Generalized Input-to-State ℓ_2-Gains of Discrete-Time Switched Linear Control Systems[*]

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Abstract

A generalized notion of input-to-state ℓ_2-gain is proposed for discrete-time switched linear control systems (SLCSs). Being a function of a certain discount factor, this generalized ℓ_2-gain provides new insight into input-to-state behaviors of the SLCSs under parameter variations. After establishing several analytical properties of the generalized ℓ_2-gain, the paper focuses on the generating function approach to the study of the generalized ℓ_2-gain. Important properties of generating functions are derived, and it is shown that their radii of convergence characterize the generalized ℓ_2-gain. Furthermore, iterative algorithms are developed for computing the generating functions with proven uniform or exponential convergence. Numerical results show that these algorithms yield efficient estimates of both the generalized and classical ℓ_2-gains.

1 Introduction

A switched linear control system (SLCS) consists of a finite number of linear control subsystems along with a switching rule that determines switchings among subsystems. Such systems find numerous applications such as power electronics and automotive control, where switchings among multiple linear controllers are exploited to improve system performance. Belonging to the general framework of hybrid control systems, the SLCSs exhibit rich dynamics in spite of simple structure in their subsystems. In particular, they demonstrate inherently nonsmooth and hybrid dynamical behaviors, and require novel analytical and numerical techniques for their study [4].

The concept of $\mathcal{L}_2$-gain (or ℓ_2-gain) plays an important role in robust control and stability analysis of control systems. Informally speaking, the $\mathcal{L}_2$-gain is the maximum output energy excited using a given input/perturbation energy and measures the disturbance attenuation of the system. This concept has been extensively studied for classical linear control systems and smooth nonlinear control systems [13, 27], and its study has been extended to the SLCSs and other hybrid systems recently [6, 7]. For instance, the $\mathcal{L}_2$-gains are addressed under the assumption of slow switchings in [5], and an LMI-based method is proposed in [28]. Other methods include the common storage function approach [8, 9, 10] and the variational approach [18]. Furthermore, the design of switching signal to achieve a certain $\mathcal{L}_2$-gain and related stability analysis are presented in [32]. For the discrete-time SLCSs, the ℓ_2-gains are characterized under dwell time constraints in [29] and their bounds are developed in [2, 16]; convergence and computation of classical or finite-horizon ℓ_2-gains

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[*] Partial preliminary results of this research were reported in the conference papers [19, 20].

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are also studied in [14, 25] respectively. Despite these advances, computation of the $\mathcal{L}_2$-gain (or ℓ_2-gain) of the SLCSs remains a difficult problem, due to switching induced combinatorial complexity. Attempts have been made toward finding nonconservative bounds of the $\mathcal{L}_2$-gains at the cost of solving certain existence problems, e.g., [8].

Motivated by robust analysis and characterization, the present paper studies the input-to-state ℓ_2-gains (or simply ℓ_2-gains) of the discrete-time SLCSs subject to system parameter variations. Specifically, two important and interconnected issues are addressed: (i) analytic issue: how does the ℓ_2-gain vary with respect to system parameters? (ii) numerical issue: how can the ℓ_2-gain be effectively computed with proven convergence under different system parameters? Note that the finiteness of the classical ℓ_2-gain of the SLCS is closely related to strong stability of the associated autonomous switched linear system, which in turn can be characterized by the joint spectral radius or a related scaling parameter; see [11] and references therein. Inspired by this observation, we consider a particular, yet nontrivial, case of parameter variations: both the system transition matrix and the control matrix of each subsystem are scaled by a varying discount factor (cf. Section 2.1). Treating the classical ℓ_2-gain as a function of this discount factor, we propose a generalized notion of the input-to-state ℓ_2-gain. This generalized notion not only includes the classical counterpart as a special case and leads to a generating function based alternative approach to compute the classical ℓ_2-gain, but also offers insight into the input-to-state behaviors of parameter perturbed SLCSs. Moreover, it is worth pointing out that many results of this paper can handle, or can be extended to handle, general system parameter variations (cf. Theorem 2.1 and Proposition 2.2).

New analytic and numerical results are developed for the following aspects of the generalized ℓ_2-gain, which constitute the main contributions of the paper:

(1) Important analytic properties of the generalized ℓ_2-gain are established. In particular, necessary and sufficient conditions are derived for the finiteness of the generalized or classical ℓ_2-gains under a suitable reachability assumption. Furthermore, it is shown that the finite generalized or classical ℓ_2-gain is (locally) continuous in system parameters.

(2) A novel generating function approach is developed for characterization and effective approximation of the generalized ℓ_2-gains, driven by the recent development for stability analysis of autonomous switched systems via this approach [11, 24]. Roughly speaking, a generating function is a power series in a discount factor with coefficients dependent on state trajectories; its radii of convergence characterize the maximum exponential growth rates of the system trajectories under different switching rules. This approach also yields effective computation of stability quantities. Towards the study of the generalized ℓ_2-gain, we introduce the controlled generating functions and show that their radii of convergence characterize the generalized ℓ_2-gain. Various analytic properties of the controlled generating functions and related quantities, e.g., the quadratic bounds and the domain of convergence, are derived and used to develop Bellman equation based iterative procedures for approximation of the generating functions with proven uniform or exponential convergence.

(3) Based on the analytic results of the controlled generating functions, finite-horizon generating function based algorithms are proposed for computation of the generalized ℓ_2-gain. While the worst-case computation of the generalized ℓ_2-gain remains (and is inherently) NP-hard (**?**), it is shown that by exploring certain relaxation techniques, the proposed algorithms yield effective and less conservative estimates of the generalized or classical ℓ_2-gains compared to the existing methods.

The paper is organized as follows. Section 2 introduces the generalized ℓ_2-gain and establishes its analytic properties. Section 3 addresses controlled generating functions and their properties, and Section 4 shows that radii of convergence of the generating functions provide complete characterization of the generalized ℓ_2-gain. As an example, one dimensional SLCSs and their generating functions are studied in Section 5. Numerical algorithms and computational results are presented in Section 6 with conclusion drawn in Section 7.

2 Generalized ℓ_2-Gain of Switched Linear Control Systems

A discrete-time switched linear control system (SLCS) has the dynamics

$$x(t+1) \;=\; A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{Z}_+ := \{0, 1, \ldots\}. \tag{1}$$

In essence, the system evolves by switching among a finite collection of linear control systems (or subsystems) $\{(A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m})\}_{i\in\mathcal{M}}$ indexed by the set $\mathcal{M} := \{1, \ldots, M\}$, with the switching governed by the switching sequence $\sigma := \{\sigma(0), \sigma(1), \ldots\}$, where $\sigma(t) \in \mathcal{M}$ for $t \in \mathbb{Z}_+$.

Denote by $x(t; \sigma, u, z)$, $t \in \mathbb{Z}_+$, the state trajectory or solution of the SLCS (1) under the switching sequence σ and the control input $u := \{u(0), u(1), \ldots\}$ starting from $x(0) = z$. For a fixed σ, the SLCS reduces to a linear time-varying system and thus $x(t; \sigma, u, z)$ is jointly linear in u and z. The reachable set of the SLCS (1) is the set of all states $x(t; \sigma, u, 0)$ that can be reached within a finite time t starting from $x(0) = 0$ under arbitrary σ and u. Unlike classical linear systems, the reachable set of a discrete-time SLCS may not be a subspace [26]. For example, a one-step finite impulse response (FIR) system satisfying $A_iB_j = 0$ for all $i, j \in \mathcal{M}$ has the reachable set $\cup_{i\in\mathcal{M}} R(B_i)$, which is in general not a subspace. Here $R(\cdot)$ denotes the range of a matrix. The following reachability assumption is imposed throughout the paper (unless otherwise indicated).

Assumption 2.1. *The reachable set of the SLCS (1), denoted by $\mathcal{R}$, spans the state space $\mathbb{R}^n$, i.e., span$(\mathcal{R}) = \mathbb{R}^n$.*

Note that Assumption 2.1 implies that at least one $B_i \neq 0$.

The autonomous switched linear system (SLS) corresponding to the SLCS (1) is obtained by setting $u \equiv 0$:

$$x(t+1) = A_{\sigma(t)}x(t), \quad \forall\, t \in \mathbb{Z}_+, \tag{2}$$

whose solution under the switching sequence σ and initial state z is denoted $x(t; \sigma, z) := x(t; \sigma, u, z)|_{u\equiv 0}$.

2.1 Generalized Input-to-State ℓ_2-Gain

Let $\ell_2(\mathbb{R}^m)$ denote all $\mathbb{R}^m$-valued sequences u with finite ℓ_2-norm $\|u\|_2 := \sqrt{\sum_{t=0}^{\infty} \|u(t)\|^2}$. The classical input-to-state ℓ_2-gain of the discrete-time SLCS (1) is defined by

$$\kappa \;:=\; \sup_{\sigma}\; \sup_{0\neq u\in\ell_2(\mathbb{R}^m)} \frac{\sqrt{\sum_{t=0}^{\infty} \|x(t+1; \sigma, u, 0)\|^2}}{\sqrt{\sum_{t=0}^{\infty} \|u(t)\|^2}}. \tag{3}$$

Given $\lambda \in \mathbb{R}_+ := [0, \infty)$, let $\|u\|_{\lambda,2} := \sqrt{\sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2}$ be the λ-discounted ℓ_2-norm of u, and $\mathcal{U}_\lambda := \{u \,|\, \|u\|_{\lambda,2} < \infty\}$. The generalized input-to-state ℓ_2-gain with a discount factor $\lambda \in \mathbb{R}_+$ is

$$\kappa(\lambda) \;:=\; \sup_{\sigma}\; \sup_{0\neq u\in\mathcal{U}_\lambda} \frac{\sqrt{\sum_{t=0}^{\infty} \lambda^t\|x(t+1; \sigma, u, 0)\|^2}}{\sqrt{\sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2}}. \tag{4}$$

In other words, $\kappa(\lambda)$ is the largest amplification factor of the λ-discounted ℓ_2-norm of state over that of input under any switching sequence. The goal of this paper is to characterize $\kappa(\lambda)$ for different $\lambda \in \mathbb{R}_+$.

Remark 2.1. The classical input-to-state ℓ_2-gain is a special case of the generalized ℓ_2-gain $\kappa(\lambda)$ with $\lambda = 1$. Conversely, $\kappa(\lambda)$ is the classical ℓ_2-gain of the following SLCS:

$$\widetilde{x}(t+1) \;=\; \widetilde{A}_{\sigma(t)}\widetilde{x}(t) + \widetilde{B}_{\sigma(t)}\widetilde{u}(t), \tag{5}$$

where $\widetilde{A}_i := \sqrt{\lambda}\, A_i$ and $\widetilde{B}_i := B_i$. In fact, under the scaled control input $\widetilde{u}(t) := \sqrt{\lambda^t} \cdot u(t)$, the solution of the new SLCS satisfies $\widetilde{x}(t+1; \sigma, \widetilde{u}, 0) = \sqrt{\lambda^t} \cdot x(t+1; \sigma, u, 0)$ for $t \in \mathbb{Z}_+$ and all σ. By the homogeneity of $\widetilde{x}(t; \sigma, \widetilde{u}, 0)$ in $\widetilde{u}$, $\kappa(\lambda) = \sup_\sigma \sup_{\|\widetilde{u}\|_2=1} \sqrt{\sum_{t=0}^{\infty} \|\widetilde{x}(t+1; \sigma, \widetilde{u}, 0)\|^2}$. $\square$

For each $k \in \mathbb{Z}_+$, the (generalized) k-horizon ℓ_2-gains, $\kappa_k(\lambda)$, of the SLCS (1) is defined as

$$\kappa_k(\lambda) := \sup_{\sigma} \sup_{0 \neq u \in \mathcal{U}_k} \frac{\sqrt{\sum_{t=0}^{k} \lambda^t \|x(t+1;\sigma,u,0)\|^2}}{\sqrt{\sum_{t=0}^{k} \lambda^t \|u(t)\|^2}}, \tag{6}$$

where $\mathcal{U}_k \subset \mathcal{U}_\lambda$ is the set of inputs u with duration at most k, namely, $u(t) \equiv 0$ for all $t > k$. Note that $\cup_{k \in \mathbb{Z}_+} \mathcal{U}_k$ is a dense subset of $\mathcal{U}_\lambda$. In what follows, we define the constant γ_0 as

$$\gamma_0 := \max_{i \in \mathcal{M}} \|B_i\|. \tag{7}$$

Here, $\|B_i\|$ denotes the largest singular value of B_i. Since at least one $B_i \neq 0$ by Assumption 2.1, we must have $\gamma_0 > 0$.

Proposition 2.1. *The generalized ℓ_2-gain $\kappa(\lambda)$ and its finite horizon couterparts $\kappa_k(\lambda)$ have the following properties:*

(1) $\kappa_k(\lambda) \uparrow \kappa(\lambda)$ as $k \to \infty$ for each $\lambda \in \mathbb{R}_+$;

(2) $\kappa_k(\lambda)$ is continuous in $\lambda \in \mathbb{R}_+$ for each $k \in \mathbb{Z}_+$, and $\kappa(\lambda)$ is lower semi-continuous in $\lambda \in \mathbb{R}_+$;

(3) At $\lambda = 0$, $\kappa(0) = \gamma_0$, where γ_0 is defined in (7).

Proof. (1) That $\kappa_k(\lambda)$ is non-decreasing in k follows as the supremum in (6) is taken over a space $\mathcal{U}_k$ that is non-decreasing in k. A similar argument shows that $\kappa_k(\lambda) \leq \kappa(\lambda)$ for any k. As a result, $\lim_{k\to\infty} \kappa_k(\lambda) \leq \kappa(\lambda)$. To show the other direction, assume first $\kappa(\lambda)$ is finite. Then for any small $\varepsilon > 0$, a control $u \in \mathcal{U}_\lambda$ and a switching sequence σ exist such that $\sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2 = 1$ and $[\kappa(\lambda)]^2 - \varepsilon \leq \sum_{t=0}^{\infty} \lambda^t \|x(t+1;\sigma,u,0)\|^2 \leq [\kappa(\lambda)]^2$. By choosing ℓ large enough, we have $\sum_{t=0}^{\ell} \lambda^t \|x(t+1;\sigma,u,0)\|^2 \geq [\kappa(\lambda)]^2 - 2\varepsilon$ while $\sum_{t=0}^{\ell} \lambda^t \|u(t)\|^2 \leq 1$; thus $[\kappa_\ell(\lambda)]^2 \geq [\kappa(\lambda)]^2 - 2\varepsilon$. As $\kappa_k(\lambda)$ is non-decreasing in k, $[\kappa(\lambda)]^2 - 2\varepsilon \leq [\kappa_k(\lambda)]^2 \leq [\kappa(\lambda)]^2$, $\forall k \geq \ell$. As $\varepsilon > 0$ is arbitrary, we have $\lim_{k\to\infty} \kappa_k(\lambda) = \kappa(\lambda)$. The case when $\kappa(\lambda) = \infty$ can be similarly proved.

(2) For any $\lambda \geq 0$, let $\widetilde{x}(t;\sigma,\widetilde{u},0)$ be the solution of the scaled SLCS (5) with subsystem matrices $\{(\sqrt{\lambda}A_i, B_i)\}_{i\in\mathcal{M}}$ and $\widetilde{u}(t) = \sqrt{\lambda^t} \cdot u(t)$. By Remark 2.1, $\kappa_k(\lambda) = \sup_\sigma \sup_{\widetilde{u} \in \mathcal{U}_k, \|\widetilde{u}\|_2 = 1} \sqrt{\sum_{t=0}^{k} \|\widetilde{x}(t+1;\sigma,\widetilde{u},0)\|^2}$ for each $k \in \mathbb{Z}_+$. Denote $\sigma_{0:k} := (\sigma(0),\ldots,\sigma(k)) \in \mathcal{M}^{k+1}$, $v := (\widetilde{u}(0),\ldots,\widetilde{u}(k)) \in \mathbb{R}^{(k+1)m}$, and define the set $\mathcal{B}_v := \{v \mid \sum_{t=0}^{k} \|\widetilde{u}(t)\|^2 = 1\}$. Then $\kappa_k(\lambda) = \max_{\sigma_{0:k}} \sup_{v \in \mathcal{B}_v} f_{\sigma_{0:k}}(\lambda, v)$ for some function $f_{\sigma_{0:k}}$ continuous in (λ, v) for each $\sigma_{0:k}$. Let $\lambda_0 \in \mathbb{R}_+$ be arbitrary. Each $f_{\sigma_{0:k}}$ is continuous, hence uniformly continuous, on the compact set $[0,\lambda_0] \times \mathcal{B}_v$. As a result, $g_{\sigma_{0:k}}(\lambda) := \sup_{v \in \mathcal{B}_v} f_{\sigma_{0:k}}(\lambda, v)$ is uniformly continuous, thus continuous, in λ on $[0,\lambda_0]$. As λ_0 is arbitrary, $g_{\sigma_{0:k}}(\cdot)$ is continuous on $\mathbb{R}_+$. Since there are finitely many $\sigma_{0:k}$, $\kappa_k(\lambda) = \max_{\sigma_{0:k}} g_{\sigma_{0:k}}(\lambda)$ is continuous in λ on $\mathbb{R}_+$. Finally, being the pointwise supremum of all continuous functions $\kappa_k(\lambda)$, $\kappa(\lambda)$ must be lower semi-continuous in λ.

(3) Since $x(1;\sigma,u,0) = B_{\sigma(0)}u(0)$ at $\lambda = 0$, $\kappa(0) = \sup_{\sigma(0)\in\mathcal{M}, u(0)\neq 0} \|B_{\sigma(0)}u(0)\|/\|u(0)\| = \gamma_0$. ☐

More properties of $\kappa(\lambda)$ under certain stability conditions are given in the next subsection.

2.2 Finiteness of Generalized ℓ_2-Gain

It is well known that an LTI system (A, B) has finite input-to-state ℓ_2-gain if and only if the autonomous system A is stable. We show that this result can be extended to the generalized ℓ_2-gain case. Recall that the autonomous SLS (2) with subsystem dynamics matrices $\{A_i\}_{i\in\mathcal{M}}$ is called exponentially stable under arbitrary switching (or absolutely exponentially stable [15]) with the parameters (C, r) if there exist constants $C > 0$ and $r \in (0, 1)$ such that its solution satisfies $\|x(t; \sigma, z)\| \le Cr^t\|z\|$, $\forall\, t \in \mathbb{Z}_+$, for all $z \in \mathbb{R}^n$ and switching sequences σ, or equivalently,

$$\|\Phi^\sigma_{t,k}\| \;\le\; Cr^{t-k}, \quad \forall\, \sigma,\, \forall\, t, k \in \mathbb{Z}_+ \text{ with } t \ge k.$$

Here, $\Phi^\sigma_{t,k}$ is the state transition matrix defined by $\Phi^\sigma_{t,k} := A_{\sigma(t-1)} \cdots A_{\sigma(k)}$ for $t > k$ and $\Phi^\sigma_{t,t} := I$.

