time on $[0, T]$. Therefore, the conditions in Proposition 19 can be greatly simplified. In fact, the sufficient conditions turn to be necessary [7, Theorem 5.6]. Specifically, x^0 is T-time locally observable if and only if $\overline{O}(H, A) \cap \overline{O}(H, A + bc^T) = \{0\}$, and

$$\begin{aligned} \overline{O}(H, A) \cap \left\{ v \mid c^T e^{At} v \leq 0, \ \forall\, t \in [0, T] \right\} &= \{0\}, \\ \overline{O}(H, A + bc^T) \cap \left\{ v \mid c^T e^{(A+bc^T)t} v \geq 0, \ \forall\, t \in [0, T] \right\} &= \{0\}. \end{aligned} \tag{16}$$

It is worth pointing out that if $T = \infty$, then, under the non-switching assumption, long-time local observability requires that the intersection of each unobservable subspace and its corresponding positively invariant cone be trivial, i.e., the intersection contains the zero vector only. Hence, long-time observability relies on the positively invariant cone of each mode. In the following, we focus on a class of bimodal CLSs to illustrate the positive invariance and observability results.

Recall that a CLS is referred to as the CLS with infinite mode switchings if for any non-equilibrium state $x^0 \in \mathbb{R}^n$, $x(\cdot, x^0)$ has infinitely many mode switchings in $[0, \infty)$ [26]. This implies that for the ith mode, the positively invariant cone $\mathcal{A}_i$ is equal to the equilibrium set of the ith mode, namely, $\mathcal{E}_i \equiv \{x \mid A_i x = 0\}$. An explicit characterization of such a CLS has been established via positive invariance in [26]. For a bimodal CLS, suppose that A is of order greater than one and (c^T, A) is an observable pair. Then the bimodal CLS has infinite mode switchings if and only if either of the following conditions holds [26, Corollary 24]:

(a) if A is of even order, then A and $A + bc^T$ have complex eigenvalues only;

(b) if A is of odd order, then A (resp. $A + bc^T$) has no nonzero real eigenvalue except the zero eigenvalue with algebraic multiplicity one and each complex eigenvalue of A (resp. $A + bc^T$) has a positive real part.

Hence, the positively invariant cones become trivial in case (a) and equal to the null spaces of the respective defining matrices in case (b). Based upon these positive invariance results and [7, Theorem 5.6], we obtain the following corollary pertaining to long-time local observability of $\xi = 0$:

Corollary 24. Consider the bimodal CLS with infinite mode switches, where A is of order greater than one and (c^T, A) is an observable pair.

(a) Let A be of even order. Then $\xi = 0$ is long-time locally observable if and only if $b \notin \overline{O}(H, A)$.

(b) Let A be of odd order. Then $\xi = 0$ is long-time locally observable if and only if $b \notin \overline{O}(H, A)$, $\mathrm{Nul}\, H \cap \mathrm{Nul}\, A = \{0\}$, and $\mathrm{Nul}\, H \cap \mathrm{Nul}\,(A + bc^T) = \{0\}$.

Proof. For case (a), it is clear that the positively invariant cone of each mode contains the zero vector only such that condition (16) holds trivially. Moreover, since the pair (c^T, A) is observable, $\overline{O}(H, A) \cap \overline{O}(c^T, A) = \{0\}$. Hence, it follows from Propositions 5.1 and 5.2 of [7] that $\overline{O}(H, A) \cap \overline{O}(H, A + bc^T) = \{0\}$ holds if and only if $b \notin \overline{O}(H, A)$.

For case (b), since $\mathcal{A}_1 = \{v \mid (A + bc^T)v = 0\} = \mathrm{Nul}\,(A + bc^T)$ and $\mathcal{A}_2 = \{v \mid Av = 0\} = \mathrm{Nul}\, A$, the condition (16) holds if and only if $\overline{O}(H, A) \cap \mathrm{Nul}\, A = \{0\}$ and $\overline{O}(H, A + bc^T) \cap \mathrm{Nul}\,(A + bc^T) = \{0\}$. The latter is further equivalent to the implications $\big[\, Hv = 0,\ v \in \mathrm{Nul} A \,\big] \Rightarrow \big[\, v = 0 \,\big]$ and $\big[\, Hv = 0,\ v \in \mathrm{Nul}(A + bc^T) \,\big] \Rightarrow \big[\, v = 0 \,\big]$. This, together with the argument for case (a), leads to the desired result. □

5 Conclusions

In this paper, we have investigated finite-time observability via directional derivative techniques and addressed long-time observability from the positive invariance perspective. These perspectives yield new observability conditions for a general CLS. Nevertheless, there remain many open issues. For example, long-time observability of a state whose corresponding trajectory has infinitely many switchings is largely unknown. A main difficulty is due to possible orbital instability along a nominal trajectory. Moreover, a further study is warranted to understand positively invariant cones and their implications to long-time dynamics of the CLS. Another interesting issue, different from the analytic perspective of the current paper, is how to design an observer for state estimation (namely, observer synthesis). Certain topological techniques may be invoked with suitable stability assumptions; see, for example, [32].

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References

[1] A. Abate, A. Tiwari, and S. Sastry. Box invariance for biologically-inspired dynamical systems. *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 5162–5167, New Orleans, LA, 2007.

[2] M. Babaali and M. Egerstedt. Observability of switched linear systems. In R. Alur and G.J. Pappas, editiors, *Hybrid Systems: Computation and Control*, LNCS 2993, Springer (Berlin 2004), pp. 48–63.

