Efficient Computation of Generalized Input-to-State $\mathcal{L}_2$-Gains of Discrete-Time Switched Linear Systems

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ABSTRACT

This paper proposes an efficient way to compute the $\mathcal{L}_2$-gain of discrete-time switched linear systems. Using the notion of generating functions, generalized versions of $\mathcal{L}_2$-gains under arbitrary switching are studied. An efficient numerical algorithm is formulated by which these generalized $\mathcal{L}_2$-gains can be estimated. The proposed method mitigates the problem of conservative bounds. Numerical examples are provided to illustrate the algorithm.

Categories and Subject Descriptors

F.2.0 [**Analysis of Algorithms and Problem Complexity**]: General; I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search—*dynamic programming, control theory*

General Terms

Algorithms, performance, theory

Keywords

L2-gains, switched linear systems, input-to-state stability, generating functions

1. INTRODUCTION

Switched linear systems form an important class of hybrid systems and are being used to model a diverse range of engineering systems [11]. Stability aspects of switched linear systems have received a lot of attention in the past decade. The survey papers [15, 12] provide a review of the results on this subject.

Estimating the input-to-state (or output) gains for switched linear systems has long been recognized as an open problem [5, 6]. Recent research in this area has focused on obtaining bounds for the $\mathcal{L}_2$-gains of the switched linear systems under various switching conditions. The analysis of $\mathcal{L}_2$-gain under slow switching was reported in [4], while [16] used an average dwell time condition. Using common storage functions, it was proved in [7, 8] that the solutions to the $\mathcal{L}_2$-gain problem of continuous-time switched linear systems can be characterized using a finite parametrization. A multiple Lyapunov function approach was used to study $\mathcal{L}_2$-gains of general switched systems in [18]. The paper [13] addressed the $\mathcal{L}_2$-gain problem by characterizing the most destabilizing switching law, leading to a sufficient condition for bounding the $\mathcal{L}_2$-gain of first order SISO systems. More recently, the variation of $\mathcal{L}_2$-gain of discrete-time switched linear systems with dwell time was studied in [1, 2]. LMI based sufficiency conditions were proposed to bound the $\mathcal{L}_2$-gains under dwell time constraints (extensible to arbitrary switching). Less conservative LMI conditions are proposed in [10], though the worst case complexity increases rapidly with the number of subsystems.

The present paper focuses on estimating the $\mathcal{L}_2$-gain numerically using an efficient algorithm. Most of the existing results tend to be either conservative or computationally expensive to verify. Using the newly introduced notion of generating functions [14, 9], we can derive necessary and sufficient conditions to characterize the input-to-state $\mathcal{L}_2$-gains of discrete-time switched linear systems under arbitrary switching. Thus an efficient iterative algorithm (based on [17]) for computing the generating functions enables us to compute bounds on the input-to-state $\mathcal{L}_2$-gains. By having a necessary and sufficient condition, the input-to-state $\mathcal{L}_2$-gains can be estimated to a desired precision while avoiding conservative bounds. An added advantage of this approach is that it enables us to study a more general version of the $\mathcal{L}_2$-gain where the input and the state energy are weighted

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by an exponential discount factor. This allows us to characterize both the trajectory growth rates and energy amplification using a single metric.

The paper is organized as follows. We describe the problem under consideration and present some preliminary results in Section 2. The notion and properties of controlled generating functions are described briefly in Section 3. Section 4 contains the main results of this paper where we formulate an iterative algorithm for the computation of the generalized $\mathcal{L}_2$-gains. A relaxed version of the algorithm that enables faster computations at the cost of slight inaccuracy is also discussed. Some numerical examples are included in Section 5 for illustrating the algorithm. Finally concluding remarks are made in Section 6.

2. PRELIMINARIES

Throughout the paper we consider discrete-time controlled switched linear systems (SLSs) with dynamics given by

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t = 0, 1, \ldots. \tag{1}$$

Here ($A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}$) are the state and input matrices indexed by $i \in \mathcal{M} := \{1, \ldots, M\}$. $\sigma(t) \in \mathcal{M}$ represents the switching law that determines the unique dynamics observed at any time t. The state and the input control input are denoted by $x(t)$ and $u(t)$ respectively.For simplicity, we often use u to denote the control input sequence $\{u(t)\}_{t=0,1,\ldots}$, and σ the switching sequence $\{\sigma(t)\}_{t=0,1,\ldots}$. We also assume that at least one of the B_i is nonzero.

Denote by $x(t; \sigma, z, u)$ the state trajectory of the controlled SLS (1) starting from the initial state $x(0) = z$ under the switching sequence σ and the control input u. For a fixed σ, system (1) becomes a linear time-varying system, whose solution $x(t; \sigma, z, u)$ is jointly linear in z and u. The reachable set $\mathcal{R}$ is defined as the set of all states that can be reached within a finite time starting from a zero initial state under arbitrary control laws. In the present paper we assume that the reachable set of SLS (1) is a subspace over $\mathbb{R}^n$ which in general might not always be the case [3].