Theorem 2.1. *Suppose the autonomous SLS (2) with subsystem matrices $\{A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching with the parameters (C, r) for some $C > 0$ and $r \in [0, 1)$. Then, the SLCS (1) with subsystem matrices $\{(A_i, B_i)\}_{i\in\mathcal{M}}$ has finite (classical) ℓ_2-gain:*

$$\kappa \;\le\; \frac{C\gamma_0}{1-r},$$

where γ_0 is defined in (7). Under Assumption 2.1, the converse also holds: if $\kappa < \infty$, then the SLS $\{A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching.

Proof. The solution of the SLCS (1) under a control input $u \in \ell_2(\mathbb{R}^m)$ and an arbitrary switching sequence σ is given by

$$x(t+1; \sigma, u, 0) \;=\; \sum_{k=0}^{t} \Phi^\sigma_{t+1,k+1} B_{\sigma(k)} u(k), \quad t \in \mathbb{Z}_+. \tag{8}$$

The stability assumption implies $\|\Phi^\sigma_{t+1,k+1}\| \;\le\; Cr^{t-k}$. Consider a sequence $w = (w_1, w_2, \ldots) \in \ell_2(\mathbb{R}^n)$ with $w_t = 0$ for all $t \ge N + 2$ for some $N \in \mathbb{Z}_+$. Then,

$$\begin{aligned}
\left|\sum_{t=0}^{\infty} (w_{t+1})^T x(t+1; \sigma, u, 0)\right| \;&=\; \left|\sum_{t=0}^{\infty} (w_{t+1})^T \sum_{k=0}^{t} \Phi^\sigma_{t+1,k+1} B_{\sigma(k)} u(k)\right| \\
\le\; \sum_{t=0}^{\infty}\sum_{k=0}^{t} C\, r^{t-k}\gamma_0 \|u(k)\| \cdot \|w_{t+1}\| \;&=\; \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} C\, r^{s}\gamma_0 \|u(k)\| \cdot \|w_{s+k+1}\| \\
\le\; C\gamma_0 \sum_{s=0}^{\infty} r^s \|u\|_2 \cdot \|w\|_2 \;&=\; \frac{C\gamma_0}{1-r}\|u\|_2 \cdot \|w\|_2.
\end{aligned}$$

By setting $w_t := x(t; \sigma, u, 0)$ for $t = 1, \ldots, N+1$, we have

$$\left(\sum_{t=0}^{N} \|x(t+1; \sigma, u, 0)\|^2\right)^{1/2} \;\le\; \frac{C\gamma_0}{1-r}\|u\|_2.$$

Letting $N \to \infty$, we have $\|x(\cdot; \sigma, u, 0)\|_2 \le C\gamma_0/(1-r) \cdot \|u\|_2$ for all σ and $u \in \ell_2(\mathbb{R}^m)$, i.e., $\kappa \le C\gamma_0/(1-r)$.

For the converse statement, assume that the SLS $\{A_i\}_{i\in\mathcal{M}}$ is not exponentially stable under arbitrary switching. Then it follows from [11, Proposition 1 and Theorem 2] that there exists a proper subspace $\mathcal{V}$ of $\mathbb{R}^n$ such that for each $z \notin \mathcal{V}$, $\sum_{t=0}^{\infty} \|x(t; \sigma_z, 0, z)\|^2 = \infty$ for some switching sequence σ_z. By Assumption 2.1, the reachable set $\mathcal{R}$ is not contained in $\mathcal{V}$, since otherwise it fails to span $\mathbb{R}^n$. Therefore, we can find one $z \in \mathcal{R} \setminus \mathcal{V}$, i.e., $z = x(\tau; \sigma', u', 0)$ for some σ' and u' and a finite time τ. A state solution to the SLCS can be constructed so that it first adopts σ' and u' to reach z at time τ, and then adopts σ_z and $u = 0$ from time τ thereon. This state solution has infinite ℓ_2 energy, even though the input u has a finite duration. Thus $\kappa = \infty$, a contradiction. □

By Remark 2.1, the following result follows immediately.

Corollary 2.1. *Suppose $\lambda \in \mathbb{R}_+$ is such that the scaled SLS $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching with the parameters (C, r) for some $C > 0$ and $r \in [0, 1)$. Then the generalized ℓ_2-gain $\kappa(\lambda)$ of the SLCS (1) satisfies $\kappa(\lambda) \le \frac{C\gamma_0}{1-r}$. Conversely, under Assumption 2.1, if $\kappa(\lambda) < \infty$, then the SLS $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching.*

The *joint spectral radius (JSR)* of the matrix set $\{A_i\}_{i\in\mathcal{M}}$ is defined as [12]:

$$\rho^* := \lim_{k\to\infty} \max\left\{ \|A_{i_1}\cdots A_{i_k}\|^{1/k} \,\middle|\, i_1,\ldots,i_k \in \mathcal{M} \right\}.$$

The SLS $\{A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching if and only if $\rho^* < 1$ [12]. Since the scaled SLS $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$ has the JSR $\sqrt{\lambda}\cdot\rho^*$, it is exponentially stable under arbitrary switching if and only if

$$\lambda < \lambda^* := (\rho^*)^{-2}. \tag{9}$$

Hence, under Assumption 2.1, Corollary 2.1 implies that $\kappa(\lambda) < \infty$ if and only if $\lambda \in [0, \lambda^*)$.

2.3 Continuity of Generalized ℓ_2-Gain

In this section, we show that $\kappa(\lambda)$ is continuous on $[0, \lambda^*)$ where λ^* is defined in (9). Denote by S the ordered matrix tuple $\big((A_i, B_i)\big)_{i\in\mathcal{M}}$. The set of all such S with $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ is a vector space denoted by $\mathcal{S}$ and endowed with a norm, e.g., $\|\cdot\|_{\mathcal{S}}$. Since the projection operator $S \mapsto A_i$ (resp. $S \mapsto B_i$) is continuous hence bounded, there exists a positive constant θ such that

$$\|A_i\| \le \theta\cdot\|S\|_{\mathcal{S}}, \quad \|B_i\| \le \theta\cdot\|S\|_{\mathcal{S}}, \quad \forall\, S \in \mathcal{S},\ i \in \mathcal{M}.$$

For any given $S \in \mathcal{S}$ and switching sequence σ, define $L_{S,\sigma}(u) := x(\cdot; \sigma, u, 0)$ as in (8) with $z = 0$. Assume that the SLS $\{A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching. Then Theorem 2.1 implies that the linear operator $L_{S,\sigma} : \ell_2(\mathbb{R}^m) \to \ell_2(\mathbb{R}^n)$ is uniformly bounded in σ, i.e.,

$$\|L(S)\| := \sup_{\sigma} \sup_{\|u\|_2=1} \|L_{S,\sigma}(u)\|_2 < \infty,$$

where $\|\cdot\|_2$ is the ℓ_2-norm on the Hilbert space $\ell_2(\mathbb{R}^m)$ or $\ell_2(\mathbb{R}^n)$. The following result[1] shows that $\|L(S)\|$ is locally Lipschitz continuous in the system parameter S.

Proposition 2.2. *Let $S = \big((A_i, B_i)\big)_{i\in\mathcal{M}} \in \mathcal{S}$ be given such that the SLS $\{A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching. Then there exist $\varepsilon > 0$ and $\mu > 0$ (dependent on S only) such that for any S', S'' within the neighborhood $\mathcal{B}_S(\varepsilon) := \{\widetilde{S} \in \mathcal{S} \mid \|\widetilde{S} - S\|_{\mathcal{S}} \le \varepsilon\}$ of S, we have*

$$\big|\|L(S')\| - \|L(S'')\|\big| \le \mu\cdot\|S' - S''\|_{\mathcal{S}}.$$

Proof. For the given S, there exist $\varepsilon > 0$, $C > 1$, and $r \in (0, 1)$ such that for any $S' \in \mathcal{B}_S(\varepsilon)$, the SLS $\{A'_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching with the parameter (C, r) (chosen uniformly in $S' \in \mathcal{B}_S(\varepsilon)$ [12, Proposition 1.4]), i.e., $\|A'_{i_1}A'_{i_2}\cdots A'_{i_k}\| \le Cr^k$ for any $i_1,\ldots,i_k \in \mathcal{M}$. Define the constant $\xi := \max_{i\in\mathcal{M}} \sup_{S'\in\mathcal{B}_S(\varepsilon)} \|B'_i\|$, which is finite as $\mathcal{B}_S(\varepsilon)$ is a bounded set.

Let $S', S'' \in \mathcal{B}_S(\varepsilon)$ and σ be arbitrary. The linear operator $T : \ell_2(\mathbb{R}^m) \to \ell_2(\mathbb{R}^n)$ defined by $T(u) := L_{S',\sigma}(u) - L_{S'',\sigma}(u)$ is bounded (hence continuous) with the operator norm $\|T\| :=$

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[1] We thank Dr. Thomas I. Seidman of University of Maryland Baltimore County for helpful discussions on this result.

$\sup_{\|u\|_2=1}\|T(u)\|_2$. We will show below that $\|T\| \le \mu \cdot \|S' - S''\|_{\mathcal{S}}$ for a positive constant μ independent of S', S'' and σ. By (8), the $(t+1)$-th entry of the sequence $T(u)$ is given by

$$\begin{aligned} T_{t+1}(u) \; &:= \; x'(t+1;\sigma,u,0) - x''(t+1;\sigma,u,0) \\ &= \sum_{k=1}^{t+1}\Big[A'_{i_1}\cdots A'_{i_{k-1}}B'_{i_k} - A''_{i_1}\cdots A''_{i_{k-1}}B''_{i_k}\Big]u(t+1-k), \end{aligned}$$

where the indices $i_1,\ldots,i_k$ are determined from the given σ, and $A'_{i_1}\cdots A'_{i_{k-1}} = A''_{i_1}\cdots A''_{i_{k-1}} = I$ if $k=1$. Then, for each $k=1,\ldots,t+1$,

$$\begin{aligned} &\Big\|A'_{i_1}\cdots A'_{i_{k-1}}B'_{i_k} - A''_{i_1}\cdots A''_{i_{k-1}}B''_{i_k}\Big\| \\ \le \;& \Big\|A'_{i_1}\cdots A'_{i_{k-1}}B'_{i_k} - A'_{i_1}\cdots A'_{i_{k-1}}B''_{i_k}\Big\| \\ &\quad + \sum_{s=0}^{k-2}\Big\|A'_{i_1}\cdots A'_{i_s}\cdot A''_{i_{s+1}}\cdots A''_{i_{k-1}}B''_{i_k} - A'_{i_1}\cdots A'_{i_{s+1}}\cdot A''_{i_{s+2}}\cdots A''_{i_{k-1}}B''_{i_k}\Big\| \\ \le \;& Cr^{k-1}\|B'_{i_k} - B''_{i_k}\| + (k-1)C^2 r^{k-2}\xi\cdot \max_{s=0,\ldots,k-2}\|A'_{i_{s+1}} - A''_{i_{s+1}}\| \\ \le \;& Cr^{k-2}\big[r + (k-1)C\xi\big]\theta\cdot\|S'-S''\|_{\mathcal{S}} \le \nu(r+\delta)^k\cdot\|S'-S''\|_{\mathcal{S}} \end{aligned}$$

for any sufficiently small $\delta > 0$ with $r+\delta < 1$ and some large $\nu > 0$ (dependent on r and δ only). The last step follows because $r^{k-2}\left[r+(k-1)C\xi\right]$ is a higher order infinitesimal than $(r+\delta)^k$ as $k\to\infty$. As a result, by using the same argument as in the proof of Theorem 2.1 (on the operator T rather than $L_{S,\sigma}$ with (C,r) replaced by $(\nu, r+\delta)$), we obtain that $\|T\| \le \mu\cdot\|S'-S''\|_{\mathcal{S}}$, where $\mu := \nu/(1-r-\delta)$. It is clear that μ is independent of σ and $S', S'' \in B_S(\varepsilon)$. Consequently,

$$\|L(S')\| \;=\; \sup_{\|u\|_2=1,\,\sigma}\|L_{S',\sigma}(u)\|_2 \le \sup_{\|u\|_2=1,\,\sigma}\Big[\|L_{S'',\sigma}(u)\|_2 + \|T(u)\|_2\Big] \le \|L(S'')\| + \mu\|S'-S''\|_{\mathcal{S}}.$$

The roles of S' and S'' can be reversed to yield a similar inequality, leading to the desired result. ☐

Recall that the scaled SLS $\{\sqrt{\lambda}A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching if and only if $\lambda\in[0,\lambda^*)$ with λ^* given in (9).

Theorem 2.2. *The generalized ℓ_2-gain $\kappa(\lambda)$ of the SLCS (1) is continuous in λ on $[0,\lambda^*)$. Furthermore, $\kappa(\lambda)$ is Lipschitz continuous in $\sqrt{\lambda}$ on any compact subset of $[0,\lambda^*)$.*

Proof. For any $\lambda \ge 0$, let $\widetilde{x}(t;\sigma,\widetilde{u},0)$ be the solution of the scaled SLCS (5) with subsystem matrices $\{(\sqrt{\lambda}A_i, B_i)\}_{i\in\mathcal{M}}$ and $\widetilde{u}(t) = \sqrt{\lambda^t}\cdot u(t)$. It follows from the discussions in Remark 2.1 that

$$\kappa(\lambda) \;=\; \sup_{\|\widetilde{u}\|_2=1,\,\sigma}\|\widetilde{x}(\cdot;\sigma,\widetilde{u},0)\|_2 = \|L(S_\lambda)\|\,,$$

where $S_\lambda := \big((\sqrt{\lambda}A_i, B_i)\big)_{i\in\mathcal{M}} \in \mathcal{S}$. At any $\lambda_0\in[0,\lambda^*)$, since the autonomous SLS $\{\sqrt{\lambda_0}A_i\}_{i\in\mathcal{M}}$ corresponding to S_{λ_0} is exponentially stable under arbitrary switching, Proposition 2.2 implies that $\kappa(\lambda)$ is locally Lipschitz continuous in S_λ, hence in $\sqrt{\lambda}$ as well, at any $\lambda_0\in[0,\lambda^*)$. This implies via an open covering argument that $\kappa(\lambda)$ is (uniformly) Lipschitz in $\sqrt{\lambda}$ on any compact subset of $[0,\lambda^*)$, e.g., $[0,\lambda_0]$. The continuity of $\kappa(\lambda)$ in λ on $[0,\lambda^*)$ then follows readily. ☐

We note that the continuity results in Proposition 2.2 and Theorem 2.2 hold without the reachability assumption.

Corollary 2.2. *For any given $\lambda_0\in(0,\lambda^*)$, $\kappa_k(\cdot)$ converges uniformly to $\kappa(\cdot)$ on $[0,\lambda_0]$ as $k\to\infty$.*

Proof. It follows from Proposition 2.1 and Theorem 2.2 that $\kappa(\cdot)$ and $\kappa_k(\cdot)$ for all k are continuous on the compact interval $[0,\lambda_0]$. Since $\kappa_k(\cdot)$ converges pointwise and monotonically to $\kappa(\cdot)$, it follows from Dini's Theorem [22, Theorem 7.13] that the convergence is uniform on $[0,\lambda_0]$. ☐

3 Generating Functions of Switched Linear Control Systems

Originally introduced in [11], generating functions are an effective tool for the stability analysis and computation of autonomous SLSs under various switching rules. In this section, we briefly review their definitions, and then extend them to the SLCSs case for studying generalized ℓ_2-gains.

3.1 Autonomous Generating Function of SLS

The (strong) generating function of the autonomous SLS (2) is the function $G_\lambda(\cdot) \in \mathbb{R}_+ \cup \{\infty\}$ defined by

$$G_\lambda(z) := \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma,z)\|^2, \ \forall z \in \mathbb{R}^n, \ \lambda \in \mathbb{R}_+. \tag{10}$$

The radius of convergence of $G_\lambda(z)$ is defined as:

$$\lambda^* := \sup\{\lambda \in \mathbb{R}_+ \,|\, G_\lambda(z) < \infty, \ \forall z \in \mathbb{R}^n\}. \tag{11}$$

Note that this is the same notation defined before in (9) via the JSR ρ^*, as the two are identical [11]. Denote $g_\lambda = \sup_{\|z\|=1} G_\lambda(z)$ and $d_\lambda = \sup_{\|z\|=1} \max_{i\in\mathcal{M}} G_\lambda(A_i z)$. Then it is shown in [11, Theorem 3] that for $\lambda \in [0,\lambda^*)$, the SLS $\{\sqrt{\lambda} A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching with the parameters (C_λ, r_λ), where $C_\lambda := \sqrt{g_\lambda}$ and $r_\lambda := \lambda d_\lambda/(1+\lambda d_\lambda) \in [0,1)$. By Theorem 2.1, this yields a (rough) bound on the generalized ℓ_2-gain: $\kappa(\lambda) \le \sqrt{g_\lambda}\gamma_0(1+\lambda d_\lambda), \ \forall \ \lambda \in [0,\lambda^*)$. See [11] for more details of the autonomous generating functions.

3.2 Controlled Generating Function of SLCS

The concept of generating functions can be extended to SLCSs. For each $\lambda, \gamma \in \mathbb{R}_+$, the (controlled) generating function $G_{\lambda,\gamma}(\cdot) \in \mathbb{R}_+ \cup \{+\infty\}$ of the SLCS (1) is defined as

$$G_{\lambda,\gamma}(z) := \sup_{u\in\mathcal{U}_\lambda,\,\sigma} \left[\sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma,u,z)\|^2 - \gamma^2 \lambda \sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2 \right] \tag{12}$$

$$= \|z\|^2 + \lambda \sup_{u\in\mathcal{U}_\lambda,\,\sigma} \sum_{t=0}^{\infty} \lambda^t \Big[\|x(t+1;\sigma,u,z)\|^2 - \gamma^2 \|u(t)\|^2 \Big], \tag{13}$$

for $z \in \mathbb{R}^n$. The signs of the two summations in (12) are chosen to coax u and σ into exciting the largest state energy using the least control energy. The restriction $u \in \mathcal{U}_\lambda$ ensures the convergence of the second summation in (12). Clearly, $G_{\lambda,\gamma}(z) \ge \|z\|^2 \ge 0$. Moreover, at $\lambda = 0$, $G_{0,\gamma}(z) = \|z\|^2$.