[3] M. Babaali and G.J. Pappas. Observability of switched linear systems in continuous time. *Hybrid Systems: Computation and Control*, LNCS 3414, Springer (Berlin 2005), pp. 103–117.

[4] C. Belta and L.C.G.J.M. Habets. Controlling of a class of nonlinear systems on rectangles. *IEEE Trans. on Automatic Control*, Vol. 51(11), pp. 1749–1759, 2006.

[5] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Trans. on Automatic Control*, Vol. 45(10), pp. 1864–1876, 2000.

[6] M.K. Çamlibel, W.P.M.H. Heemels and J.M. Schumacher. Algebraic necessary and sufficient conditions for the controllability of conewise linear systems. *IEEE Trans. on Automatic Control*, Vol. 53(3), pp. 762–774, 2008.

[7] M.K. Çamlibel, J.S. Pang, and J. Shen. Conewise linear systems: non-Zenoness and observability. *SIAM Journal on Control and Optimization*, Vol. 45(5), pp. 1769–1800, 2006.

[8] F.H. Clarke. *Optimization and Nonsmooth Analysis*. Classics in applied mathematics, No. 5, SIAM press, 1990.

[9] P. Collins and J.H. van Schuppen. Observability of piecewise affine hybrid systems. In R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control*, LNCS 2993, Springer (Berlin 2004), pp. 250–264.

[10] E.F. Costa and J.B.R. do Val. On the observability and detectability of continuous-time jump linear systems. *SIAM Journal on Control and Optimization*, Vol. 41(4), pp. 1295–1314, 2002.

[11] H. de Jong. Modeling and simulation of genetic regulatory systems: a literature review. *Journal of Computational Biology*, Vol. 9(1), pp. 67–103, 2002.

[12] D. Del Vecchio, R.M. Murray, and E. Klavins. Discrete state estimators for systems on lattice. *Automatica*, Vol. 42(2), pp. 271–285, 2006.

[13] F. Facchinei and J.S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer-Verlag, New York, 2003.

[14] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM Journal on Applied Mathematics*, Vol. 60(4), pp. 1234–1269, 2000.

[15] J.P. Hespanha, D. Liberzon, D. Angeli, and E.D. Sontag. Nonlinear norm-observability notions and stability of switched systems. *IEEE Trans. on Automatic Control*, Vol. 50(2), pp. 154–168, 2005.

[16] R. Hermann and A.J. Krener. Nonlinear controllability and observability. *IEEE Trans. on Automatic Control*, Vol. 22(5), pp. 728–740, 1977.

[17] T. Hu, L. Ma, and Z. Lin. Stabilization of switching systems via composite quadratic functions. *IEEE Trans. on Automatic Control*, Vol.53(11), pp. 2571–2585, 2008.

[18] N. Kazantzis, and C. Kravaris. Nonlinear observer design using Lyapunov's auxiliary theorem. *Systems and Control Letter*, Vol. 34(5), pp. 241–247, 1998.

[19] H. Khalil. *Nonlinear Systems*. 2nd Edition, Prentice Hall, 1996.

[20] R.K. Miller and A.N. Michel. Asymptotic stability of systems: results involving the system topology. *SIAM Journal on Control and Optimization*, Vol. 18(2), pp. 181–190, 1980.

[21] J.S. Pang and J. Shen. Strongly regular differential variational systems. *IEEE Trans. on Automatic Control*, Vol. 52(2), pp. 242–255, 2007.

[22] J.S. Pang and D. Stewart. Solution dependence on initial conditions in differential variational inequalities. *Mathematical Programming, Series B*, Vol. 116, pp. 429–460, 2009.

[23] S. Scholtes. *Introduction to Piecewise Differentiable Equations*. Habilitation thesis, Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, 1994.

[24] J.M. Schumacher. Complementarity systems in optimization. *Mathematical Programming, Series B*, Vol. 101, pp. 263–295, 2004.

[25] J. Shen. Positive invariance of constrained affine dynamics and its applications to hybrid systems and safety verification. Revision submitted, 2009.

[26] J. Shen, L. Han, and J.S. Pang. Switching and stability properties of conewise linear systems *ESAIM: Control, Optimisation, and Calculus of Variations*, DOI: 10.1051/cocv/2009021, 2009.

[27] J. Shen and J.S. Pang. Linear complementarity systems: Zeno states. *SIAM Journal on Control and Optimization*, Vol. 44(3), pp. 1040–1066, 2005.

[28] J. Shen and J.S. Pang. Linear complementarity systems with singleton properties: non-Zenoness. *Proceedings of 2007 American Control Conference*, pp. 2769–2774, New York, 2007.

[29] Special issue on systems biology. *IEEE Trans. on Automatic Control*, Vol. 53(1), 2008.

[30] A. Tiwari, J. Fung, R. Bhattacharya, and R.M. Murray. Polyhedral cone invariance applied to rendezvous of multiple agents. *Proceedings of the 43rd IEEE Conference on Decision and Control*, pp. 165–170, Bahamas, 2004.

[31] R. Vidal, A. Chiuso, S. Soatto, and S. Sastry. Observability of linear hybrid systems. In O. Maler and A. Pnueli, editors, *Hybrid Systems: Computation and Control*, LNCS 2623, Springer (Berlin 2003), pp. 526–539.

[32] M. Xiao. The global existence of nonlinear observers with linear error dynamics: a topological point of view. *Systems and Control Letters*, Vol. 55(10), pp. 849–858, 2006.