The dynamics of the corresponding autonomous SLS are given by

$$x(t+1) = A_{\sigma(t)}x(t), \quad t = 0, 1, \ldots. \tag{2}$$

Denote by $x(t; \sigma, z)$ the solution to (2) starting from $x(0) = z$ under the switching sequence σ. Then $x(t; \sigma, z)$ is exactly the solution $x(t; \sigma, z, u)$ to the controlled SLS (1) with $u = 0$.

2.1 Generalized $\mathcal{L}_2$-Gain

We are concerned with the estimation of the following generalized input-to-state gains (introduced in [14]).

$$[\kappa(\lambda)]^2 := \sup_{\sigma} \sup_{0\neq u\in\mathcal{U}_c} \frac{\sum_{t=0}^{\infty} \lambda^t \|x(t+1; \sigma, 0, u)\|^2}{\sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2}, \tag{3}$$

where $\lambda \in \mathbb{R}_+ := [0, \infty)$ is a discount factor, and $\mathcal{U}_c$ is the space of all u with finite duration (identically zero after a finite time). The classical definition of $\mathcal{L}_2$-gain (denoted by κ) is obtained by setting $\lambda = 1$ in (3).

The above definition of the generalized $\mathcal{L}_2$-gain captures both the worst-case energy amplification and the worst-case trajectory growth rates. A finite generalized $\mathcal{L}_2$-gain not only implies convergence of the state trajectories but also bounds their rate of decay. In this sense, the generalized $\mathcal{L}_2$-gain can be used as a single metric representing information of two important factors in SLS stability. Furthermore the generalized $\mathcal{L}_2$-gain of a SLS can also be viewed as the classical $\mathcal{L}_2$-gain of a scaled version of the SLS. To see this consider the following SLS obtained by scaling (1):

$$\tilde{x}(k+1) = \tilde{A}_{\sigma(k)}\tilde{x}(k) + \tilde{B}_{\sigma(k)}\tilde{u}(k),$$

where $\tilde{A}_i = \sqrt{\lambda} \cdot A_i, \tilde{B}_i = \sqrt{\lambda} \cdot B_i$, with the same initial condition $\tilde{x}(0) = z$. Applying the transformed control input law $\tilde{u}(k) = \sqrt{\lambda^k} \cdot u(k)$ results in a transformed state trajectory $\tilde{x}(k) = \sqrt{\lambda^k} \cdot x(k)$. Writing the definition of the classical gain for the scaled SLS, we immediately obtain the generalized $\mathcal{L}_2$-gain of the original SLS.

2.2 Properties

We report some relevant properties of the generalized $\mathcal{L}_2$-gain. The proofs can be found in [14].

Proposition 1. *The $\mathcal{L}_2$-gain $\kappa(\lambda)$ as a function of $\lambda \in \mathbb{R}_+$ has the following properties:*

1. *At $\lambda = 0$, $\kappa(0) = \max_{i\in\mathcal{M}} \sigma_{\max}(B_i)$, where $\sigma_{\max}(B_i)$ denotes the largest singular value of B_i;*
2. *$\kappa(\lambda)$ is a lower semi-continuous function in $\lambda \in \mathbb{R}_+$.*
3. *$\kappa(\lambda)$ is a non-decreasing function in λ.*

3. GENERATING FUNCTIONS

In [14], the concept of controlled generating functions is introduced and some of their useful properties are derived. For each $\lambda, \gamma \in \mathbb{R}_+$, the strong generating function $G_{\lambda,\gamma} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ of the SLS (1) is defined as

$$G_{\lambda,\gamma}(z) := \sup_{\sigma,u\in\mathcal{U}_c} \left[\sum_{t=0}^{\infty} \lambda^t \|x(t; \sigma, z, u)\|^2 - \gamma^2\lambda \sum_{t=0}^{\infty} \lambda^t \|u(t)\|^2 \right] \tag{4}$$

$$= \|z\|^2 + \lambda \cdot \sup_{\sigma,u\in\mathcal{U}_c} \sum_{t=0}^{\infty} \lambda^t \left[\|x(t+1; \sigma, z, u)\|^2 - \gamma^2\|u(t)\|^2\right] \tag{5}$$

for $\lambda, \gamma \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$. This definition allows the choice of the hybrid control law (σ, u) to excite the largest state energy with limited input energy. A finite horizon version can also be defined as follows,

$$G_{\lambda,\gamma,k}(z) := \sup_{\sigma,u\in\mathcal{U}_k} \left[\sum_{t=0}^{k} \lambda^t \|x(t; \sigma, z, u)\|^2 - \gamma^2\lambda \sum_{t=0}^{k-1} \lambda^t \|u(t)\|^2 \right] \tag{6}$$

$$= \|z\|^2 + \lambda \cdot \sup_{\sigma,u\in\mathcal{U}_k} \sum_{t=0}^{k-1} \lambda^t \left[\|x(t+1; \sigma, z, u)\|^2 - \gamma^2\|u(t)\|^2\right]. \tag{7}$$

The properties of strong generating functions have been investigated and reported in [14]. We present some relevant ones here without proofs.