For $k \in \mathbb{Z}_+$ and $\lambda, \gamma \in \mathbb{R}_+$, the k-horizon generating function $G_{\lambda,\gamma,k}(\cdot)$ is defined as

$$G_{\lambda,\gamma,k}(z) := \sup_{\sigma,\,u} \left[\sum_{t=0}^{k} \lambda^t \|x(t;\sigma,u,z)\|^2 - \gamma^2 \lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2 \right] \tag{14}$$

$$= \|z\|^2 + \lambda \cdot \sup_{\sigma,u} \sum_{t=0}^{k-1} \lambda^t \Big[\|x(t+1;\sigma,u,z)\|^2 - \gamma^2 \|u(t)\|^2 \Big], \tag{15}$$

for $z \in \mathbb{R}^n$, with the understanding that $G_{\lambda,\gamma,0}(\cdot) = \|\cdot\|^2$.

The following proposition shows that $G_{\lambda,\gamma}(z)$ can be approximated by $G_{\lambda,\gamma,k}(z)$ when k is large.

Proposition 3.1. *For each fixed $\lambda, \gamma \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$, $G_{\lambda,\gamma,k}(z) \uparrow G_{\lambda,\gamma}(z)$ as $k \to \infty$.*

Proof. It follows directly from the definitions (12) and (14) that $G_{\lambda,\gamma,k}(z)$ is non-decreasing in k and $G_{\lambda,\gamma,k}(z) \leq G_{\lambda,\gamma}(z)$. Consider first the case when $G_{\lambda,\gamma}(z) < \infty$. For any $\varepsilon > 0$, there exist $u_\varepsilon \in \mathcal{U}_\lambda$ and a switching sequence σ_ε such that

$$G_{\lambda,\gamma}(z) \;\geq\; \sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma_\varepsilon, u_\varepsilon, z)\|^2 - \gamma^2\lambda \sum_{t=0}^{\infty} \lambda^t \|u_\varepsilon(t)\|^2 \;\geq\; G_{\lambda,\gamma}(z) - \varepsilon.$$

Since $u_\varepsilon \in \mathcal{U}_\lambda$ and $G_{\lambda,\gamma}(z) < \infty$, both summations above converge. Thus, we can find $k_\varepsilon \in \mathbb{Z}_+$ large enough such that

$$\sum_{t=0}^{k_\varepsilon} \lambda^t \|x(t;\sigma_\varepsilon, u_\varepsilon, z)\|^2 - \gamma^2\lambda \sum_{t=0}^{k_\varepsilon - 1} \lambda^t \|u_\varepsilon(t)\|^2 \geq G_{\lambda,\gamma}(z) - 2\varepsilon.$$

This shows $G_{\lambda,\gamma,k_\varepsilon}(z) \geq G_{\lambda,\gamma}(z) - 2\varepsilon$; hence $G_{\lambda,\gamma,k}(z) \uparrow G_{\lambda,\gamma}(z)$ as $k \to \infty$ as $\varepsilon > 0$ is arbitrary.

Consider $G_{\lambda,\gamma}(z) = \infty$ next. For any $M > 0$, there exist $u_M \in \mathcal{U}_\lambda$ and a switching sequence σ_M such that $\sum_{t=0}^{\infty} \lambda^t \|x(t;\sigma_M, u_M, z)\|^2 - \gamma^2\lambda \sum_{t=0}^{\infty} \lambda^t \|u_M(t)\|^2 \geq 2M$. If the first summation is finite, it follows from a similar argument as in the first case that there exists $k_M \in \mathbb{Z}_+$ such that $G_{\lambda,\gamma,k_M}(z) \geq \sum_{t=0}^{k_M} \lambda^t \|x(t;\sigma_M, u_M, z)\|^2 - \gamma^2\lambda \sum_{t=0}^{k_M - 1} \lambda^t \|u_M(t)\|^2 \geq M$. Otherwise, as $u_M \in \mathcal{U}_\lambda$, the same inequality holds for some k_M large enough. This shows that $G_{\lambda,\gamma,k}(z) \uparrow \infty$ as $k \to \infty$. ☐

The following proposition establishes some basic properties of the generating functions.

Proposition 3.2. *The controlled generating function $G_{\lambda,\gamma}(\cdot)$ and its finite-horizon counterparts $G_{\lambda,\gamma,k}(\cdot)$ for $\lambda, \gamma \in \mathbb{R}_+$ and $k \in \mathbb{Z}_+$ have the following properties.*

1. *(Homogeneity): $G_{\lambda,\gamma}(\cdot)$ and $G_{\lambda,\gamma,k}(\cdot)$ are both homogeneous of degree two, i.e., $G_{\lambda,\gamma}(\alpha z) = \alpha^2 G_{\lambda,\gamma}(z)$ and $G_{\lambda,\gamma,k}(\alpha z) = \alpha^2 G_{\lambda,\gamma,k}(z)$, $\forall z \in \mathbb{R}^n$, $\forall \alpha \in (0,\infty)$. Thus, $G_{\lambda,\gamma}(0) \in \{0, \infty\}$.*

2. *(Bellman Equation): For all $z \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$,*

$$G_{\lambda,\gamma,k+1}(z) \;=\; \|z\|^2 + \lambda \cdot \sup_{i \in \mathcal{M}, v \in \mathbb{R}^m} \big[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v) \big], \tag{16}$$

$$G_{\lambda,\gamma}(z) \;=\; \|z\|^2 + \lambda \cdot \sup_{i \in \mathcal{M}, v \in \mathbb{R}^m} \big[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma}(A_i z + B_i v) \big]. \tag{17}$$

3. *(Monotonicity): For any $z \in \mathbb{R}^n$, $G_{\lambda,\gamma}(z)$ and $G_{\lambda,\gamma,k}(z)$ are non-increasing in $\gamma \in \mathbb{R}_+$ (for a fixed λ) and non-decreasing in $\lambda \in \mathbb{R}_+$ (for a fixed γ).*

4. *(Sub-additivity): $\sqrt{G_{\lambda,\gamma}(z)}$ and $\sqrt{G_{\lambda,\gamma,k}(z)}$ for each $k \in \mathbb{Z}_+$ are sub-additive in z:*

$$\begin{aligned} \sqrt{G_{\lambda,\gamma,k}(z_1 + z_2)} &\leq \sqrt{G_{\lambda,\gamma,k}(z_1)} + \sqrt{G_{\lambda,\gamma,k}(z_2)}, \\ \sqrt{G_{\lambda,\gamma}(z_1 + z_2)} &\leq \sqrt{G_{\lambda,\gamma}(z_1)} + \sqrt{G_{\lambda,\gamma}(z_2)}, \quad \forall\, z_1, z_2 \in \mathbb{R}^n \end{aligned}$$

5. *(Convexity): $\sqrt{G_{\lambda,\gamma}(z)}$ and $\sqrt{G_{\lambda,\gamma,k}(z)}$ for each $k \in \mathbb{Z}_+$ are convex functions of z on $\mathbb{R}^n$.*

6. *(Invariant Subspace): $\mathcal{G}_{\lambda,\gamma} := \{z \in \mathbb{R}^n \,|\, G_{\lambda,\gamma}(z) < \infty\}$ is a subspace of $\mathbb{R}^n$ invariant under subsystem dynamics, i.e., $A_i \mathcal{G}_{\lambda,\gamma} + B_i \mathbb{R}^m \subseteq \mathcal{G}_{\lambda,\gamma}$, $\forall i$.*

7. *(Lower Bound): $G_{\lambda,\gamma}(z) \geq G_\lambda(z)$, $\forall\, z \in \mathbb{R}^n$, where $G_\lambda(z)$ is the autonomous generating function in (10).*

Proof. Let $\lambda, \gamma \in \mathbb{R}_+$ and $k \in \mathbb{Z}_+$ be arbitrary.

(1) The homogeneity property follows directly from $x(t; \sigma, \alpha u, \alpha z) = \alpha \cdot x(t; \sigma, u, z)$, $\forall\, t$.

(2) Partition $u \in \mathcal{U}_k$ as $u = (v, u')$ for $v \in \mathbb{R}^m$, $u' \in \mathcal{U}_{k-1}$, and let $\sigma = (i, \sigma')$ with $i \in \mathcal{M}$. By (14), we have

$$\begin{aligned} G_{\lambda,\gamma,k+1}(z) &= \sup_{\substack{i\in\mathcal{M}\\ v\in\mathbb{R}^m}} \left\{ \|z\|^2 - \gamma^2\lambda\|v\|^2 + \lambda \cdot \sup_{\sigma', u'} \left[\sum_{t=0}^{k} \lambda^t \cdot \|x(t; \sigma', u', A_i z + B_i v)\|^2 - \gamma^2\lambda \sum_{t=0}^{k-1} \lambda^t \|u'(t)\|^2 \right] \right\} \\ &= \|z\|^2 + \sup_{i\in\mathcal{M},\, v\in\mathbb{R}^m} \Big[-\gamma^2\lambda\|v\|^2 + \lambda\, G_{\lambda,\gamma,k}(A_i z + B_i v) \Big]. \end{aligned}$$

The Bellman equation for $G_{\lambda,\gamma}(z)$ can be proved similarly.

(3) By Proposition 3.1, we need only to prove the monotonicity property for $G_{\lambda,\gamma,k}(z)$. Monotonicity of $G_{\lambda,\gamma,k}(z)$ in γ is obvious from (14). We prove its monotonicity in λ by induction. At $k = 0$, $G_{\lambda,\gamma,0}(z) = \|z\|^2$ is clearly non-decreasing in λ. Suppose this is the case for $G_{\lambda,\gamma,k}(z)$ for some $k \in \mathbb{Z}_+$. Then for any $\lambda > \lambda' \geq 0$, the Bellman equation and the induction hypothesis imply that

$$G_{\lambda,\gamma,k+1}(z) \;\geq \|z\|^2 + \lambda' \sup_{i\in\mathcal{M}, v\in\mathbb{R}^m} \Big[-\gamma^2\|v\|^2 + G_{\lambda',\gamma,k}(A_i z + B_i v) \Big] = G_{\lambda',\gamma,k+1}(z).$$

By induction, $G_{\lambda,\gamma,k}(z)$ is non-decreasing in λ, $\forall\, k \in \mathbb{Z}_+$.

(4) Denote $\mathbf{u} := \big[u(0)^T \cdots u(k-1)^T\big]^T \in \mathbb{R}^{nk}$. Define

$$f_\sigma(z) \;:=\; \sup_u \left[\sum_{t=0}^{k} \lambda^t \|x(t; \sigma, u, z)\|^2 - \gamma^2\lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2 \right] := \sup_{\mathbf{u}} \begin{bmatrix} z \\ \mathbf{u} \end{bmatrix}^T \begin{bmatrix} Q_{zz} & Q_{zu} \\ Q_{uz} & Q_{uu} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{u} \end{bmatrix} \tag{18}$$

for some symmetric matrices $Q_{zz} \succeq I_n \in \mathbb{R}^{n\times n}$ and $Q_{uu} \in \mathbb{R}^{nk\times nk}$, and matrices $Q_{zu} = Q_{uz}^T \in \mathbb{R}^{n\times nk}$. Noting that $G_{\lambda,\gamma,k}(z) = \sup_\sigma f_\sigma(z)$, for the sub-additivity of $\sqrt{G_{\lambda,\gamma,k}(z)}$, it suffices to show that, for any fixed σ,

$$\sqrt{f_\sigma(z_1 + z_2)} \;\leq\; \sqrt{f_\sigma(z_1)} + \sqrt{f_\sigma(z_2)}, \quad \forall\, z_1, z_2 \in \mathbb{R}^n. \tag{19}$$

Let $\lambda_{\max}(Q_{uu})$ be the largest eigenvalue of Q_{uu}. Consider the following three cases:
a) If $\lambda_{\max}(Q_{uu}) > 0$, then (19) holds as $f_\sigma(z) \equiv \infty$.
b) If $\lambda_{\max}(Q_{uu}) < 0$, then the supremum in (18) is achieved by $\mathbf{u} = -Q_{uu}^{-1} Q_{uz} z$; and $\sqrt{f_\sigma(z)} = \big[z^T(Q_{zz} - Q_{zu} Q_{uu}^{-1} Q_{uz})z\big]^{1/2}$ with $Q_{zz} - Q_{zu} Q_{uu}^{-1} Q_{uz} \succ 0$ is a norm hence satisfies (19).
c) If $\lambda_{\max}(Q_{uu}) = 0$, then the null space $\mathcal{N}(Q_{uu}) \neq \{0\}$. If $Q_{uz} z \notin R(Q_{uu}) = \mathcal{N}(Q_{uu})^\perp$, there exists $\mathbf{u}_1 \in \mathcal{N}(Q_{uu})$ with $\mathbf{u}_1^T Q_{uz} z > 0$. By letting $\mathbf{u} = \alpha \mathbf{u}_1$ for arbitrarily large $\alpha > 0$, we obtain that $f_\sigma(z) = \infty$. If $Q_{uz} z \in R(Q_{uu})$, i.e., $Q_{uz} z = Q_{uu} \mathbf{u}_0$ for some $\mathbf{u}_0 \in \mathbb{R}^{nk}$, then

$$\begin{aligned} f_\sigma(z) \;&=\; \sup_{\mathbf{u}} \big(z^T Q_{zz} z + 2\mathbf{u}^T Q_{uu} \mathbf{u_0} + \mathbf{u}^T Q_{uu} \mathbf{u} \big) \\ &=\; z^T Q_{zz} z - \mathbf{u}_0^T Q_{uu} \mathbf{u}_0 \;=\; z^T (Q_{zz} - Q_{uz}^T Q_{uu}^\dagger Q_{uz}) z < \infty. \end{aligned}$$

Here, $Q_{uu}^\dagger$ denotes the Moore-Penrose pseudo inverse of Q_{uu}, and $Q_{zz} - Q_{uz}^T Q_{uu}^\dagger Q_{uz} \succ 0$ as $Q_{zz} \succ 0$ and $Q_{uu}^\dagger \preceq 0$. To sum up, $\sqrt{f_\sigma(z)}$ defines a norm on the subspace $Q_{uz}^{-1}[R(Q_{uu})]$, and is infinite everywhere else. As a result, (19) still holds.
This proves (19), hence the sub-additivity of $\sqrt{G_{\lambda,\gamma,k}(\cdot)}$ and $\sqrt{G_{\lambda,\gamma}(\cdot)}$ by Proposition 3.1.

(5) For any $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$, by the sub-additivity and homogeneity properties,

$$\sqrt{G_{\lambda,\gamma}(\alpha_1 z_1 + \alpha_2 z_2)} \leq \sqrt{G_{\lambda,\gamma}(\alpha_1 z_1)} + \sqrt{G_{\lambda,\gamma}(\alpha_2 z_2)} = \alpha_1 \sqrt{G_{\lambda,\gamma}(z_1)} + \alpha_2 \sqrt{G_{\lambda,\gamma}(z_2)}.$$

This shows the convexity of $\sqrt{G_{\lambda,\gamma}(\cdot)}$. The convexity of $\sqrt{G_{\lambda,\gamma,k}(\cdot)}$ can be proved similarly.

(6) That $\mathcal{G}_{\lambda,\gamma}$ is a subspace follows from the sub-additivity property of $\sqrt{G_{\lambda,\gamma}(\cdot)}$. Its invariance to subsystem dynamics follows from the Bellman equation (17).

(7) By setting $u = 0$, the right hand side of (12) reduces to the definition (10). $\square$

3.3 Quadratic Bound

Denote by $\mathbb{S}^{n-1} := \{z \in \mathbb{R}^n \mid \|z\| = 1\}$ the unit sphere. For each $\lambda, \gamma \in \mathbb{R}_+$, define

$$g_{\lambda,\gamma} := \sup_{z\in\mathbb{S}^{n-1}} G_{\lambda,\gamma}(z) \in \mathbb{R}_+ \cup \{+\infty\}. \tag{20}$$

By homogeneity, $G_{\lambda,\gamma}(z)$ admits the following tight bound:

$$G_{\lambda,\gamma}(z) \le g_{\lambda,\gamma}\|z\|^2, \quad \forall\, z \in \mathbb{R}^n. \tag{21}$$

We next study $g_{\lambda,\gamma}$ as a function of $(\lambda, \gamma) \in \mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}_+$. Obviously $g_{\lambda,\gamma}$ is non-decreasing in λ and non-increasing in γ by the monotonicity of $G_{\lambda,\gamma}(z)$; and $g_{0,\gamma} = 1, \forall\, \gamma \in \mathbb{R}_+$.

Proposition 3.3. *Both $g_{\lambda,\gamma}$ and $G_{\lambda,\gamma}(0)$ are lower semi-continuous functions of $(\lambda, \gamma) \in \mathbb{R}^2_+$.*

Proof. Let $\phi_{\sigma,u,z,k}(\lambda,\gamma) := \left[\sum_{t=0}^{k} \lambda^t \|x(t;\sigma,u,z)\|^2 - \gamma^2\lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2\right]$. Then

$$g_{\lambda,\gamma} = \sup_{z\in\mathbb{S}^{n-1}} \sup_{k\in\mathbb{Z}_+} \sup_{\sigma} \sup_{u\in\mathcal{U}_{k-1}} \phi_{\sigma,u,z,k}(\lambda,\gamma), \qquad G_{\lambda,\gamma}(0) = \sup_{k\in\mathbb{Z}_+} \sup_{\sigma} \sup_{u\in\mathcal{U}_{k-1}} \phi_{\sigma,u,0,k}(\lambda,\gamma), \tag{22}$$

expressing each as the supremum of a family of continuous functions of λ and γ. $\square$

For the next property, we first introduce the following lemma [1, pp. 119, Ex. 3.32].

Lemma 3.1. *Let $f, h : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ be two (extended valued) convex non-decreasing functions. Then their product $f \cdot h : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is also a convex function.*

Proposition 3.4. *$g_{\lambda,\gamma}$ is a convex function of $\lambda \in \mathbb{R}_+$ for each fixed $\gamma \in \mathbb{R}_+$, and a convex function of $\gamma^2 \in \mathbb{R}_+$ for each fixed $\lambda \in \mathbb{R}_+$.*

Proof. To show the convexity of $g_{\lambda,\gamma}$ in λ when γ is fixed, we first show by induction that for each $k \in \mathbb{Z}_+$, $G_{\lambda,\gamma,k}(z)$ is a convex function of λ for fixed $\gamma \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$. This is trivially true at $k = 0$ since $G_{\lambda,\gamma,0}(z) = \|z\|^2$ is constant in λ. Suppose it holds for $k = 0, 1, \ldots, k_*$. From the Bellman equation (17), $G_{\lambda,\gamma,k_*+1}(z) = \|z\|^2 + \lambda \cdot h(\lambda)$, where the function $h : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is given by $h(\lambda) := \sup_{i\in\mathcal{M}, v\in\mathbb{R}^m} \left[-\gamma^2\|v\|^2 + G_{\lambda,\gamma,k_*}(A_i z + B_i v)\right]$. It follows from the induction hypothesis and monotone property of $G_{\lambda,\gamma,k_*}(z)$ that the term inside the brackets, and hence $h(\lambda)$ itself, is convex and non-decreasing in λ. Applying Lemma 3.1 to the two functions $f(\lambda) := \lambda$ and $h(\lambda)$, we deduce that $G_{\lambda,\gamma,k_*+1}(z)$ is convex in λ. By induction, $G_{\lambda,\gamma,k}(z)$ are convex in λ for all $k \in \mathbb{Z}_+$. It follows from Proposition 3.1 that $G_{\lambda,\gamma}(z)$ is convex in λ. Finally, using the definition (20), we conclude that $g_{\lambda,\gamma}$ is convex in λ. Moreover, by (22), $g_{\lambda,\gamma}$ is the supremum of a family of affine functions of γ^2. Hence the second statement holds. $\square$

It should be noted that $g_{\lambda,\gamma}$ is in general not convex in (λ, γ^2), or convex in γ. More properties of $g_{\lambda,\gamma}$ will be presented in Section 3.4.

3.4 Domain of Convergence

This subsection is concerned with the set of $(\lambda, \gamma) \in \mathbb{R}_+^2$ at which $G_{\lambda,\gamma}(z)$ is finite for all z. This set plays a crucial role in characterizing the generalized ℓ_2-gain $\kappa(\lambda)$.

Definition 3.1. *The domain of convergence (DOC) of the generating function $G_{\lambda,\gamma}(z)$ is the set*

$$\Omega := \{(\lambda,\gamma) \in \mathbb{R}_+^2 \,|\, G_{\lambda,\gamma}(z) < \infty, \ \forall z \in \mathbb{R}^n\} \subset \mathbb{R}_+^2.$$

The open domain of convergence (open DOC) is defined to be the interior of Ω, i.e., $\Omega^\circ := int(\Omega)$.

Since $G_{\lambda,\gamma}(z) = \|z\|^2 < \infty$ at $\lambda = 0$, the DOC always contains the nonnegative γ-axis $\Lambda_0 := \{(0,\gamma) \,|\, \gamma \in \mathbb{R}_+\}$. Denote by Ω_+ the nontrivial part of DOC with Λ_0 removed:

$$\Omega_+ := \Omega \setminus \Lambda_0 = \Omega \cap (\mathbb{R}_{++} \times \mathbb{R}_+), \quad \text{where } \ \mathbb{R}_{++} := (0,\infty).$$

In the following, the DOC may refer to either Ω or Ω_+.

Some properties of the DOC are straightforward. For example, due to the monotonicity property of $G_{\lambda,\gamma}(z)$ in (λ,γ), if (λ,γ) belongs to the DOC, so does any (λ',γ') with $\lambda' \le \lambda$ and $\gamma' \ge \gamma$. Further properties of the DOC can be derived by utilizing the following finiteness tests.

Proposition 3.5. *For any fixed $\lambda, \gamma \in \mathbb{R}_+$, the following statements are equivalent:*

1) $G_{\lambda,\gamma}(z) < \infty$, $\forall\, z \in \mathbb{R}^n$;

2) $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is a norm, and thus continuous, on $\mathbb{R}^n$;

3) $g_{\lambda,\gamma} < \infty$;

4) $G_{\lambda,\gamma}(0) = 0$.

Moreover, if further $\lambda > 0$, then each of the above statements is equivalent to the following statement:

5) $G_{\lambda,\gamma}(B_i v) \le \gamma^2 \|v\|^2$ for all $i \in \mathcal{M}$ and all $v \in \mathbb{R}^m$.

Proof. 1) ⇒ 2): If 1) holds, then $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is finite on $\mathbb{R}^n$, positive homogeneous of degree one, and sub-additive by Proposition 3.2. Therefore, it defines a norm on $\mathbb{R}^n$.

2) ⇒ 3): Suppose 2) holds. Since $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is a norm, it and $G_{\lambda,\gamma}(\cdot)$ are continuous function on $\mathbb{R}^n$. Thus, their values on the compact set $\mathbb{S}^{n-1}$ must be bounded.

3) ⇒ 4): If $g_{\lambda,\gamma} < \infty$, then by (21), $0 \le G_{\lambda,\gamma}(0) \le g_{\lambda,\gamma}\|0\|^2 = 0$.

4) ⇒ 1): Suppose $G_{\lambda,\gamma}(0) = 0$. If $\lambda = 0$, then $G_{\lambda,\gamma}(z) = \|z\|^2 < \infty$, $\forall\, z$. Assume $\lambda > 0$ in the following. Then (17) evaluated at $z = 0$ yields $\sup_{i\in\mathcal{M},\, v\in\mathbb{R}^m}[G_{\lambda,\gamma}(B_i v) - \gamma^2\|v\|^2] = 0$, i.e., $G_{\lambda,\gamma}(B_i v) \le \gamma^2\|v\|^2$, $\forall\, i \in \mathcal{M}$, $v \in \mathbb{R}^m$. This shows that $G_{\lambda,\gamma}(z) < \infty$ at all z reachable from $x(0) = 0$ in one time step. Next let z be in the reachable set from $x(0) = 0$ in one step. The Bellman equation (17) implies

$$\sup_{i\in\mathcal{M},\, v\in\mathbb{R}^m} \big[-\gamma^2\|v\|^2 + G_{\lambda,\gamma}(A_i z + B_i v)\big] \;=\; \frac{G_{\lambda,\gamma}(z) - \|z\|^2}{\lambda}.$$

Therefore, for any $i \in \mathcal{M}$ and any $v \in \mathbb{R}^m$, $G_{\lambda,\gamma}(A_i z + B_i v) \;\le\; \lambda^{-1}\big(G_{\lambda,\gamma}(z) - \|z\|^2\big) + \gamma^2\|v\|^2 < \infty$. This shows that $G_{\lambda,\gamma}(\cdot)$ is finite on the reachable set from $x(0) = 0$ in two steps. By induction, we deduce that $G_{\lambda,\gamma}(z) < \infty$ for all z in the reachable set of the SLCS. Since the reachable set spans $\mathbb{R}^n$ by Assumption 2.1 and $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is sub-additive by Proposition 3.2, $G_{\lambda,\gamma}(\cdot)$ is finite on $\mathbb{R}^n$.

4) ⇔ 5): Let $\lambda > 0$. The equivalence of 4) and 5) follows directly from (17) by setting $z = 0$. $\square$

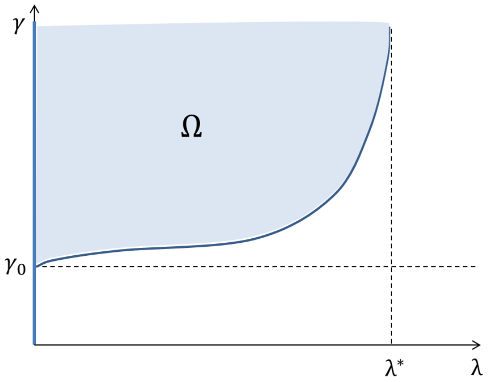

Figure 1: A generic plot of the DOC of the generating function $G_{\lambda,\gamma}(\cdot)$.

Additional observations on the DOC can be made based on the above proposition.

(1) *The lower bound of* Ω_+. Since $G_{\lambda,\gamma}(z) \geq \|z\|^2$, for $(\lambda, \gamma) \in \Omega_+$, by letting $z = B_i v$ we must have $\gamma^2\|v\|^2 \geq \|B_i v\|^2$, $\forall i \in \mathcal{M}$, $v \in \mathbb{R}^m$, i.e., $\gamma \geq \gamma_0$ with γ_0 defined in (7). Thus, the set Ω_+ is bounded from below by the horizontal line $\mathbb{R}_{++} \times \{\gamma_0\}$.

(2) *The right asymptotic bound of* Ω. Consider any $\lambda \in (0, \lambda^*)$, where λ^* is the radius of convergence of the autonomous generating function defined in (11). Since the SLS $\{\sqrt{\lambda} A_i\}_{i\in\mathcal{M}}$ is exponentially stable under arbitrary switching, $\kappa(\lambda)$ is finite by Theorem 2.1. Thus for all γ large enough, specifically $\gamma \geq \kappa(\lambda)$, we have $G_{\lambda,\gamma}(0) = 0$ by its definition in (13) and hence $(\lambda, \gamma) \in \Omega_+$. On the other hand, if $\lambda \geq \lambda^*$, $G_\lambda(z) = \infty$ for some z, so is $G_{\lambda,\gamma}(z)$ for any γ by property 7) of Proposition 3.2. In summary, the DOC is bounded from the right by the vertical line $\{\lambda^*\} \times \mathbb{R}_+$ and can be arbitrarily close to that line as $\lambda \to \lambda^*$.

(3) *The interior of* Ω. The argument in (2) above also implies that Ω must have nonempty interior.

Proposition 3.6. *The DOC* Ω *is a closed subset of* $\mathbb{R}^2_+$.

Proof. By Proposition 3.5, the DOC can be written as $\Omega = \{(\lambda, \gamma) \in \mathbb{R}^2_+ \mid G_{\lambda,\gamma}(0) \leq 0\}$, which is a sublevel set of the lower semi-continuous function $G_{\lambda,\gamma}(0)$ (cf. Proposition 3.3). Thus Ω is closed [21, pp. 51]. □

As a result, $g_{\lambda,\gamma}$ is finite not only in the interior Ω° but also on the boundary $\partial\Omega$ of the DOC. A generic plot of the DOC is shown in Fig. 1, with more to be given in Section 5. It should be mentioned that in general Ω_+ may not be convex.

When restricted on the DOC, the quadratic bound $g_{\lambda,\gamma}$ defined in (20) enjoys more favorable properties, in addition to the generic ones derived in Section 3.3.

Proposition 3.7. *The function* $g_{\lambda,\gamma} : \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ *has the following properties.*

(1) It is finite everywhere on Ω*, including on its boundary;*

(2) It is continuous on the interior Ω° *of* Ω*;*

(3) If at least one $A_i \neq 0$*, then for any fixed* $\gamma \in \mathbb{R}_+$*,* $g_{\lambda,\gamma}$ *is strictly increasing in* λ *on* Ω*.*

Proof. Statement (1) follows directly from the definition of the DOC and Proposition 3.6.

For Statement (2), we first show that on Ω, $g_{\lambda,\gamma}$ is continuous in each of the two variables when the other is fixed. For example, let $\gamma \in \mathbb{R}_+$ be fixed and let $\lambda_r := \sup\{\lambda \,|\, (\lambda,\gamma) \in \Omega\}$. We show that the function $h(\lambda) := g_{\lambda,\gamma}$ is continuous on $[0,\lambda_r]$ as follow. By Proposition 3.4, $h(\lambda)$ is convex and finite on $[0,\lambda_r]$, which implies its continuity on the open interval $(0,\lambda_r)$. The continuity of $h(\lambda)$ at λ_r follows from its being monotonically non-decreasing and lower semicontinuous. At $\lambda = 0$, $h(0) \le \liminf_{\lambda\downarrow 0} h(\lambda) = \lim_{\lambda\downarrow 0} h(\lambda) := \eta$ by the monotonicity of h. It is noted that the strict inequality $h(0) < \eta$ is impossible since otherwise, it follow from the monotonicity and continuity of h on the open interval $(0,\varepsilon)$ for a small $\varepsilon > 0$ that for any $z \in (0,\varepsilon)$, $h(z) \ge \eta$ and the line segment joining the points $(0,h(0))$ and $(z,h(z))$ would be below the point $(z',h(z')) \in \mathbb{R}^2_+$ for some $z' > 0$ sufficiently close to 0. This contradicts the convexity of h. Thus, $h(\cdot)$ is continuous at $\lambda = 0$. Similarly, we can show that $g_{\lambda,\gamma}$ is continuous in γ^2 (hence in γ) for a fixed λ whenever $(\lambda,\gamma) \in \Omega$. Now let $(\lambda',\gamma') \in \Omega^\circ$ be arbitrary, and let $\big((\lambda_s,\gamma_s)\big)$ be a sequence in Ω converging to (λ',γ'). Due to the lower semicontinuity of $g_{\lambda,\gamma}$, we have $g_{\lambda',\gamma'} \le \liminf_{s\to\infty} g_{\lambda_s,\gamma_s}$. For any $\varepsilon > 0$ small enough such that $(\lambda',\gamma'-\varepsilon) \in \Omega^\circ$, $\gamma_s \ge \gamma' - \varepsilon$ for all s sufficiently large. Thus, by the monotonicity property of $g_{\lambda,\gamma}$, $\limsup_{s\to\infty} g_{\lambda_s,\gamma_s} \le \limsup_{s\to\infty} g_{\lambda_s,\gamma'-\varepsilon} = g_{\lambda',\gamma'-\varepsilon}$, where the equality follows from the continuity of $g_{\lambda,\gamma'-\varepsilon}$ in λ. Since $\varepsilon > 0$ is arbitrary, $g_{\lambda',\gamma'-\varepsilon}$ is arbitrarily close to $g_{\lambda',\gamma'}$ due to the continuity of $g_{\lambda',\gamma}$ in γ. This shows $\limsup_{s\to\infty} g_{\lambda_s,\gamma_s} \le g_{\lambda',\gamma'}$, hence $\lim_{s\to\infty} g_{\lambda_s,\gamma_s} = g_{\lambda',\gamma'}$.

To prove Statement (3), recall that for a given $\gamma \in \mathbb{R}_+$, $[0,\lambda_r] \times \{\gamma\}$ is a maximal horizontal line segment contained in Ω. Assume without loss of generality that $\lambda_r > 0$. Since $g_{\lambda,\gamma}$ is convex in λ (see Proposition 3.4), the function $(g_{\lambda,\gamma} - g_{0,\gamma})/\lambda$ is non-decreasing in λ on $(0,\lambda_r]$. Thus, to show that $g_{\lambda,\gamma}$ is strictly increasing on $[0,\lambda_0]$, it suffices to show that $g_{\varepsilon,\gamma} > g_{0,\gamma} = 1$ for all $\varepsilon > 0$ small enough. Assume $A_{i_0} \ne 0$ for some $i_0 \in \mathcal{M}$. Then there exists $z_0 \in \mathbb{S}^{n-1}$ such that $A_{i_0} z_0 \ne 0$. By setting $z = z_0$, $u \equiv 0$, and $\sigma = (i_0, i_0, \ldots)$ in (12), we have $G_{\varepsilon,\gamma}(z_0) \ge \|z_0\|^2 + \varepsilon \|A_{i_0} z_0\|^2 + \cdots > 1$; hence $g_{\varepsilon,\gamma} > 1$. This completes the proof of Statement (3). □

Example 3.1 (One-Step FIR System)**.** Consider the case excluded in Statement 3) of Proposition 3.7 where $A_i = 0$ for all $i \in \mathcal{M}$. The resulting SLCS is a one-step FIR system as $A_i B_j = 0$, $\forall i,j \in \mathcal{M}$. In this case, the SLCS has solutions $x(t;\sigma,u,z) = B_{\sigma(t-1)} u(t-1)$, $\forall\, t \in \mathbb{N}$. Therefore,

$$G_{\lambda,\gamma}(z) \;=\; \|z\|^2 + \sum_{t=0}^{\infty} \lambda^{t+1} \sup_{\sigma(t),u(t)} \Big[-u^T(t) Q_{\sigma(t)} u(t) \Big], \tag{23}$$

where $Q_i := \gamma^2 I - B_i^T B_i$, $i \in \mathcal{M}$. If $\gamma \ge \gamma_0$ with γ_0 given in (7), then $Q_{\sigma(t)} \succeq 0, \forall\, t$; (23) thus implies $G_{\lambda,\gamma}(z) = \|z\|^2, \forall\, z$. If $\gamma < \gamma_0$, then at least one Q_i has a negative eigenvalue and $G_{\lambda,\gamma}(z) = \infty, \forall z$, for all $\lambda > 0$. In summary, the DOC is $\Omega_+ = \mathbb{R}_{++} \times [\gamma_0, \infty)$ and $g_{\lambda,\gamma} \equiv 1$ on Ω. □

It is shown in Proposition 3.1 that $G_{\lambda,\gamma,k}(\cdot) \uparrow G_{\lambda,\gamma}(\cdot)$ as $k \to \infty$ pointwise on $\mathbb{R}^n$. The following result shows that the convergence is uniform on the unit sphere if $(\lambda,\gamma) \in \Omega$.

Proposition 3.8. *Suppose* $(\lambda,\gamma) \in \Omega$. *Then* $G_{\lambda,\gamma,k}(\cdot) \uparrow G_{\lambda,\gamma}(\cdot)$ *uniformly on* $\mathbb{S}^{n-1}$.

Proof. As $k \to \infty$, the monotonically increasing sequence of continuous functions $\big(G_{\lambda,\gamma,k}\big)$ converges pointwise on $\mathbb{S}^{n-1}$ to $G_{\lambda,\gamma}$. Since $(\lambda,\gamma) \in \Omega$, it follows from Proposition 3.5 that $\sqrt{G_{\lambda,\gamma}(\cdot)}$, and hence $G_{\lambda,\gamma}(\cdot)$, is continuous on $\mathbb{S}^{n-1}$. Therefore, by Dini's Theorem [22, Theorem 7.13], $\big(G_{\lambda,\gamma,k}\big)$ converges uniformly to $G_{\lambda,\gamma}$ on $\mathbb{S}^{n-1}$. □

A stronger convergence result will be established in Theorem 3.1 and Corollary 3.1 when (λ,γ) is in the interior of Ω.

3.5 Approximation of Generating Functions

We uncover some structures of $G_{\lambda,\gamma,k}(\cdot)$ on the interior of the DOC, i.e., Ω°, which will be exploited to develop numerical algorithms for computing the generating function $G_{\lambda,\gamma}$.