Proposition 2. *For any $\lambda, \gamma \in \mathbb{R}_+$, the strong generating function $G_{\lambda,\gamma}(\cdot)$ and its k-horizon version $G_{\lambda,\gamma,k}(\cdot)$ for any $k \in \mathbb{N}$ have the following properties*

1. (**Homogeneity**): $G_{\lambda,\gamma}(\cdot)$ and $G_{\lambda,\gamma,k}(\cdot)$ are both homogeneous of degree two, i.e., for any nonzero $\alpha \in \mathbb{R}$, $G_{\lambda,\gamma}(\alpha z) = \alpha^2 G_{\lambda,\gamma}(z)$ and $G_{\lambda,\gamma,k}(\alpha z) = \alpha^2 G_{\lambda,\gamma,k}(z)$, $\forall z \in \mathbb{R}^n$. Thus, $G_{\lambda,\gamma}(0) \in \{0, \infty\}$.

2. (**Bellman Equation**): For all $z \in \mathbb{R}^n$,
$$G_{\lambda,\gamma,k+1}(z) = \|z\|^2 + \lambda \cdot \sup_{i \in \mathcal{M}, v \in \mathbb{R}^m} \big[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v)\big],$$
$$G_{\lambda,\gamma}(z) = \|z\|^2 + \lambda \cdot \sup_{i \in \mathcal{M}, v \in \mathbb{R}^m} \big[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma}(A_i z + B_i v)\big]. \quad (8)$$

3. (**Monotonicity**): For any $z \in \mathbb{R}^n$, $G_{\lambda,\gamma}(z)$ and $G_{\lambda,\gamma,k}(z)$ are non-increasing in $\gamma \in \mathbb{R}_+$ (for fixed λ); and non-decreasing in $\lambda \in \mathbb{R}_+$ (for fixed γ).

4. (**Convergence**) $G_{\lambda,\gamma,k}(z) \uparrow G_{\lambda,\gamma}(z)$ as $k \to \infty$.

The idea of defining trajectory dependent power series called generating functions is adopted from [9], where exponential stability of autonomous SLS was characterized using autonomous generating functions $G_\lambda : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ defined as

$$G_\lambda(z) := \sup_\sigma \sum_{t=0}^{\infty} \lambda^t \|x(t; \sigma, z)\|^2, \quad \forall z \in \mathbb{R}^n. \qquad (9)$$

3.1 Radius of Convergence

The monotonicity of $G_{\lambda,\gamma}(z)$ enables us to define a radius of convergence for the generating function. For $\gamma \in \mathbb{R}_+$, the *radius of convergence* of the generating function $G_{\lambda,\gamma}(z)$ (on $\mathbb{R}^n$) is defined as

$$\lambda^*(\gamma) := \sup\{\lambda \,|\, G_{\lambda,\gamma}(z) < \infty, \ \forall z \in \mathbb{R}^n\}.$$

More generally, for the reachable subspace $\mathcal{R}$ of $\mathbb{R}^n$ the radius of convergence of $G_{\lambda,\gamma}(z)$ on $\mathcal{R}$ is defined as

$$\lambda^*_{\mathcal{R}}(\gamma) := \sup\{\lambda \,|\, G_{\lambda,\gamma}(z) < \infty, \ \forall z \in \mathcal{R}\}.$$

Note that $\mathcal{R}$ is always invariant under the subsystem dynamics and is assumed to be a subspace. This assumption on the reachable set is not too restrictive since reachability is a generic property—any randomly generated SLS is reachable with probability one.

Proposition 3. *The radius of convergence $\lambda^*_{\mathcal{R}}(\gamma)$ of the generating function on the reachable subspace $\mathcal{R}$ as a function of $\gamma \in \mathbb{R}_+$ has the following properties.*

1. $\lambda^*_{\mathcal{R}}(\gamma) \equiv 0$ *for* $0 \le \gamma < \max_{i \in \mathcal{M}} \sigma_{\max}(B_i)$;
2. $\lambda^*_{\mathcal{R}}(\gamma)$ *is a non-decreasing function of* γ *for* $\gamma \ge \max_{i \in \mathcal{M}} \sigma_{\max}(B_i)$;

We now show how the $\mathcal{L}_2$-gain can be characterized by the radius of convergence of $G_{\lambda,\gamma}(z)$ on $\mathcal{R}$. As we will observe, this result provides a necessary and sufficient condition for the $\mathcal{L}_2$-gain to lie below a given value based on the convergence of the generating function. Ultimately we can utilize this result for bounding the $\mathcal{L}_2$-gain provided the generating function can be efficiently tested for convergence. This result and its proof have been reported in [14].