Define the following set

$$\mathcal{W} := \left\{ (\lambda,\gamma) \in \mathbb{R}^2_+ \,\middle|\, \lambda > 0,\, \gamma^2 > \sup_{i\in\mathcal{M}, v\in\mathbb{S}^{m-1}} G_{\lambda,\gamma}(B_i v) \right\}.$$

Lemma 3.2. *The following holds: $\Omega^\circ \subseteq \mathcal{W} \subseteq \Omega_+$, where Ω_+ is the part of Ω with $\lambda > 0$.*

Proof. That $\mathcal{W} \subseteq \Omega_+$ is a direct consequence of Statement 5) of Proposition 3.5. To show $\Omega^\circ \subseteq \mathcal{W}$, assume $(\lambda,\gamma) \in \Omega^\circ$, which satisfies $\lambda > 0$. If there exist $i_0 \in \mathcal{M}$ and $v_0 \in \mathbb{S}^{m-1}$ such that $G_{\lambda,\gamma}(B_{i_0}v_0) \geq \gamma^2$, then for any sufficiently small $\varepsilon > 0$, $G_{\lambda,\gamma-\varepsilon}(B_{i_0}v_0) \geq G_{\lambda,\gamma}(B_{i_0}v_0) \geq \gamma^2 > (\gamma-\varepsilon)^2$. By 5) of Proposition 3.5, $(\lambda,\gamma-\varepsilon) \notin \Omega$ for small $\varepsilon > 0$, contradicting the assumption $(\lambda,\gamma) \in \Omega^\circ$. Thus, $G_{\lambda,\gamma}(B_i v) < \gamma^2$, $\forall\, i \in \mathcal{M}$, $v \in \mathbb{S}^{m-1}$, i.e., $(\lambda,\gamma) \in \mathcal{W}$. ☐

Denote by $\mathbb{P}$ the set of all $n \times n$ symmetric positive definite matrices. Whenever a matrix $P \in \mathbb{P}$, we write $P \succ 0$. Define the following subset of $\mathbb{P}$ for any fixed $(\lambda,\gamma) \in \mathcal{W}$:

$$\mathbb{P}_{\lambda,\gamma} := \{P \in \mathbb{P} \,|\, G_{\lambda,\gamma}(z) \geq z^T P z,\ \forall\, z \in \mathbb{R}^n\}.$$

Obviously, $\mathbb{P}_{\lambda,\gamma}$ is nonempty since it contains the identity matrix I. For any $P \in \mathbb{P}_{\lambda,\gamma}$ with $(\lambda,\gamma) \in \mathcal{W}$, we have $\gamma^2 > G_{\lambda,\gamma}(B_i v) \geq v^T B_i^T P B_i v$ for any $v \in \mathbb{S}^{m-1}$, and hence $\gamma^2 I - B_i^T P B_i \succ 0$, $\forall\, i \in \mathcal{M}$. Thus the following mapping $\rho_{\lambda,\gamma,i} : \mathbb{P}_{\lambda,\gamma} \to \mathbb{P}$ is well defined for each $i \in \mathcal{M}$:

$$\rho_{\lambda,\gamma,i}(P) := I + \lambda A_i^T P A_i + \lambda A_i^T P B_i \big(\gamma^2 I - B_i^T P B_i\big)^{-1} B_i^T P A_i, \quad \forall\, P \in \mathbb{P}_{\lambda,\gamma}. \tag{24}$$

This mapping is called the *Riccati mapping* of the subsystem $i \in \mathcal{M}$ of the SLCS (1).

Lemma 3.3. *For any $(\lambda,\gamma) \in \mathcal{W}$ and $i \in \mathcal{M}$, the Riccati mapping $\rho_{\lambda,\gamma,i} : \mathbb{P}_{\lambda,\gamma} \to \mathbb{P}_{\lambda,\gamma}$ is a self mapping of $\mathbb{P}_{\lambda,\gamma}$.*

Proof. Let $(\lambda,\gamma) \in \mathcal{W}$ and $P \in \mathbb{P}_{\lambda,\gamma}$ be arbitrary. Then $G_{\lambda,\gamma}(\cdot)$ is finite everywhere. Using the Bellman equation (17) and the definition of $\mathbb{P}_{\lambda,\gamma}$, we obtain, for any $z \in \mathbb{R}^n$,

$$G_{\lambda,\gamma}(z) \geq \sup_{i\in\mathcal{M},\, v\in\mathbb{R}^m} \Big[\|z\|^2 - \lambda\cdot\gamma^2\|v\|^2 + \lambda\cdot(A_i z + B_i v)^T P (A_i z + B_i v) \Big] = \max_{i\in\mathcal{M}} z^T \rho_{\lambda,\gamma,i}(P) z,$$

where the last step follows by choosing the optimal $v^* = (\gamma^2 I - B_i^T P B_i)^{-1} B_i^T P A_i z$ for each i in the supremum. This shows that $\rho_{\lambda,\gamma,i}(P) \in \mathbb{P}_{\lambda,\gamma}$, $\forall\, i \in \mathcal{M}$. ☐

Let $(\lambda,\gamma) \in \mathcal{W}$. The set-valued switched Riccati mapping $\rho_{\lambda,\gamma,\mathcal{M}} : 2^{\mathbb{P}_{\lambda,\gamma}} \to 2^{\mathbb{P}_{\lambda,\gamma}}$ is defined as

$$\rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{A}) := \{\rho_{\lambda,\gamma,i}(P) \,|\ i \in \mathcal{M} \text{ and } P \in \mathcal{A}\}, \qquad \forall\, \mathcal{A} \subseteq \mathbb{P}_{\lambda,\gamma},$$

which maps $\mathcal{A}$ to $\rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{A})$, both of which are subsets of $\mathbb{P}_{\lambda,\gamma}$. The following sequence of subsets of $\mathbb{P}_{\lambda,\gamma}$, called the Switched Riccati Sets (SRSs), can be generated recursively:

$$\mathcal{H}_0 := \{I\}, \quad \text{and} \quad \mathcal{H}_{k+1} := \rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H}_k), \quad \forall\, k \in \mathbb{Z}_+. \tag{25}$$

It turns out that $\mathcal{H}_k$'s completely characterize the finite-horizon generating functions $G_{\lambda,\gamma,k}(\cdot)$.

Proposition 3.9. *Suppose $(\lambda,\gamma) \in \mathcal{W}$. The following hold:*

(1) $G_{\lambda,\gamma,k}(z) = \max_{P\in\mathcal{H}_k} z^T P z$*,* $\forall\, z \in \mathbb{R}^n$*,* $k \in \mathbb{Z}_+$*;*

(2) For each $z \in \mathbb{R}^n$*, the supremum in the Bellman equation (17) can be achieved by some* $i_*(z) \in \mathcal{M}$ *and* $v_*(z) \in \mathbb{R}^m$*, namely,* $G_{\lambda,\gamma}(z) = \|z\|^2 - \lambda\cdot\gamma^2\|v_*(z)\|^2 + \lambda\cdot G_{\lambda,\gamma}(A_{i_*(z)}z + B_{i_*(z)}v_*(z))$*, with the properties that* $i_*(z)$ *and* $v_*(z)$ *are homogeneous along the ray directions:* $i_*(\alpha z) = i_*(z)$ *and* $v_*(\alpha z) = \alpha \cdot v_*(z)$ *for all* $\alpha > 0$ *and* $z \in \mathbb{R}^n$*; and that* $v_*(z)$ *is uniformly bounded in* z*:* $\|v_*(z)\| \le K_v\|z\|$*,* $\forall\ z \in \mathbb{R}^n$*, for some finite constant* $K_v \in \mathbb{R}_+$ *independent of* z*;*

(3) For any $z \in \mathbb{R}^n$ *and any* $k \in \mathbb{Z}_+$*, the supremum in the Bellman equation (16) can be achieved by some* $i_*^k(z) \in \mathcal{M}$ *and* $v_*^k(z) \in \mathbb{R}^m$ *that have the same homogeneous and uniformly bounded properties (and the same constant* K_v *as well) as* $i_*(z)$ *and* $v_*(z)$ *in (1).*

Proof. (1) We prove this by induction on k. The case $k = 0$ is trivial as $\mathcal{H}_0 = \{I\}$ and $G_{\lambda,\gamma,0}(z) = \|z\|^2$. Suppose $G_{\lambda,\gamma,j}(z) = \max_{P\in\mathcal{H}_j} z^T P z$ for $j = 0, 1, \dots, k-1$. By (16), we have

$$\begin{aligned}
G_{\lambda,\gamma,k}(z) &= \|z\|^2 + \sup_{i\in\mathcal{M},\, v\in\mathbb{R}^m} \left[-\lambda\gamma^2\|v\|^2 + \lambda\cdot G_{\lambda,\gamma,k-1}(A_i z + B_i v)\right] \\
&= \sup_{i\in\mathcal{M},\, v\in\mathbb{R}^m,\, P\in\mathcal{H}_{k-1}} \left[\|z\|^2 - \lambda\gamma^2\|v\|^2 + \lambda\cdot(A_i z + B_i v)^T P (A_i z + B_i v)\right] \\
&= \sup_{i\in\mathcal{M},\, P\in\mathcal{H}_{k-1}} z^T \rho_{\lambda,\gamma,i}(P) z = \sup_{P'\in\mathcal{H}_k} z^T P' z, \quad \forall\, z \in \mathbb{R}^n.
\end{aligned}$$

Note that in deriving the last two equalities, we choose the (z-dependent) optimal switching and control as

$$\left(i_*^k, P_*^k\right) := \operatorname*{arg\,max}_{i\in\mathcal{M},\, P\in\mathcal{H}_{k-1}} \left[z^T\rho_{\lambda,\gamma,i}(P)z\right], \qquad v_*^k := \left(\gamma^2 I - B_{i_*^k}^T P_*^k B_{i_*^k}\right)^{-1} B_{i_*^k}^T P_*^k A_{i_*^k} z. \tag{26}$$

The desired result then follows from the induction principle.

(2) Since $(\lambda, \gamma) \in \mathcal{W}$, there exists a small $\varepsilon > 0$ such that $\gamma - \varepsilon > 0$ and $G_{\lambda,\gamma}(B_i v) \le (\gamma-\varepsilon)^2$, $\forall\, v \in \mathbb{S}^{m-1}$, $i \in \mathcal{M}$. Define the finite constant $K_v := \varepsilon^{-1}\cdot\max_{i\in\mathcal{M}}\sup_{z\in\mathbb{S}^{n-1}}\sqrt{G_{\lambda,\gamma}(A_i z)}$. Then for any $z \in \mathbb{R}^n$ and $i \in \mathcal{M}$, whenever $v \notin \mathcal{V}_z := \{v \in \mathbb{R}^m \mid \|v\| \le K_v\|z\|\}$,

$$\begin{aligned}
-\gamma^2\|v\|^2 + G_{\lambda,\gamma}(A_i z + B_i v) &\le -\gamma^2\|v\|^2 + \left(\sqrt{G_{\lambda,\gamma}(A_i z)} + \sqrt{G_{\lambda,\gamma}(B_i v)}\right)^2 \\
&\le -\gamma^2\|v\|^2 + \left(\varepsilon K_v\|z\| + (\gamma-\varepsilon)\|v\|\right)^2 < 0,
\end{aligned}$$

where the sub-additivity of $\sqrt{G_{\lambda,\gamma}(\cdot)}$ is used in the first step. Thus, the supremum in the Bellman equation (17) cannot be achieved by v outside $\mathcal{V}_z$. As $G_{\lambda,\gamma}(\cdot)$ is continuous and $\mathcal{V}_z$ is compact, the supremum must be achieved by some $v_*(z) \in \mathcal{V}_z$ and some $i \in \mathcal{M}$. The homogeneity of $i_*(z)$ and $v_*(z)$ along rays follows immediately by noting that equation (17) is positive homogeneous of degree two in z and v.

(3) This follows by replacing $G_{\lambda,\gamma}$ with $G_{\lambda,\gamma,k}$ in the above proof of (2). □

The first statement of Proposition 3.9 shows that, for $(\lambda, \gamma) \in \mathcal{W}$, each finite horizon generating function is piecewise quadratic and continuous in z and can be represented by a finite set $\mathcal{H}_k$ of matrices generated by an iterative algorithm in (25). Furthermore, the optimal state sequence $x_*^k(t)$ and control sequence $u_*^k(t)$ over the horizon $t = 0, \dots, k-1$ achieving $G_{\lambda,\gamma,k}(z)$ can be obtained from the solution of a closed-loop system adopting in sequence the optimal state feedback switching and control policy $\left(i_*^{k-t}(\cdot), v_*^{k-t}(\cdot)\right)$ given in (26) for $t = 0, 1, \dots, k-1$. In what follows, we discuss this representation for the infinite horizon case.

Let $(\lambda,\gamma)\in\mathcal{W}$. Then $i_*(\cdot)$ and $v_*(\cdot)$ defined in the second statement of Proposition 3.9 specify a state feedback switching and control policy for the SLCS (1). Denote by $\widehat{x}(t)$ the resulting state trajectory under this policy starting from the initial state $z\in\mathbb{R}^n$, namely, $\widehat{x}(0)=z$, and

$$\widehat{x}(k+1) \;=\; A_{\sigma_{*,z}(k)}\widehat{x}(k)+B_{\sigma_{*,z}(k)}u_{*,z}(k), \quad \forall\, k\in\mathbb{Z}_+,$$

where $\sigma_{*,z}(k)=i_*(\widehat{x}(k))$, $u_{*,z}(k)=v_*(\widehat{x}(k))$. The switching and control sequences $\sigma_{*,z}(\cdot)$ and $u_{*,z}(\cdot)$ above are in general dependent on z. In the case this dependency needs to be highlighted, we write $\widehat{x}(\cdot)$ explicitly as $\widehat{x}(\cdot;\sigma_{*,z},u_{*,z},z)$. Under $(\sigma_{*,z},u_{*,z})$, the supremum in the Bellman equation (17) is exactly achieved at each step along $\widehat{x}(\cdot)$, i.e., for each $k\in\mathbb{Z}_+$,

$$\begin{aligned} G_{\lambda,\gamma}(z) \;&=\; G_{\lambda,\gamma}(\widehat{x}(0)) \;=\; \|\widehat{x}(0)\|^2-\gamma^2\lambda\|u_{*,z}(0)\|^2+\lambda\cdot G_{\lambda,\gamma}(\widehat{x}(1)) \;=\; \cdots \\ &=\; \sum_{t=0}^{k}\lambda^t\Big[\|\widehat{x}(t)\|^2-\gamma^2\lambda\|u_{*,z}(t)\|^2\Big]+\lambda^{k+1}G_{\lambda,\gamma}(\widehat{x}(k+1)). \end{aligned} \tag{27}$$

In contrast, when a generic switching-control sequence (σ,u) is applied, the resulting state trajectory $x(\cdot;\sigma,u,z)$ satisfies only the inequality, i.e., for each $k\in\mathbb{Z}_+$,

$$G_{\lambda,\gamma}(z) \;\geq\; \sum_{t=0}^{k}\lambda^t\Big[\|x(t;\sigma,u,z)\|^2-\gamma^2\lambda\|u(t)\|^2\Big]+\lambda^{k+1}G_{\lambda,\gamma}(x(k+1;\sigma,u,z)). \tag{28}$$

For the above reason, we refer to $(\sigma_{*,z},u_{*,z})$ as a *Bellman switching-control sequence* for a given initial state z. Note that such sequences may not be unique. The results developed in the next proposition hold uniformly for all possible Bellman switching-control sequences.

Proposition 3.10. *Let $(\lambda,\gamma)\in\Omega^\circ$ and let $(\sigma_{*,z},u_{*,z})$ be any choice of the Bellman switching-control sequences for an arbitrary initial state $z\in\mathbb{R}^n$. Then the following hold:*

(1) $u_{*,z}\in\mathcal{U}_\lambda$*;*

(2) $\sqrt{\lambda^t}\cdot\widehat{x}(t;\sigma_{*,z},u_{*,z},z)\to 0$ *as* $t\to\infty$*, and* $(\sigma_{*,z},u_{*,z})$ *achieves the supremum in the definition (12) of* $G_{\lambda,\gamma}(z)$*, namely,* $G_{\lambda,\gamma}(z)=\sum_{t=0}^{\infty}\lambda^t\|\widehat{x}(t;\sigma_{*,z},u_{*,z},z)\|^2-\gamma^2\lambda\sum_{t=0}^{\infty}\lambda^t\|u_{*,z}(t)\|^2$*;*

(3) The trajectories $\sqrt{\lambda^t}\cdot\widehat{x}(t;\sigma_{*,z},u_{*,z},z)$ *are uniformly bounded for all* $z\in\mathbb{S}^{n-1}$*;*

(4) The $\sqrt{\lambda}$*-discounted state trajectories of the SLCS (1), i.e.,* $\sqrt{\lambda^t}\cdot\widehat{x}(t;\sigma_{*,z},u_{*,z},z)$*, are (strongly) asymptotically stable at the origin.*

Proof. For notational simplicity, we denote $\widehat{x}(\cdot;\sigma_{*,z},u_{*,z},z)$ by $\widehat{x}(\cdot)$ when necessary in this proof.

(1) Note that (27) holds. Since $(\lambda,\gamma)\in\Omega^\circ$, we can find a small $\varepsilon>0$ such that $(\lambda,\gamma-\varepsilon)\in\Omega^\circ$. By applying the inequality (28) and then comparing it with (27), we have, for each $k\in\mathbb{Z}_+$,

$$\begin{aligned} G_{\lambda,\gamma-\varepsilon}(z) \;&\geq\; \sum_{t=0}^{k}\lambda^t\Big[\|\widehat{x}(t)\|^2-(\gamma-\varepsilon)^2\lambda\|u_{*,z}(t)\|^2\Big]+\lambda^{k+1}\cdot G_{\lambda,\gamma}(\widehat{x}(k+1)) \\ &=\; G_{\lambda,\gamma}(z)+\varepsilon(2\gamma-\varepsilon)\lambda\cdot\sum_{t=0}^{k}\lambda^t\|u_{*,z}(t)\|^2. \end{aligned} \tag{29}$$

We conclude that $u_{*,z}\in\mathcal{U}_\lambda$, for otherwise the right hand side of the above inequality tends to infinity as $k\to\infty$, contradicting the fact that $G_{\lambda,\gamma-\varepsilon}(z)$ is finite due to $(\lambda,\gamma-\varepsilon)\in\Omega^\circ$.

(2) Applying (27) and the fact that $u_{*,z} \in \mathcal{U}_\lambda$, we have, for each $k \in \mathbb{Z}_+$,

$$\sum_{t=0}^{k} \lambda^t \|\widehat{x}(t)\|^2 \;\le\; G_{\lambda,\gamma}(z) + \gamma^2 \lambda \sum_{t=0}^{k} \lambda^t \|u_{*,z}(t)\|^2 \le\; G_{\lambda,\gamma}(z) + \gamma^2\lambda \sum_{t=0}^{\infty} \lambda^t \|u_{*,z}(t)\|^2 < \infty.$$

Hence, $\sum_{t=0}^{\infty} \lambda^t \|\widehat{x}(t)\|^2 < \infty$, which implies $\lim_{t\to\infty} \sqrt{\lambda^t} \cdot \widehat{x}(t) = 0$. By letting $k \to \infty$ in (27) and noting that $\lambda^{k+1} G_{\lambda,\gamma}(\widehat{x}(k+1)) \le g_{\lambda,\gamma} \cdot \lambda^{k+1} \|\widehat{x}(k+1)\|^2 \to 0$, we obtain the second result.