Proposition 4. (***$\mathcal{L}_2$-gain characterization***) *For $\lambda > 0$ and $\gamma \in \mathbb{R}_+$, the following statements are equivalent:*

1. $\kappa(\lambda) \le \gamma$, *where $\kappa(\lambda)$ is the generalized $\mathcal{L}_2$-gain defined in (3);*
2. $\lambda \le \lambda^*_{\mathcal{R}}(\gamma)$, *where $\mathcal{R}$ is the reachable subspace of the SLS (1) from the origin.*
3. $G_{\lambda,\gamma}(\cdot) < \infty$

In other words, the generalized $\mathcal{L}_2$-gain $\kappa(\lambda)$ and the radius of convergence $\lambda^*_{\mathcal{R}}(\gamma)$ are (generalized) inverse functions of each other. This property implies that the strong generating function $G_{\lambda,\gamma}(\cdot)$ is finite if and only if the value of γ is greater than generalized $\mathcal{L}_2$-gain $\kappa(\lambda)$. Based on the convergence of the strong generating function, it is thus possible to bound $\kappa(\lambda)$ in terms of γ. Since the condition is both necessary and sufficient, one can use a bisection type algorithm to accurately estimate the $\kappa(\lambda)$.

For systems whose reachable set is not a subspace, it is still possible to define a (restricted) generating function and a corresponding radius of convergence by restricting the SLS to a subspace invariant under the subsystem dynamics. Such a subspace would always be contained within the reachable set. The generalized $\mathcal{L}_2$-gain of the restricted SLS would be the worst case weighted energy gain among all the state trajectories starting in the invariant subspace. Using (restricted) generating functions, it is possible to estimate the generalized $\mathcal{L}_2$-gain for the restricted system using similar principles as discussed above.

4. NUMERICAL COMPUTATION

In this section we present methods to compute the generating functions numerically. A numerical method to compute the generating function enables us to use Proposition 4 to estimate the generalized $\mathcal{L}_2$-gains. We first prove that any finite horizon generating functions can be represented by a piecewise quadratic function and utilize this structure to formulate an iterative algorithm for efficient computation.

We begin by observing that the generating function (or k-horizon generating function) can be thought of as an infinite (or finite) horizon performance index that needs to be maximized over all possible switching and control laws. Since the generating function has quadratic terms, this optimization is equivalent to the discrete-time switched linear quadratic regulator (DSLQR) problem. One obvious way of achieving the maximum is through Dynamic Programming (DP). The Bellman equation in Proposition 2 gives the value iteration procedure for computing the strong generating function through DP. However the lack of an analytic closed form for the generating function necessitates numerical methods for finding the supremum at each step. This introduces inaccuracies and limits the computational efficiency of this procedure. Thus a more efficient method for computing the strong generating function is needed.

It has been shown in [17] that the finite horizon value function of a DSLQR problem is piecewise quadratic and can be completely characterized by a finite number of positive semidefinite (p.s.d) matrices. An efficient algorithm for computing the value function based on this characterization was also discussed. We now restate the relevant results here in notation consistent with our framework.

Denote by $\mathcal{A}$ the set (convex cone) of all $n \times n$ symmetric positive definite matrices. For the SLS given in (1), the Riccati Mapping $\rho_{\lambda,\gamma,i}(P) : \mathcal{A} \to \mathcal{A}$ of subsystem $i \in \mathcal{M}$, with $\lambda > 0$ and sufficiently large $\gamma > 0$, is defined as

$$\rho_{\lambda,\gamma,i}(P) := I + \lambda \cdot A_i^T P A_i + \lambda \cdot A_i^T P B_i \left(\gamma^2 I - B_i^T P B_i\right)^{-1} B_i^T P A_i. \quad (10)$$

For any subset $\mathcal{H}$ of $\mathcal{A}$, the Switched Riccati Mapping $\rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H})$ is defined by

$$\rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H}) = \{\rho_{\lambda,\gamma,i}(P) : \text{for some } i \in \mathcal{M} \text{ and } P \in \mathcal{H}\}.$$

Starting with the initial condition $\mathcal{H}_0 = \{I\}$, we generate a sequence of sets through the iteration $\mathcal{H}_{k+1} = \rho_{\lambda,\gamma,\mathcal{M}}(\mathcal{H}_k)$. These sets are called the Switched Riccati Sets (SRSs) and have the property of completely characterizing $G_{\lambda,\gamma,k}(\cdot)$ as follows.

PROPOSITION 5. *For all $k \in \mathbb{N} \cup \{0\}$,*

$$G_{\lambda,\gamma,k}(z) = \max_{P \in \mathcal{H}_k} z^T P z. \quad (11)$$

PROOF. The proposition can be proved by induction. For $k = 0$ the statement (11) holds as

$$G_{\lambda,\gamma,0} = \|z\|^2 = \max_{P \in \{I\}} z^T P z.$$

If (11) holds for some $k \geq 0$, then from the Bellman's equation in Proposition 2,

$$\begin{aligned} G_{\lambda,\gamma,k+1}(z) &= \|z\|^2 + \lambda \sup_{i \in \mathcal{M}, v \in \mathbb{R}^n} \left[-\gamma^2 \|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v)\right], \\ &= z^T z + \lambda \sup_{i \in \mathcal{M}, v \in \mathbb{R}^n, P \in \mathcal{H}_k} \Big[-\gamma^2 v^T v \\ &\qquad + (A_i z + B_i v)^T P (A_i z + B_i v)\Big], \\ &= \sup_{i \in \mathcal{M}, P \in \mathcal{H}_k, v \in \mathbb{R}^n} \Big[z^T \left(I + \lambda \cdot A_i^T P A_i\right) z \\ &\qquad + \lambda \cdot v^T \left(-\gamma^2 I + B_i^T P B_i\right) v \\ &\qquad + 2\lambda \cdot v^T B_i^T P A_i z\Big]. \end{aligned}$$

The expression inside the brackets is quadratic in v. The supremum can be found (provided γ is sufficiently large) as,

$$\begin{aligned} G_{\lambda,\gamma,k+1} &= \sup_{i \in \mathcal{M}, P \in \mathcal{H}_k} z^T \Big(I + \lambda \cdot A_i^T P A_i \\ &\qquad + \lambda \cdot A_i^T P B_i \left[\gamma^2 I - B_i^T P B_i\right]^{-1} B_i^T P A_i\Big) z, \\ &= \max_{i \in \mathcal{M}, P \in \mathcal{H}_{k+1}} z^T P z, \\ &= \max_{P \in \mathcal{H}_{k+1}} z^T P z. \end{aligned}$$

This proves the theorem. □

Remark 1. Proposition 5 states that the finite horizon generating functions can be exactly represented by a finite number of positive definite matrices thus removing the need for gridding the state space during computation. Also by Proposition 2, the strong generating function can be computed as a pointwise limit of piecewise quadratic functions.

Remark 2. Finite parametrization of a common storage function was derived in [8] for bounding the $\mathcal{L}_2$-gain of a continuous-time SLS. Our contribution is to explicitly derive the relation for discrete-time systems and (as we will demonstrate in the Section 4.1) generate an effective iterative algorithm to compute bounds on the $\mathcal{L}_2$-gain.

Estimating generalized $\mathcal{L}_2$-gains from the strong generating function involves checking the convergence of the generating functions. To do this, it might be essential to compute the finite horizon generating function $G_{\lambda,\gamma,k}$ over a large time horizon k. This might be impracticable as the number of matrices in $\mathcal{H}_k$ required to represent $G_{\lambda,\gamma,k}$ increases exponentially with k. Applying the Switched Riccati Mapping to this exponentially increasing family of matrices forms the major computational bottleneck in computing generating functions and thus $\mathcal{L}_2$-gains. A more efficient way of managing both memory requirements and computational time is to prune redundant matrices which do not lead to a supremum at each step, as detailed in [17]. This idea is introduced in the context of generating functions in the next section.

4.1 Algorithm for Computing $G_{\lambda,\gamma}(z)$

We now present an algorithm for computing the strong generating functions using the characterization presented in Proposition 5. The algorithm is a specialization of the general one in [17] to the strong generating functions with paralle development. The key idea of the algorithm is the removal of all those matrices which do not contribute to the maximum in (11) for any $z \in \mathbb{R}^n$. To this end we introduce the idea of algebraic redundant matrices in the present context.

Definition 1. A matrix $\hat{P} \in \mathcal{H} \subset \mathcal{A}$ is called redundant w.r.t H if for any $z \in \mathbb{R}^n$, there exists a matrix $P \in \mathcal{H}$ such that $P \neq \hat{P}$ and $z^T P z \geq z^T \hat{P} z$.