(3) Fix a sufficiently small $\varepsilon > 0$ such that $(\lambda, \gamma - \varepsilon) \in \Omega^\circ$. For any $z \in \mathbb{S}^{n-1}$ and associated Bellman sequence $(\sigma_{*,z}, u_{*,z})$, the inequality (29) holds, which implies that

$$\sum_{t=0}^{\infty} \lambda^t \|u_{*,z}(t)\|^2 \;\le\; \frac{G_{\lambda,\gamma-\varepsilon}(z) - G_{\lambda,\gamma}(z)}{\varepsilon(2\gamma-\varepsilon)\lambda} \;\le\; M_u \;:=\; \frac{g_{\lambda,\gamma-\varepsilon}}{\varepsilon(2\gamma-\varepsilon)\lambda}.$$

This, together with the results in Statement (2), implies that for each $z \in \mathbb{S}^{n-1}$,

$$\left\| \sqrt{\lambda^t} \cdot \widehat{x}(t) \right\|^2 \le \sum_{t=0}^{\infty} \lambda^t \|\widehat{x}(t)\|^2 \le g_{\lambda,\gamma} + \gamma^2 \lambda M_u, \;\; \forall\, t \in \mathbb{Z}_+,$$

where the upper bound is a constant independent of $z \in \mathbb{S}^{n-1}$ and the associated $(\sigma_{*,z}, u_{*,z})$.

(4) As $\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)$ is homogeneous in z, the asymptotic stability follows directly from the convergence property in Statement (2) and the uniform boundedness property in Statement (3), regardless of z and $(\sigma_{*,z}, u_{*,z})$. ☐

In the following, strong exponential stability of the state trajectories $\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)$ is further proved.

Theorem 3.1. *Let $(\lambda, \gamma) \in \Omega^\circ$. Then the $\sqrt{\lambda}$-discounted state trajectories $\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)$ of the SLCS (1) under any Bellman switching-control policy $(\sigma_{*,z}, u_{*,z})$ are (strongly) exponentially stable at the origin, i.e., there exist constants $\rho > 0$ and $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$ and $(\sigma_{*,z}, u_{*,z})$, $\|\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)\| \le \rho \cdot r^t \|z\|, \forall\, t \in \mathbb{Z}_+$.*

Proof. Fix $(\lambda, \gamma) \in \Omega^\circ$. Motivated by [23, Theorem 3], we first show that the closed-loop system under any $(\sigma_{*,z}, u_{*,z})$ is (strongly) uniformly asymptotically stable at the origin, i.e., for any given constants $\delta > 0$ and $c \in (0, 1)$, there exists $T_{\delta,c} \in \mathbb{Z}_+$ such that $\|\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)\| \le c\delta$, $\forall\, t \ge T_{\delta,c}$ under any $(\sigma_{*,z}, u_{*,z})$, whenever $\|z\| \le \delta$. We prove this assertion by contradiction. Suppose it fails for some given $\delta > 0$ and $c \in (0, 1)$. Then there exist a sequence of initial states (z_k) with $\|z_k\| \le \delta$ for each k, a sequence of corresponding Bellman switching-control sequences $(\sigma_{*,z_k}, u_{*,z_k})$, and a strictly increasing sequence of times (t_k) with $\lim\limits_{k\to\infty} t_k = \infty$ such that $\|\sqrt{\lambda^{t_k}} \cdot \widehat{x}(t_k; \sigma_{*,z_k}, u_{*,z_k}, z_k)\| > c\,\delta$ for all $k \in \mathbb{Z}_+$. By Proposition 3.10, there exist two constants $\varrho \ge \delta$ and $\mu \in (0, \delta)$ satisfying the following two conditions:

- (i) $\|\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z_k}, u_{*,z_k}, z_k)\| \le \varrho$, $\forall\, t \in \mathbb{Z}_+$, $\forall\, k$;
- (ii) $\|\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)\| \le c\,\delta$, $\forall\, t \in \mathbb{Z}_+$ under any $(\sigma_{*,z}, u_{*,z})$, whenever $\|z\| < \mu$.

The condition (ii) implies that for each k, $\|\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z_k}, u_{*,z_k} z_k)\| \ge \mu$ for $t = 0, 1, \ldots, t_k$. Since $\mu \le \|z_k\| \le \delta$ for each k, a subsequence of (z_k), which we assume without loss of generality to be (z_k) itself, converges to some z_* satisfying $\mu \le \|z_*\| \le \delta$. Since the index set $\mathcal{M}$ is finite, there also exist a subsequence of (z_k), which is again assumed to be (z_k) itself, and $j_0 \in \mathcal{M}$ such that

$$G_{\lambda,\gamma}(z_k) \;=\; \|z_k\|^2 - \lambda \cdot \gamma^2 \|u_{*,z_k}(0)\|^2 + \lambda \cdot G_{\lambda,\gamma}\big(A_{j_0} z_k + B_{j_0} u_{*,z_k}(0)\big), \quad \forall\, k. \tag{30}$$

It follows from Statement (2) of Proposition 3.9 that the sequence $\big(u_{*,z_k}(0)\big)$ is bounded; hence there exists a subsequence of it, assumed to be $(u_{*,z_k}(0))$ itself without loss of generality, that converges to some $v_*(0) \in \mathbb{R}^m$. By letting $k \to \infty$ in (30) and noting the continuity of $G_{\lambda,\gamma}(\cdot)$, we obtain

$$G_{\lambda,\gamma}(z_*) \;=\; \|z_*\|^2 - \lambda \cdot \gamma^2 \|v_*(0)\|^2 + \lambda \cdot G_{\lambda,\gamma}\big(A_{j_0} z_* + B_{j_0} v_*(0)\big).$$

In other words, $(j_0, v_*(0))$ achieves the supremum in the Bellman equation (17) for z_*. Based on condition (ii) and the construction of (t_k), we must have $\|\sqrt{\lambda} \cdot \widehat{x}(1; \sigma_{*,z_k}, u_{*,z_k} z_k)\| \ge \mu$ for each $k \ge 1$. Letting $k \to \infty$, this shows that $\widehat{x}_*(1) := A_{j_0} z_* + B_{j_0} v_*(0)$ satisfies $\|\sqrt{\lambda} \cdot \widehat{x}_*(1)\| \ge \mu$.

By condition (i), for a fixed $t \ge 1$, any subsequence of $\big(\sqrt{\lambda^t} \cdot \widehat{x}(t; \sigma_{*,z_k}, u_{*,z_k}, z_k)\big)$ has a convergent subsequence. Using this result and Statement (2) of Proposition 3.9 (to bound the corresponding $\big(u_{*,z_k}(t)\big)$ at each fixed $t \ge 1$) and repeating a similar argument for $t = 0$ as above, we obtain via induction a switching sequence $\sigma_* = (j_0, j_1, \ldots)$ and a control sequence $v_* = (v_*(0), v_*(1), \ldots)$ such that: (i) (σ_*, v_*) is a Bellman switching-control sequence for the initial state z_*; and (ii) the resulting state trajectory $\widehat{x}_*(\cdot)$, which starts from $\widehat{x}_*(0) = z_*$ and is recursively defined by $\widehat{x}_*(t+1) = A_{\sigma_*(t)} \widehat{x}_*(t) + B_{\sigma_*(t)} v_*(t)$, $t \in \mathbb{Z}_+$, satisfies $\|\sqrt{\lambda^t} \cdot \widehat{x}_*(t)\| \ge \mu$ for each $t \in \mathbb{Z}_+$. However, this contradicts the convergence property in Statement (2) of Proposition 3.10. Consequently, the uniform asymptotic stability holds.

Having proved the uniform asymptotic stability, by using the homogeneity of $\widehat{x}(t; \sigma_{*,z}, u_{*,z}, z)$ in z and employing a similar argument in [13, Theorem 4.11], we can easily establish the (strong) exponential stability at the origin. $\square$

The next corollary shows that the exponential stability of the closed-loop system of the SLCS (1) under $(\sigma_{*,z}, u_{*,z})$ leads to exponential convergence of $\big(G_{\lambda,\gamma,k}\big)$ to $G_{\lambda,\gamma}$ on the unit sphere. This result is a cornerstone for numerical approximation of the generating function $G_{\lambda,\gamma}$ discussed in Section 6.

Corollary 3.1. *Let $(\lambda, \gamma) \in \Omega^\circ$. Then the sequence of finite-horizon generating functions $\big(G_{\lambda,\gamma,k}\big)$ converges uniformly exponentially on $\mathbb{S}^{n-1}$ to $G_{\lambda,\gamma}$ as $k \to \infty$, i.e., there exist constants $\varrho > 0$ and $\eta \in [0, 1)$ such that $|G_{\lambda,\gamma}(z) - G_{\lambda,\gamma,k}(z)| \;\le\; \varrho \cdot \eta^k$, $\forall\, z \in \mathbb{S}^{n-1}$, $k \in \mathbb{Z}_+$.*

Proof. It follows from (27) and Theorem 3.1 that, for any Bellman switching-control sequence $(\sigma_{*,z}, u_{*,z})$ associated with $z \in \mathbb{S}^{n-1}$ and any $k \in \mathbb{Z}_+$,

$$\begin{aligned} G_{\lambda,\gamma}(z) \;&=\; \sum_{t=0}^{k} \lambda^t \Big[\|\widehat{x}(t; \sigma_{*,z}, u_{*,z}, z))\|^2 - \gamma^2 \lambda \|u_{*,z}(t)\|^2 \Big] + \lambda^{k+1} G_{\lambda,\gamma}\big(\widehat{x}(k+1; \sigma_{*,z}, u_{*,z}, z)\big) \\ &\le\; G_{\lambda,\gamma,k}(z) + g_{\lambda,\gamma}(\rho\, r^{k+1})^2. \end{aligned}$$

The corollary thus holds with $\varrho := g_{\lambda,\gamma}(\rho r)^2$ and $\eta := r^2$. $\square$

Note that if (λ, γ) lies on the boundary of Ω, then the above results may fail. Indeed, it is shown in Section 5.1 that for $(\lambda, \gamma) \in \partial\Omega$, (i) the convergence of $\big(G_{\lambda,\gamma,k}\big)$ to $G_{\lambda,\gamma}$ is slower than exponential (cf. Fact 1), and (ii) there is no a Bellman switching-control sequence $(\sigma_{*,z}, u_{*,z})$ with $u_{*,z} \in \mathcal{U}_\lambda$ that achieves the supremum in $G_{\lambda,\gamma}(z)$ (cf. Fact 2).

4 Characterizing Generalized ℓ_2-Gain via Generating Functions

In this section, radii of convergence of the generating functions are introduced and shown to completely characterize the generalized ℓ_2-gain of the SLCS. These results provide the key connection between the generalized ℓ_2-gain in Section 2 and the generating functions in Section 3.

The radius of convergence of a power series $\sum_{t=0}^{\infty} \lambda^t a_t$ is the supremum of all $\lambda > 0$ for which the series converges. Viewing $G_{\lambda,\gamma}(z)$ in (15) as a power series in λ, a similar notion is defined.

Definition 4.1. *The radius of convergence of the generating function $G_{\lambda,\gamma}(z)$ is defined as*

$$\lambda^*(\gamma) := \sup\{\lambda \in \mathbb{R}_+ \,|\, G_{\lambda,\gamma}(z) < \infty,\ \forall z \in \mathbb{R}^n\},\ \forall\, \gamma \in \mathbb{R}_+.$$

Alternatively, $\lambda^*(\gamma) := \sup\{\lambda \,|\, g_{\lambda,\gamma} < \infty\}$. In view of the DOC, $\lambda^*(\gamma) = \sup\{\lambda \,|\, (\lambda, \gamma) \in \Omega\}$, i.e., $(\lambda^*(\gamma), \gamma) \in \mathbb{R}^2_+$ is on the right boundary of Ω. Since Ω is closed, $(\lambda^*(\gamma), \gamma) \in \Omega$ and $g_{\lambda^*(\gamma),\gamma} < \infty$ for each γ; see Proposition 3.7.

Proposition 4.1. *The radius of convergence $\lambda^*(\gamma)$ for $\gamma \geq 0$ has the following properties.*

(1) $\lambda^*(\gamma) \equiv 0$ for $0 \leq \gamma < \gamma_0$ where γ_0 is defined in (7);

(2) $\lambda^*(\gamma)$ is non-decreasing in γ;

(3) $\lambda^*(\gamma)$ is upper semicontinuous in γ.

Proof. Statement (1) is obvious as we have shown that the DOC Ω_+ is bounded from below by the horizontal line $\mathbb{R}_+ \times \{\gamma_0\}$. Statement (2) follows as $g_{\lambda,\gamma}$ is non-increasing in γ. Statement (3) is due to the fact that the hypograph of the function $\lambda^*(\gamma)$, namely $\{(\lambda, \gamma) \in \mathbb{R}^2_+ \,|\, \lambda \leq \lambda^*(\gamma)\}$, is a closed set (in fact, it is exactly the DOC Ω). ☐

Example 4.1. For the one-step FIR system with all $A_i = 0$, it has been shown in Example 3.1 that the DOC is $\Omega_+ = \mathbb{R}_{++} \times [\gamma_0, \infty)$. As a result, the radius of convergence is

$$\lambda^*(\gamma) = \begin{cases} 0 & \text{if } 0 \leq \gamma < \gamma_0 \\ \infty & \text{if } \gamma \geq \gamma_0, \end{cases}$$

which is upper semicontinuous but not continuous in γ. ☐

The graph of the function $\lambda^*(\gamma)$ describes the right boundary of the DOC. By a change of perspective, the same boundary can be viewed from bottom up as the graph of the function $\gamma^*(\lambda)$ defined as follows,

$$\gamma^*(\lambda) := \inf\{\gamma \in \mathbb{R}_+ \,|\, g_{\lambda,\gamma} < \infty\}, \quad \forall\, \lambda \in \mathbb{R}_+. \tag{31}$$

Or equivalently, $\gamma^*(\lambda) := \inf\{\gamma \in \mathbb{R}_+ \,|\, (\lambda, \gamma) \in \Omega\}$.

Proposition 4.2. *$\gamma^*(\lambda)$ has the following properties.*

(1) $\gamma^*(0) = 0$ and $\gamma^*(\lambda) \geq \gamma_0$ for $\lambda > 0$;

(2) $\gamma^*(\lambda)$ is non-decreasing in λ;

(3) $\gamma^*(\lambda)$ is lower semicontinuous in λ.

Proof. We deduce (1) by noting that Ω contains the nonnegative γ-axis and that Ω_+ is bounded from below by the horizontal line $\mathbb{R}_+ \times \{\gamma_0\}$. Statement (2) follows as $g_{\lambda,\gamma}$ is non-decreasing in λ. To prove (3), we note that the epigraph of $\gamma^*(\lambda)$, i.e., $\{(\lambda, \gamma) \,|\, \gamma \geq \gamma^*(\lambda)\}$, is the closed set Ω. ☐

Example 4.2. For the one-step FIR system whose DOC is given by $\Omega_+ = \mathbb{R}^0_+ \times [\gamma_0, \infty)$, we have

$$\gamma^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0 \\ \gamma_0 & \text{if } \lambda > 0 \end{cases},$$

which is lower semicontinuous but not continuous in λ. ☐

As $\gamma^*(\lambda)$ and $\lambda^*(\gamma)$ are different characterizations of the same boundary of the DOC, it is not surprising that they are (generalized) inverse of each other, as is shown below.

Proposition 4.3. *The two functions $\gamma^*(\lambda)$ for $\lambda > 0$ and $\lambda^*(\gamma)$ for $\gamma > 0$ satisfy*

$$\begin{aligned} \gamma^*(\lambda) &= \inf\{\gamma > 0 : \lambda^*(\gamma) \geq \lambda\}, \quad \forall\, \lambda > 0, \\ \lambda^*(\gamma) &= \sup\{\lambda > 0 : \gamma^*(\lambda) \leq \gamma\}, \quad \forall\, \gamma > 0. \end{aligned}$$

Proof. For any $\lambda > 0$, the condition $\lambda^*(\gamma) \geq \lambda$ is equivalent to $(\lambda, \gamma) \in \Omega_+$, which in turn is equivalent to $\gamma \geq \gamma^*(\lambda)$. This proves the first identity. Similarly, for any $\gamma > 0$, assuming $\lambda > 0$, the condition $\gamma^*(\lambda) \leq \gamma$ is equivalent to $(\lambda, \gamma) \in \Omega_+$, which in turn is equivalent to $\lambda^*(\gamma) \geq \lambda$. This proves the second identity. ☐

It turns out that $\gamma^*(\lambda)$ for $\lambda > 0$ is exactly the generalized ℓ_2-gain $\kappa(\lambda)$ that we focus on.

Theorem 4.1. *The generalized ℓ_2-gain $\kappa(\lambda)$ of the SLCS (1) satisfies $\kappa(\lambda) = \gamma^*(\lambda)$ for all $\lambda > 0$. As a result,*

$$\begin{aligned} \kappa(\lambda) &= \inf\{\gamma > 0 : \lambda^*(\gamma) \geq \lambda\}, \quad \forall\, \lambda > 0, \\ \lambda^*(\gamma) &= \sup\{\lambda > 0 : \kappa(\lambda) \leq \gamma\}, \quad \forall\, \gamma > 0. \end{aligned}$$

Proof. Suppose $\lambda > 0$ is arbitrary. It suffices to show that the two conditions $\gamma \geq \kappa(\lambda)$ and $\gamma \geq \gamma^*(\lambda)$ are equivalent. By definition (4), $\gamma \geq \kappa(\lambda)$ if and only if

$$\sum_{t=0}^{\infty} \lambda^t \|x(t+1; \sigma, u, 0)\|^2 \leq \gamma^2 \sum_{t=0}^{\infty} \lambda^t \, \|u(t)\|^2, \quad \forall\, \sigma, \forall\, u \in \mathcal{U}_\lambda.$$

By (13), the above inequality holds if and only if $G_{\lambda,\gamma}(0) = 0$. By Proposition 3.5, the latter is equivalent to $g_{\lambda,\gamma} < \infty$, i.e., $(\lambda, \gamma) \in \Omega$. Finally, $(\lambda, \gamma) \in \Omega$ is equivalent to $\gamma \geq \gamma^*(\lambda)$ by the definition (31) of $\gamma^*(\lambda)$. This proves the first statement. The rest follows from Proposition 4.3. ☐

Theorem 4.1 connects the generating functions with the generalized ℓ_2-gain of the SLCSs. It implies that, to determine $\kappa(\lambda)$, it suffices to characterize the DOC, or more precisely, the right/bottom boundary of the DOC. As an example, using Theorem 4.1 and Proposition 4.2, we arrive at the following result that is not obvious from the definition (4).