If a matrix $\hat{P}$ is redundant w.r.t a SRS $\mathcal{H}_k$, then the generating function $G_{\lambda,\gamma,k}$ can be represented exactly using the set $\mathcal{H}_k \setminus \{\hat{P}\}$. This implies $\hat{P}$ can be removed without causing any error. Hence to maintain ease of computation, we should remove as many redundant matrices as possible at each step of computing the SRSs. However, testing redundancy is in itself a challenging problem. From a geometric viewpoint, any matrix $\hat{P} \in \mathcal{H}_k \subset \mathcal{A}$ represents a unique ellipsoid $\{z \in \mathbb{R}^n : z^T \hat{P} z \leq 1\}$. A matrix $\hat{P}$ is redundant w.r.t $\mathcal{H}_k$ if and only if its corresponding ellipsoid completely covers the intersection of the ellipsoids for matrices in $\mathcal{H}_k \setminus \{\hat{P}\}$. This leads to an easily verifiable sufficient condition given in the following lemma.

LEMMA 1. *$\hat{P} \in \mathcal{H}_k$ is redundant w.r.t $\mathcal{H}_k$ if there exist non-negative constants $\{\alpha_i\}_{i=1}^{|\mathcal{H}_k|-1}$ such that $\sum_{i=1}^{|\mathcal{H}_k|-1} \alpha_i P_i \succeq \hat{P}$, where $\{P_i\}_{i=1}^{|\mathcal{H}_k|-1}$ is an enumeration of $\mathcal{H}_k \setminus \{\hat{P}\}$.*

PROOF. The proof follows from the fact that $\sum_{i=1}^{|\mathcal{H}_k|-1} \alpha_i P_i$ represents an ellipsoid containing the intersection of all the ellipsoids represented by matrices $\{P_i\}_{i=1}^{|\mathcal{H}_k|-1}$. □

The condition stated in Lemma 1 can be tested using convex optimization techniques. Most redundant matrices can be eliminated this way leading to less computation time. Algorithm 1 summarizes the idea of pruning based on Lemma 1.

ALGORITHM 1. *1. Initialize $k := 0$, $\mathcal{H}_0 = \{I\}$;*

2. *Initialize* $H_{k+1} = \emptyset$

3. *repeat for every* $i \in \mathcal{M}$

   - *repeat for all* $\hat{P} \in \mathcal{H}_k$
   - *Compute* $P_i = \rho_{\lambda,\gamma,i}(\hat{P})$.
   - $\mathcal{H}_{k+1} = \mathcal{H}_{k+1} \cup P_i$.
   - *end repeat*

4. *end repeat*

5. *If any* $P \in \mathcal{H}_{k+1}$ *satisfies the condition of Lemma 1 w.r.t* $\mathcal{H}_{k+1}$, *then* $\mathcal{H}_{k+1} = \mathcal{H}_{k+1} \setminus \{P\}$;

6. *Set* $G_{\lambda,\gamma,k+1}(z) = \max_{P \in \mathcal{H}_{k+1}} z^T P z$;

7. $k := k+1$;

8. *Iterate till* $G_{\lambda,\gamma,k}(\cdot)$ *converges within tolerance (or appears to diverge);*

It is to be noted that the sets $\mathcal{H}_k$ returned in Algorithm 1 contain only the non-redundant matrices from the actual SRS generated by the Riccatti equation. However, they are functionally equivalent to the SRS as they both define the same generating function and hence we denote these sets by the same notation.

Algorithm 1 alleviates the computational burden incurred in applying the Switched Riccati Mapping for an exponentially growing number of matrices at the cost of introducing a Linear Matrix Inequality(LMI) computation(which has a complexity polynomial in the state space dimension) for redundancy check at each step. The algorithm proves most efficient for SLS comprising of a large number of subsystems with a small state-space dimension where the number of redundant matrices is typically large enough to offset the redundancy check complexity. Additionally, it is also possible to prune redundant matrices only when the Switched Riccati Set becomes large enough to warrant it, leading to more efficient computation.

The above idea of redundancy can be further relaxed to reduce computational complexity (number of matrices required to describe the generating function) at the expense of accuracy. We describe such an relaxation algorithm in the next section.

4.2 Relaxation Algorithm for Approximate Computations

We begin by modifying the definition of redundancy to allow for slight errors in characterizing the strong generating functions.

Definition 2. A matrix $\hat{P} \in \mathcal{H}_k$ is called ϵ-redundant with respect to $\mathcal{H}_k$ if $\forall z \in \mathbb{R}^n \exists P \in \mathcal{H}_k \setminus \hat{P}$ such that $z^T P z \geq z^T \hat{P} z - \epsilon \|z\|^2$.

By the same reasoning as in Lemma 1, a sufficient condition for ϵ-redundancy can be given as follows.

LEMMA 2. *$\hat{P} \in \mathcal{H}_k$ is ϵ-redundant w.r.t $\mathcal{H}_k$ if there exist non-negative constants $\{\alpha_i\}_{i=1}^{|\mathcal{H}_k|-1}$ such that $\sum_{i=1}^{|\mathcal{H}_k|-1} \alpha_i P_i + \epsilon \cdot I \succeq \hat{P}$, where $\{P_i\}_{i=1}^{|\mathcal{H}_k|-1}$ is an enumeration of $\mathcal{H}_k \setminus \{\hat{P}\}$ and I is an Identity matrix of appropriate dimension.*

If we denote by $\mathcal{H}^\epsilon$ the set formed by removing all the ϵ-redundant matrices from a set $\mathcal{H}$, the following relationship follows from definition 2.

$$\max_{P \in \mathcal{H}} z^T P z - \epsilon \|z\|^2 \leq \max_{P \in \mathcal{H}^\epsilon} z^T P z \leq \max_{P \in \mathcal{H}} z^T P z. \quad (12)$$

Hence pruning the ϵ-redundant matrices from a SRS $\mathcal{H}_k$ introduces an error of at most $\epsilon \|z\|^2$ in the representation of the generating function.

Lemma 2 can be incorporated into Algorithm 1 to compute relaxed subsets $\mathcal{H}_k^\epsilon$ of the SRS's $\mathcal{H}_k$ iteratively. The sets $\mathcal{H}_k^\epsilon$ can be used to define approximations of the strong generating functions as follows,

$$G^\epsilon_{\lambda,\gamma,k}(z) := \max_{P \in \mathcal{H}_k^\epsilon} z^T P z. \quad (13)$$

Here, $G^\epsilon_{\lambda,\gamma,k}$ is an under approximation of the finite horizon generating function $G_{\lambda,\gamma,k}$ (as $\mathcal{H}_k^\epsilon \subseteq \mathcal{H}_k$). Since errors will be introduced in the representation of the generating functions at each iteration, it is desired that the cumulative effects of these errors are bounded. This will ensure that the numerical value of the infinite horizon generating function will be close to the actual value. The following proposition gives a bound on the error incurred.

PROPOSITION 6. *For all $k \geq 0$ and $z \in \mathbb{R}^n$, the following condition holds:*

$$G^\epsilon_{\lambda,\gamma,k}(z) \geq G_{\lambda,\gamma,k}(z) - \epsilon \sum_{t=0}^{k} \lambda^t \|x(t; \sigma^*_{k,z}, z, u^*_{k,z})\|^2, \quad (14)$$

where $u^*_{k,z} \in \mathcal{U}_k$ and $(\sigma^*_{k,z}, u^*_{k,z})$ is the hybrid control law achieving the maximum in the definition of $G_{\lambda,\gamma,k}(z)$.

PROOF. The proof is by induction. For $k = 0$, $G^\epsilon_{\lambda,\gamma,k}(z) = G_{\lambda,\gamma,k}(z) = \|z\|^2$. Assume the statement is true for some $k \geq 0$. We shall show it is true for $k+1$ as well.

Define $\tilde{G}^\epsilon_{\lambda,\gamma,k+1}(z)$ as follows.

$$\tilde{G}^\epsilon_{\lambda,\gamma,k+1}(z) = \|z\|^2 + \lambda \max_{i,v} \left\{ -\gamma^2 \|v\|^2 + G^\epsilon_{\lambda,\gamma,k}(A_i z + B_i u) \right\}. \quad (15)$$

As in Theorem 5, it can be proved that $\tilde{G}^\epsilon_{\lambda,\gamma,k+1}(z) = \max_{P \in \rho_{\lambda,\gamma,M}(\mathcal{H}_k^\epsilon)} z^T P z$. It follows from (12) that

$$\tilde{G}^\epsilon_{\lambda,\gamma,k+1}(z) - \epsilon \|z\|^2 \leq G^\epsilon_{\lambda,\gamma,k+1}(z) \leq \tilde{G}^\epsilon_{\lambda,\gamma,k+1}(z). \quad (16)$$

From (15) and the induction hypothesis (14) we have,

$$\tilde{G}^\epsilon_{\lambda,\gamma,k+1}(z) \geq \|z\|^2 + \lambda \max_{i,v} \{ -\gamma^2 \|v\|^2 + G_{\lambda,\gamma,k}(A_i z + B_i v) - \epsilon \sum_{t=0}^{k} \lambda^t \|x(t; \sigma^*_{k, A_i z + B_i v}, A_i z + B_i v, u^*_{k, A_i z + B_i v})\|^2 \}. \quad (17)$$