Corollary 4.1. *The generalized ℓ_2-gain $\kappa(\lambda)$ is a non-decreasing function of $\lambda \in \mathbb{R}_+$.*

The following example shows that $\kappa(\lambda)$ may *not* be strictly increasing even if at least one $A_i \neq 0$.

Example 4.3. Consider the following SLCS with a constant $\theta > 1$ that satisfies Assumption 2.1:

$$A_1 = 0, \quad B_1 = \theta \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For all $P \in \mathbb{R}^{2\times 2}$ such that $B_i^T P B_i \prec \gamma^2 I$, the Riccati mapping for subsystem $i \in \{1, 2\}$ is

$$\rho_{\lambda,\gamma,i}(P) = I + \lambda A_i^T P A_i + \lambda A_i^T P B_i \big(\gamma^2 I - B_i^T P B_i\big)^{-1} B_i^T P A_i.$$

For a diagonal matrix $P = \mathrm{diag}(p_1, p_2)$, it is easy to see that (1) $\rho_{\lambda,\gamma,1}(P) \equiv I$ if $\theta^2 p_1 < \gamma^2$; and (2) $\rho_{\lambda,\gamma,2}(P) = \mathrm{diag}(1, 1 + \lambda p_1)$ if $p_2 < \gamma^2$. Assume $\gamma > \theta$ and $\gamma > \sqrt{1+\lambda}$, and let $P_\lambda := \mathrm{diag}(1, 1+\lambda)$. Then the Riccati iteration of the SLCS is given by: $\mathcal{H}_0 = \{I\}$, $\mathcal{H}_1 = \rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H}_0) = \{I, P_\lambda\}$, $\mathcal{H}_2 = \rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H}_1) = \{I, P_\lambda, \rho_{\lambda,\gamma,1}(P_\lambda), \rho_{\lambda,\gamma,2}(P_\lambda)\} = \{I, P_\lambda\}$, and $\mathcal{H}_k = \{I, P_\lambda\}, \forall\, k = 3, 4, \ldots$. Note that I is redundant because $I \preceq P_\lambda$. Thus $G_{\lambda,\gamma}(z) = G_{\lambda,\gamma,k}(z) = z^T P_\lambda z \Rightarrow g_{\lambda,\gamma} = 1 + \lambda < \infty$. This implies that the interior of DOC contains the intersection of $\Omega_1^\circ = \{(\lambda, \gamma) \,|\, \gamma > \theta\}$ and $\Omega_2^\circ = \{(\lambda, \gamma) \,|\, \gamma > \sqrt{1+\lambda}\}$, where Ω_1° and Ω_2° are the interiors of the DOC of subsystem 1 and subsystem 2, respectively. Therefore, $\Omega^\circ = \{(\lambda, \gamma) \,|\, \gamma > \max(\sqrt{1+\lambda}, \theta)\}$. This shows that $\kappa(\lambda) = \max(\sqrt{1+\lambda}, \theta)$, $\forall \lambda > 0$, which remains constant for all $\lambda \in (0, \theta^2 - 1)$. ☐

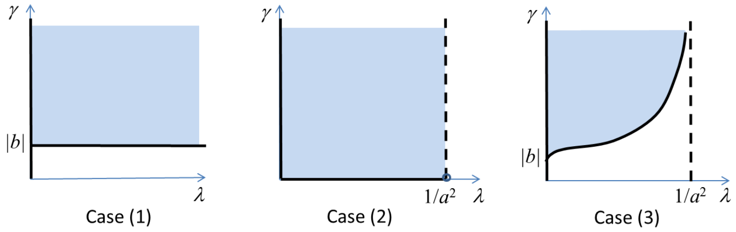

Figure 2: DOC of linear system (32) for the three cases in Proposition 5.1.

5 One-dimensional Switched Linear Control Systems

We study a special class of SLCSs, i.e., one-dimensional (1-D) SLCSs or the SLCSs on $\mathbb{R}$.

5.1 One-dimensional Linear Control System

First consider the 1-D (non-switched) linear control system for some $a, b \in \mathbb{R}$:

$$x(t+1) \;=\; ax(t) + bu(t), \quad t \in \mathbb{Z}_+. \tag{32}$$

For any $\lambda, \gamma \in \mathbb{R}_+$, its controlled generating function and finite horizon counterparts are all quadratic: $G_{\lambda,\gamma}(z) = g_{\lambda,\gamma} z^2$, $G_{\lambda,\gamma,k}(z) = g_k z^2$, $\forall\, z \in \mathbb{R}$, $k \in \mathbb{Z}_+$. In the following we focus on the nontrivial case $\lambda > 0$. The Bellman equation implies

$$g_{k+1} z^2 = z^2 + \lambda \cdot \sup_{v \in \mathbb{R}} \left[-\gamma^2 v^2 + g_k (az + bv)^2 \right], \; \forall\, z \in \mathbb{R} \tag{33}$$

$$\Rightarrow \quad g_{k+1} = F_{a,b}(g_k), \;\; k = 0, 1, \ldots, \text{ with } g_0 = 1. \tag{34}$$

Here, $F_{a,b} : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and convex with $F_{a,b}(0) = 1$. The generating function $G_{\lambda,\gamma}(\cdot)$ is finite if the sequence (g_k) generated from (34) converges, i.e., $g_k \uparrow g_{\lambda,\gamma} < \infty$, or equivalently, if the graph of $F_{a,b}(g)$ intersects that of the identify function $\mathrm{id}(g) := g$ at some $g > 0$.

Proposition 5.1. *Given $\lambda > 0$, the DOC Ω_+ of the generating function $G_{\lambda,\gamma}(z) = g_{\lambda,\gamma} z^2$ for the system (32) and the corresponding $g_{\lambda,\gamma}$ are characterized as follows.*

(1) If $a = 0$ and $b \neq 0$, then $\Omega_+ = \{(\lambda,\gamma) \in \mathbb{R}^2_+ \,|\, \gamma \geq |b|, \lambda > 0\}$, on which $g_{\lambda,\gamma} \equiv 1$.

(2) If $a \neq 0$ and $b = 0$[2], then $\Omega_+ = \{(\lambda,\gamma) \,|\, 0 < \lambda < 1/a^2\}$, on which $g_{\lambda,\gamma} = 1/(1 - a^2\lambda)$.

(3) If $a \neq 0, b \neq 0$, then $\Omega_+ = \{(\lambda,\gamma) \,|\, \gamma > |b|,\, 0 < \lambda \leq (\gamma - |b|)^2/(a^2\gamma^2)\}$, on which

$$g_{\lambda,\gamma} = \frac{c_0 - \sqrt{(c_0)^2 - 4b^2\gamma^2}}{2b^2}, \tag{35}$$

where $c_0 := b^2 + \gamma^2 - a^2\lambda\gamma^2$.

The proof of Proposition 5.1 is given in Appendix. Fig. 2 plots the DOC for the above three cases. Note that Ω is not closed in Case (2). This does not contradict Proposition 3.6, since Assumption 2.1 fails in this case.

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[2] Strictly speaking, the system in this case does not satisfy the reachability assumption (Assumption 2.1). It is included here for the sake of completeness.

Corollary 5.1. *For the three cases in Proposition 5.1, the radius of convergence and the generalized ℓ_2-gain of system (32) are given by*

(1) $\lambda^*(\gamma) = \begin{cases} 0 & \text{if } 0 \le \gamma < |b| \\ \infty & \text{if } \gamma \ge |b|, \end{cases}$ *and* $\kappa(\lambda) = |b|,\ \forall \lambda \ge 0$*;*

(2) $\lambda^*(\gamma) = \frac{1}{a^2},\ \forall \gamma \ge 0,$ *and* $\kappa(\lambda) = \begin{cases} 0 & \text{if } 0 \le \lambda < \frac{1}{a^2} \\ \infty & \text{if } \lambda \ge \frac{1}{a^2}; \end{cases}$

(3) $\lambda^*(\gamma) = \begin{cases} 0 & \text{if } 0 \le \gamma \le |b| \\ \frac{(\gamma-|b|)^2}{a^2\gamma^2} & \text{if } \gamma > |b|, \end{cases}$ *and* $\kappa(\lambda) = \begin{cases} \frac{|b|}{1-\sqrt{\lambda}|a|} & \text{if } 0 \le \lambda < \frac{1}{a^2} \\ \infty & \text{if } \lambda \ge \frac{1}{a^2}. \end{cases}$

We present two interesting facts of $G_{\lambda,\gamma}$ when (λ, γ) is on the boundary of Ω as follows. Consider Case (3) with $a \neq 0, b \neq 0$. For a given $\gamma > |b|$, let $\lambda = \lambda^*$ be the $\lambda^*(\gamma)$ given in Corollary 5.1. Plugging it into (35), we obtain $g_{\lambda^*,\gamma} = \gamma/|b| < \infty$. Denote by (g_k) the sequence generated in (34), which is strictly increasing and converges to $g_{\lambda^*,\gamma}$ as $k \to \infty$. Then $G_{\lambda^*,\gamma,k}(z) = g_k\|z\|^2$.

Fact 1: The convergence rate of the sequence (g_k) to $g_{\lambda^*,\gamma}$ is slower than exponential. This is because the linearized system of (34) around its equilibrium point $g_{\lambda^*,\gamma}$ is marginally stable, i.e., $\frac{d}{dg}F_{a,b}(g_{\lambda^*,\gamma}) = 1$. Geometrically, the graphs of the function $1+(a^2\lambda\gamma^2 g)/(\gamma^2 - b^2 g)$ and the identity function are tangential at $g_{\lambda^*,\gamma}$. In contrast, when $0 < \lambda < \lambda^*$, (g_k) converges to $g_{\lambda,\gamma}$ exponentially fast as shown in Corollary 3.1.

Fact 2: As the time horizon k increases, the "optimal" input sequence and the resulting state sequence that achieve the supremum in the definition (14) of k-horizon generating function both have unbounded energy. To see this, we first find the optimal control u^* achieving $G_{\lambda^*,\gamma,k}(z) = g_k\|z\|^2$ for a horizon $k \ge 1$. From the Bellman equation (33) with $\lambda = \lambda^*$ and k replaced by $k-1$, we see that starting from $x^*(0) = z$, the optimal control $u^*(0) = abg_{k-1}/(\gamma^2 - b^2 g_{k-1})\, x^*(0)$. Plugging this into (32) yields $x^*(1) = a\gamma^2/(\gamma^2 - b^2 g_{k-1})x^*(0)$. Thus, the k-horizon optimal u^* and x^* are $u^*(t) = abg_{k-t-1}(\gamma^2 - b^2 g_{k-t-1})x^*(t)$ and $x^*(t+1) = a\gamma^2/(\gamma^2 - b^2 g_{k-1-t})x^*(t)$ for $t = 0, 1, \ldots, k-1$. The energy of the state sequence x^* up to time k, $J_k(x^*) = \sum_{t=0}^{k}(\lambda^*)^t|x^*(t)|^2$, is given by

$$J_k(x^*) \;=\; z^2 \sum_{t=0}^{k}\prod_{s=0}^{t-1}\left(\frac{\gamma/|b| - 1}{\gamma/|b| - g_{k-1-s}/g_{\lambda^*,\gamma}}\right)^2,$$

which tends to infinity as $k \to \infty$ since $g_k \to g_{\lambda^*,\gamma}$. Similarly, we can show that the energy of the control sequence, $J_k(u^*)$, tends to ∞ as $k \to \infty$. Thus, while both $J_k(x^*)$ and $J_k(u^*)$ approach ∞ as $k \to \infty$, their difference is bounded and approaches the finite value $G_{\lambda^*,\gamma}(z)$.

5.2 One-dimensional SLCS

Consider the SLCS on $\mathbb{R}$ with each $a_i, b_i \in \mathbb{R}$:

$$x(t+1) \;=\; a_{\sigma(t)}x(t) + b_{\sigma(t)}u(t), \quad t \in \mathbb{Z}_+,\ \sigma(t) \in \mathcal{M}. \tag{36}$$

The generating functions of the SLCS (36) remain quadratic, i.e., $G_{\lambda,\gamma}(z) = g_{\lambda,\gamma}z^2$, $G_{\lambda,\gamma,k}(z) = g_k z^2$, $\forall\, z \in \mathbb{R}$. It follows from the Bellman equation that g_k is recursively defined by

$$g_{k+1} = \max_{i\in\mathcal{M}} F_{a_i,b_i}(g_k), \quad g_0 = 1, \tag{37}$$

where $F_{a_i,b_i} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined in (34) with (a, b) replaced by (a_i, b_i). The generating function is finite if and only if the above iteration converges, or equivalently, the graph of the convex function

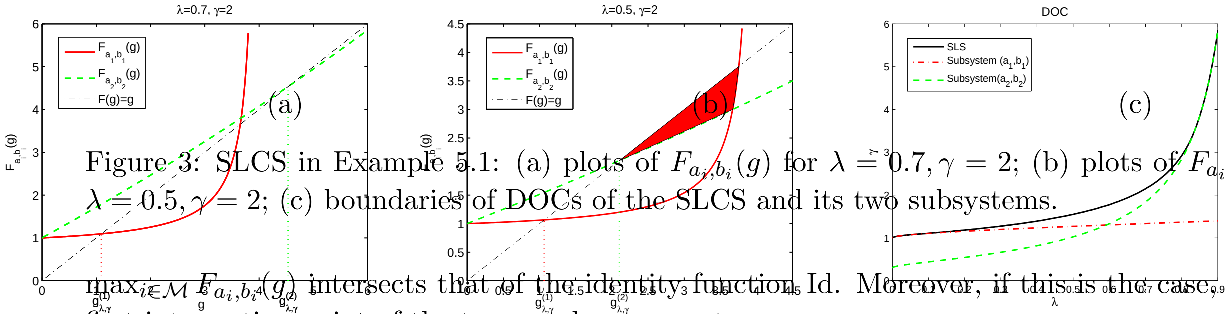


Figure 3: SLCS in Example 5.1: (a) plots of $F_{a_i,b_i}(g)$ for $\lambda = 0.7, \gamma = 2$; (b) plots of $F_{a_i,b_i}(g)$ for $\lambda = 0.5, \gamma = 2$; (c) boundaries of DOCs of the SLCS and its two subsystems.

$\max_{i\in\mathcal{M}} F_{a_i,b_i}(g)$ intersects that of the identity function Id. Moreover, if this is the case, then the first intersection point of the two graphs occurs at $g = g_{\lambda,\gamma}$.

Denote by $g^{(i)}_{\lambda,\gamma} \in \mathbb{R}_+ \cup \{+\infty\}$, $i \in \mathcal{M}$, the final value of iteration (33) with (a, b) replaced by (a_i, b_i), which characterizes the generating function of the i-th subsystem without switching. Clearly, $g_{\lambda,\gamma} \geq \max_{i\in\mathcal{M}} g^{(i)}_{\lambda,\gamma}$. It is possible that a strict inequality holds. Due to the convexity of $F_{a_i,b_i}(\cdot)$, this is only possible if $g_{\lambda,\gamma} = \infty$ while $g^{(i)}_{\lambda,\gamma} < \infty$ for all $i \in \mathcal{M}$.

Example 5.1. Consider a 1-D SLCS with two subsystems: $a_1 = 0.3$, $b_1 = 1$, $a_2 = 1$, $b_2 = 0.3$. In the first two figures of Fig. 3, we plot the functions $F_{a_i,b_i}(g)$, $i = 1, 2$, for two different choices of λ, γ. In subfigure (a) where $\lambda = 0.7$ and $\gamma = 2$, the graph of either function intersects that of Id (dashed dotted line) while the graph of their maximum does not. Thus, without switching both subsystems have finite generating functions while the SLCS has an infinite one: $g_{\lambda,\gamma} > \max_{i=1,2} g^{(i)}_{\lambda,\gamma}$. In subfigure (b), we set $\lambda = 0.5$ and $\gamma = 2$, in which case $g_{\lambda,\gamma} = \max_{i=1,2} g^{(i)}_{\lambda,\gamma} < \infty$. $\square$

Determining the convergence of (37), and if so, the limit $g_{\lambda,\gamma}$, is achieved by solving the following optimization problem:

$$\text{minimize } g \text{ subject to } F_{a_i,b_i}(g) \leq h \leq g, \ \forall\, i \in \mathcal{M}. \tag{38}$$

The feasible set of (g, h) in (38) is the part of the epigraph of $\max_{i\in\mathcal{M}} F_{a_i,b_i}(g)$ on and below the graph of the identity function (e.g., the shaded region in Fig. 3(b)). The solution of (38) is exactly $g_{\lambda,\gamma}$ if the feasible set is nonempty. Otherwise $g_{\lambda,\gamma} = \infty$. Note that each constraint $F_{a_i,b_i}(g) \leq h$ can be reduced to linear and/or second order cone constraints:

1. In the case $a_i = 0$: $g \leq \gamma^2/b_i^2$ and $h \geq 1$;
2. In the case $a_i \neq 0$, $b_i = 0$: $h \geq 1 + a_i^2\lambda g$;
3. In the case $a_i \neq 0$, $b_i \neq 0$: $0 \leq g \leq \gamma^2/b_i^2$, $h \geq 1$, and $\left\| \begin{bmatrix} b_i^2(g+h) + c_1 \\ 2a_i\gamma^2\sqrt{\lambda} \end{bmatrix} \right\| \leq b_i^2(h-g) + c_2$, where $c_1 = a_i^2\lambda\gamma^2 - b_i^2 - \gamma^2$ and $c_2 = a_i^2\lambda\gamma^2 - b_i^2 + \gamma^2$.

Thus (38) is a second order cone program that can be solved efficiently by many of the existing LMI tools.

Using the software tool SeDuMi, the boundaries of the DOCs for the SLCS in Example 5.1 (solid line) and for its two subsystems (dashdot and dash lines, respectively) are computed and are depicted in Fig. 3(c). Clearly, the DOC of the SLCS is a proper subset of the intersection of its two subsystems' DOCs. Furthermore, it is noted that the DOCs are not convex.

6 Computation of Generating Functions

In this section, the results established in Section 3 are exploited to develop numerical algorithms for computing the generating functions, and hence the generalized ℓ_2-gain of the SLCS (1).

6.1 Numerical Algorithms

As shown in Proposition 3.8 and Corollary 3.1, if $(\lambda,\gamma)\in\Omega$, the finite-horizon generating functions $G_{\lambda,\gamma,k}(\cdot)$ converge uniformly on the unit sphere to the infinite-horizon generating function $G_{\lambda,\gamma}(\cdot)$ as $k\to\infty$; and the convergence is uniformly exponential when $(\lambda,\gamma)\in\Omega^\circ$. Thus, a strategy to compute $G_{\lambda,\gamma}(\cdot)$ within a given accuracy is to compute $G_{\lambda,\gamma,k}(\cdot)$ for sufficiently large k. By Proposition 3.9, each $G_{\lambda,\gamma,k}(\cdot)$ is represented by a finite set $\mathcal{H}_k$ of positive definite matrices: $G_{\lambda,\gamma,k}(z)=\sup_{P\in\mathcal{H}_k} z^TPz$, where $\mathcal{H}_k$ is generated through the switched Riccati iteration procedure in (25). This yields a basic algorithm for computing $\mathcal{H}_k$, hence $G_{\lambda,\gamma,k}$.

A drawback of this basic algorithm is that its complexity measured by the size of the set $\mathcal{H}_k$ grows exponentially as k increases: $|\mathcal{H}_k|=|\mathcal{M}|^k$, where $|\mathcal{M}|$ is the number of subsystems of the SLCS. To reduce complexity, a technique originally proposed in [30] is employed. Note that not all matrices in $\mathcal{H}_k$ are useful in the representation $G_{\lambda,\gamma,k}(z)=\sup_{P\in\mathcal{H}_k} z^TPz$. A matrix $P\in\mathcal{H}_k$ is *redundant* if it can be removed without affecting the representation, or more precisely, if for any $z\in\mathbb{R}^n$, we can find $P'\in\mathcal{H}_k$ with $P'\neq P$ such that $z^TPz\le z^TP'z$. A sufficient (though not necessary) condition for $P\in\mathcal{H}_k$ to be redundant is

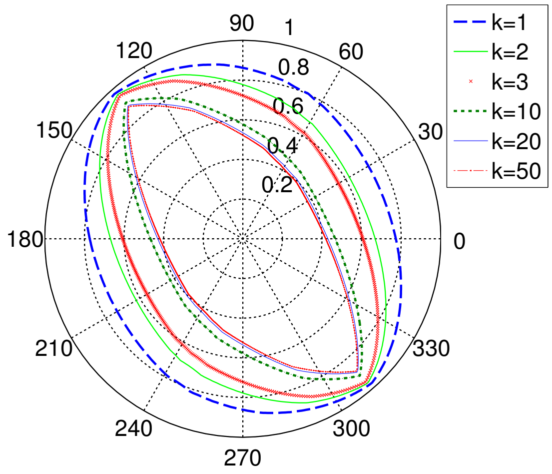
$$P \preceq \sum_{P'\in\mathcal{H}_k,\,P'\neq P} \alpha_{P'}P' \tag{39}$$

for some nonnegative constants $\alpha_{P'}$ that sum up to one. This condition can be formulated as an LMI feasibility problem and checked efficiently by convex optimization software. By incorporating this technique, the basic switched Riccati iteration is enhanced and summarized in Algorithm 1.

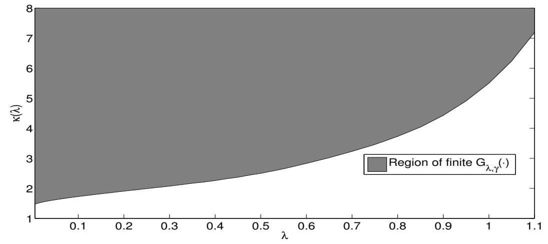
Algorithm 1 (Enhanced Switched Riccati Iteration)

```
$k \leftarrow 0$ and $\mathcal{H}_0 \leftarrow \{I\}$;
repeat
  $k \leftarrow k+1$;
  $\mathcal{H}_k \leftarrow \rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H}_{k-1})$;
  for $P \in \mathcal{H}_k$ do
    if $P$ satisfies condition (39) then
      $\mathcal{H}_k \leftarrow \mathcal{H}_k \setminus \{P\}$;
    end if
  end for
until prescribed stopping criteria are met
Return $\mathcal{H}_k$ and $G_{\lambda,\gamma,k}(z) = \sup_{P\in\mathcal{H}_k} z^TPz$, $\forall\, z \in \mathbb{R}^n$.
```

The computational complexity of Algorithm 1 may still be prohibitive, especially when the state space dimension n and the number of subsystems are large. Inspired by the idea of relaxed dynamic programming [17], complexity can be further reduced (though at the expense of accuracy) by removing those matrices that are *almost* redundant at each iteration. For a given small $\varepsilon>0$, a matrix P in $\mathcal{H}_k$ is called *ε-redundant* ([30, 31]) if its relaxed version, $P-\varepsilon I$, is redundant in $\mathcal{H}_k$. Again, an easy-to-check sufficient condition for ε-redundancy is given by (39) with the matrix P on the left hand side replaced by $P-\varepsilon I$. Denote by $\mathcal{H}_k^\varepsilon$ the remaining set after removing



(a)



(b)

(c)

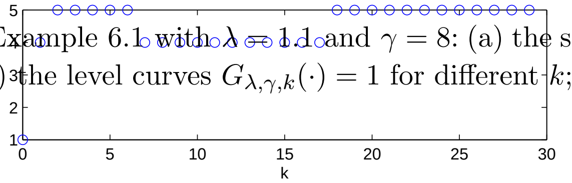


Figure 4: Results of Algorithm 1 on Example 6.1 with $\lambda = 1.1$ and $\gamma = 8$: (a) the size of matrix set $\mathcal{H}_k$ vs. the number of iterations k; (b) the level curves $G_{\lambda,\gamma,k}(\cdot) = 1$ for different k; (c) the DOC Ω (shaded region).

all the ε-redundant matrices from $\mathcal{H}_k$. Then, $\max_{P \in \mathcal{H}_k} z^T P z - \varepsilon\|z\|^2 \le \max_{P \in \mathcal{H}_k^{\varepsilon}} z^T P z \le \max_{P \in \mathcal{H}_k} z^T P z$. Thus, pruning ε-redundant matrices at each iteration leads to an under approximation error of the generating function by at most $\epsilon\|z\|^2$. This yields a new algorithm, i.e., Algorithm 2.

Algorithm 2 (Relaxed Switched Riccati Iteration)

```
$k \leftarrow 0$ and $\widetilde{\mathcal{H}}_0 \leftarrow \{I\}$;
repeat
    $k \leftarrow k+1$;
    $\widetilde{\mathcal{H}}_k \leftarrow \rho_{\lambda,\gamma,\mathcal{M}}\left(\widetilde{\mathcal{H}}_{k-1}\right)$;
    $\widetilde{\mathcal{H}}_k \leftarrow \widetilde{\mathcal{H}}_k^{\varepsilon}$ where $\widetilde{\mathcal{H}}_k^{\varepsilon}$ is obtained by removing $\varepsilon$-redundant matrices from $\widetilde{\mathcal{H}}_k$;
until prescribed stopping criteria are met
Return $\widetilde{\mathcal{H}}_k$ and $\widetilde{G}_{\lambda,\gamma,k}(z) = \sup_{P \in \widetilde{\mathcal{H}}_k} z^T P z, \forall z \in \mathbb{R}^n$.
```

Since errors are introduced at each step and accumulate with the iteration, the functions $\widetilde{G}_{\lambda,\gamma,k}(\cdot)$ returned by Algorithm 2 are in general only under approximations of the true generating functions $G_{\lambda,\gamma,k}(\cdot)$. It is shown in [31] that, by a suitable choice of ε, the relaxation technique can lead to a significant reduction in the size of the matrix sets while maintaining a prescribed accuracy when solving the switched LQR problem. A similar error analysis can be extended to Algorithm 2, but detailed discussions are beyond the scope of the present paper and will be reported in future.

6.2 Numerical Examples

Numerical examples are presented as follows to illustrate the proposed algorithms.

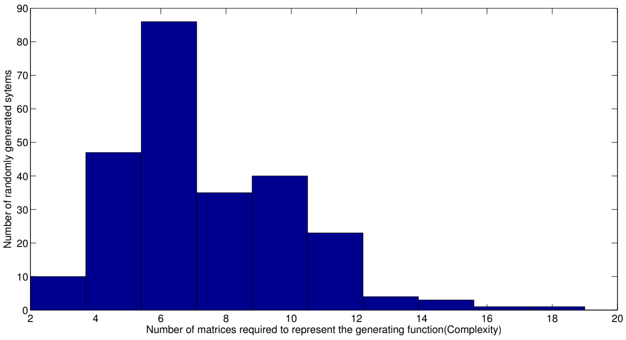


Figure 5: Distribution of $|\mathcal{H}_k|$ in equilibrium for random SLCSs on $\mathbb{R}^3$.

Example 6.1. Consider the following SLCS:

$$A_1 = \begin{bmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}; \qquad A_2 = \begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

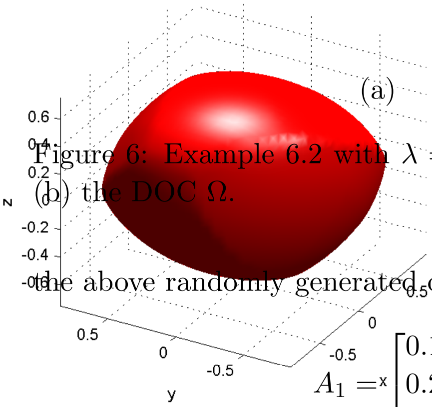
$$A_3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix},\ B_3 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}; \qquad A_4 = \begin{bmatrix} \frac{1}{6} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix},\ B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

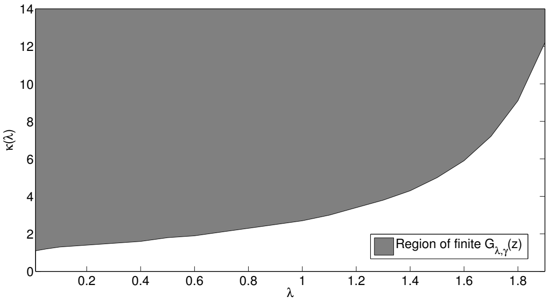
which can be verified to satisfy the reachability assumption. Fig. 4 shows the results of applying Algorithm 1 to compute the generating functions $G_{\lambda,\gamma,k}(\cdot)$ for $\lambda = 1.1$ and $\gamma = 8$ and various k. Although in theory the size of matrix sets $\mathcal{H}_k$ could grow exponentially fast with the maximum size being 4^k, the actual size after removing redundant matrices never exceeds five, as can be seen from Fig. 4(a). This indicates the effectiveness of the complexity reduction technique in Algorithm 1. In Fig. 4(b), we plot the level curves $G_{\lambda,\gamma,k}(\cdot) = 1$ for different k. As k increases, the level curves shrink to an equilibrium curve away from the origin. As a result, $G_{\lambda,\gamma,k}(\cdot)$ is finite and hence $(\lambda, \gamma) \in \Omega$ for $\lambda = 1.1$ and $\gamma = 8$.

By repeating the above procedure for different combinations of λ and γ, the DOC Ω is found and plotted as the shaded region in Figure 4(c). By Theorem 4.1, the generalized ℓ_2-gain $\kappa(\lambda)$ and the radius of convergence $\lambda^*(\gamma)$ can be obtained from the boundary of Ω. For instance, at $\lambda = 1$, the (classical) ℓ_2-gain $\kappa(1)$ of the SLCS is approximately 5.8. By using a bisection type algorithm, a finer estimate of $\kappa(1)$ within arbitrary precision can be further achieved. In comparison, sufficient conditions based approaches (e.g., [3, 16]) provide only conservative estimates of the ℓ_2-gain. $\square$

In more extensive tests, a set of SLCSs on $\mathbb{R}^3$ are randomly generated, each with three stable single-input subsystems. Algorithm 1 is applied to compute the generating functions up to the horizon $k = 75$ for $\lambda = 0.75, \gamma = 15$. Only those SLCSs for which the computation converges at $k = 75$ are kept, and a total of 250 such SLCSs are generated. Fig. 5 displays the number of matrices required to characterize the generating function $G_{\lambda,\gamma,75}(\cdot)$. We observe that a maximum of 19 matrices are needed among the 250 cases while a majority of the cases require fewer than 8 matrices. Running on an Intel Core2Duo desktop and using SeDuMi, it takes typically 3–15 minutes to compute $k = 75$ iterations of the generating functions for a randomly generated SLCS.

Example 6.2. To demonstrate the effectiveness of relaxation, consider the following SLCS from





(a) (b)

Figure 6: Example 6.2 with $\lambda = 0.75$ and $\gamma = 15$: (a) the level surface $\{z \in \mathbb{R}^3 \,|\, G_{\lambda,\gamma,75}(z) = 1\}$; (b) the DOC Ω.

the above randomly generated ones with the highest complexity:

$$A_1 = \begin{bmatrix} 0.1515 & 0.2351 & 0.3763 \\ 0.2696 & 0.3257 & 0.1295 \\ 0.0822 & 0.2374 & 0.2100 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.9981 \\ 0.1132 \\ 0.3316 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 0.1719 & 0.1846 & 0.1186 \\ 0.2420 & 0.2792 & 0.3645 \\ 0.0066 & 0.3453 & 0.1936 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.6511 \\ 0.2015 \\ 0.7880 \end{bmatrix};$$

$$A_3 = \begin{bmatrix} 0.3249 & 0.0105 & 0.1955 \\ 0.0499 & 0.1833 & 0.3756 \\ 0.2664 & 0.0009 & 0.4905 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0.2872 \\ 0.0415 \\ 0.6339 \end{bmatrix}.$$

For $\lambda = 0.75$ and $\gamma = 15$, Algorithm 1 requires 19 matrices for representing the generation function after $k = 75$ iterations, and the level curve $\{z : G_{\lambda,\gamma,75}(z) = 1\}$ is plotted in Fig. 6(a). In comparison, Algorithm 2 with the relaxation parameter $\varepsilon = 10^{-3}$ requires only 14 matrices. This number is further reduced to 9 if $\varepsilon = 10^{-2}$. In both the cases, the maximum error incurred due to relaxation is less than 10^{-3} on the unit sphere in $\mathbb{R}^3$. The DOC of the SLCS is shown in Fig. 6(b). $\square$

7 Conclusion

In this paper, a generalized input-to-state ℓ_2-gain is introduced for the discrete-time SLCS, and a generating function based approach is proposed for analysis and computation of the generalized ℓ_2-gain. It is shown that the radii of convergence of the generating function characterize the ℓ_2-gain, and effective algorithms are developed for computing the generating function and the ℓ_2-gain with proven convergence.

Many interesting issues of the ℓ_2-gain of the SLCS remain unsolved and call for further research; examples include analysis and computation of the ℓ_2-gain subject to general system parameter variations, extended notions and properties of the ℓ_2-gain under different switching rules, and input-output ℓ_2-gain. The generating function approach proposed in this paper has shed lights on these issues, which will be addressed in detail in the future.

8 Appendix: Proof of Proposition 5.1

Proof. Case (1): If $a = 0$, then the conclusions follow directly from the fact that (33) is reduced to

$$g_{k+1} = F_{a,b}(g_k) = \begin{cases} 1 & \text{if } \gamma^2 \geq b^2 g_k \\ \infty & \text{if } \gamma^2 < b^2 g_k. \end{cases}$$

Case (2): If $a \neq 0$ and $b = 0$, then (33) implies $g_{k+1} = F_{a,b}(g_k) = 1 + a^2\lambda g_k$, which converges to a finite value $1/(1 - a^2\lambda)$ if and only if $a^2\lambda < 1$.

Case (3): If $a \neq 0$ and $b \neq 0$, then (33) can be written as

$$g_{k+1} = F_{a,b}(g_k) = \begin{cases} 1 + \frac{a^2\lambda\gamma^2 g_k}{\gamma^2 - b^2 g_k} & \text{if } g_k < \frac{\gamma^2}{b^2} \\ \infty & \text{if } g_k \geq \frac{\gamma^2}{b^2}. \end{cases} \tag{40}$$

For the conclusion on Ω_+, we only need to show that the iteration (40) converges if and only if

$$\gamma > |b| \quad \text{and} \quad 0 < \lambda \leq (\gamma - |b|)^2/(a^2\gamma^2). \tag{41}$$

For the necessity of (41), suppose (g_k) converges to $g_\infty < \infty$. Then we must have $\gamma > |b|$ for otherwise $g_1 = \infty$. In addition, g_∞ must be a positive solution to the equation

$$g_\infty = 1 + \frac{a^2\lambda\gamma^2 g_\infty}{\gamma^2 - b^2 g_\infty} \quad \Longleftrightarrow \quad b^2 g_\infty^2 - (b^2 + \gamma^2 - a^2\lambda\gamma^2) g_\infty + \gamma^2 = 0. \tag{42}$$

This is possible only if $b^2 + \gamma^2 - a^2\lambda\gamma^2 \geq 0$ and $(b^2 + \gamma^2 - a^2\lambda\gamma^2)^2 \geq 4b^2\gamma^2$. These imply $a^2\lambda\gamma^2 \leq (\gamma - |b|)^2$, i.e., the second part of condition (41). (Here $\lambda > 0$ is by assumption.)

To show the sufficiency of condition (41), we first prove the following claim under (41) by induction: the monotone sequence (g_k) generated by the iteration (40) is bounded, i.e., $g_k \leq \gamma/|b|$, $\forall\, k$. Clearly, the claim is trivially true for $k = 0$ as $g_0 = 1$ and $\gamma/|b| > 1$ by (41). Suppose it holds for some $k \in \mathbb{Z}_+$, i.e., $g_k \leq \gamma/|b|$. Then it follows from (41) that $(\gamma - |b|)\gamma \geq (\gamma - |b|)|b| + a^2\lambda\gamma^2$ implies $\left(\frac{\gamma}{|b|} - 1\right)\gamma^2 \geq \left[\left(\frac{\gamma}{|b|} - 1\right) b^2 + a^2\lambda\gamma^2\right] g_k$, which further implies $\frac{\gamma}{|b|} \geq 1 + \frac{a^2\lambda\gamma^2 g_k}{\gamma^2 - b^2 g_k} = g_{k+1}$. Note that in the last step, we use the fact that $\gamma^2 - b^2 g_k > 0$, which follows from $g_k \leq \gamma/|b| < \gamma^2/b^2$. This proves the claim. As a result, (g_k) converges (to some $g_\infty < r/|b|$). Finally, suppose (λ, γ) satisfies condition (41). Then the limit g_∞ (i.e., $g_{\lambda,\gamma}$) of (g_k) is the smallest positive root of equation (42), which is exactly given by (35). ☐

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