Partition the optimal switching sequence $\sigma^*_{k+1,z} = (\sigma, \sigma')$ and $u^*_{k+1,z} = (u, u')$ with $u' \in \mathcal{U}_k$ and $\sigma \in \mathcal{M}$. Then $x(1; \sigma^*_{k+1,z}, z, u^*_{k+1,z}) = A_\sigma z + B_\sigma u$. Therefore, by the Bellman's principle the k-horizon trajectory starting from $A_\sigma z + B_\sigma u$ will coincide with the last k steps of the $k+1$-trajectory starting from z. This implies

$$\sigma' = \sigma^*_{k, A_\sigma z + B_\sigma u}, \quad u' = u^*_{k, A_\sigma z + B_\sigma u}, \quad \text{and}$$
$$x(t+1; \sigma^*_{k+1,z}, z, u^*_{k+1,z}) = x(t; \sigma', A_\sigma z + B_\sigma u, u').$$

Also, $G_{\lambda,\gamma,k+1}(z) = \|z\|^2 + \lambda\left\{-\gamma^2\|u\|^2 + G_{\lambda,\gamma,k}(A_\sigma z + B_\sigma u)\right\}$ from the optimality of the trajectory (Bellman's Equation). Choosing $i = \sigma$ and $v = u$, we have

$$\begin{aligned}\tilde{G}^{\epsilon}_{\lambda,\gamma,k+1}(z) \geq \|z\|^2 + \lambda\Big\{&-\gamma^2\|u\|^2 + G_{\lambda,\gamma,k}(A_\sigma z + B_\sigma u) \\ &- \epsilon\sum_{t=0}^{k}\lambda^t\|x(t;\sigma', A_\sigma z + B_\sigma u, u')\|^2\Big\}, \\ \geq G_{\lambda,\gamma,k+1}(z) &- \epsilon\sum_{t=1}^{k+1}\|x(t;\sigma^*_{k+1,z}, z, u^*_{k+1,z})\|^2.\end{aligned}$$

Combining with (16), we have

$$G^{\epsilon}_{\lambda,\gamma,k+1}(z) \geq G_{\lambda,\gamma,k+1}(z) - \epsilon\sum_{t=0}^{k+1}\|x(t;\sigma^*_{k+1,z}, z, u^*_{k+1,z})\|^2.$$

Thus the statement holds for $k+1$ as well. □

Thus the error from successive approximations is bounded by a fraction of the state energy of the SLS. If $G_{\lambda,\gamma}(z) < \infty$, then the state energy will also be bounded and hence relaxation can be used to compute the infinite horizon generating function to any desired accuracy by choosing a sufficiently small tolerance ϵ.

5. NUMERICAL EXAMPLES

Example 1. We consider the SLS with the following subsystems.

$$A_1 = \begin{bmatrix}\frac{1}{2} & \frac{2}{5}\\ \frac{1}{3} & \frac{1}{3}\end{bmatrix}, B_1 = \begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}; \qquad A_2 = \begin{bmatrix}\frac{3}{5} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{4}\end{bmatrix}, B_2 = \begin{bmatrix}0\\ 1\end{bmatrix};$$

$$A_3 = \begin{bmatrix}\frac{1}{3} & \frac{1}{2}\\ \frac{1}{3} & \frac{1}{4}\end{bmatrix}, B_3 = \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}; \qquad A_4 = \begin{bmatrix}\frac{1}{6} & \frac{1}{5}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix}, B_4 = \begin{bmatrix}1\\ 1\end{bmatrix}.$$

The dynamics were selected randomly from a set of proper fractions to ensure stability under arbitrary switching and reachability. Algorithm 1 is used to compute the generating function $G_{\lambda,\gamma,k}(\cdot)$ for $\lambda = 1.1$ and $\gamma = 8$. The computations indicate that the matrices required to represent the generating function $G_{\lambda,\gamma,k}$ do not change significantly after $k = 50$ iterations. Hence convergence can be inferred. Also the number of matrices required to describe the generating function remains constant at 5 instead of growing exponentially. Due to the small number of matrices involved, no relaxation was required to manage complexity. The following 5 matrices were sufficient to characterize the generating function.

$$G_{\lambda\gamma,k} = \max\{z^T P_1 z, z^T P_2 z, z^T P_3 z, z^T P_4 z, z^T P_5 z\},$$

where

$$P_1 = \begin{bmatrix}5.6223 & 2.4604\\ 2.4604 & 2.3101\end{bmatrix}, \quad P_2 = \begin{bmatrix}1.7701 & 1.2813\\ 1.2813 & 3.1464\end{bmatrix},$$

$$P_3 = \begin{bmatrix}4.2323 & 2.7597\\ 2.7597 & 3.3596\end{bmatrix}, \quad P_4 = \begin{bmatrix}5.7712 & 2.5317\\ 2.5317 & 2.3610\end{bmatrix},$$

$$P_5 = \begin{bmatrix}2.7774 & 2.1657\\ 2.1657 & 3.6759\end{bmatrix}.$$

Figure 1 depicts the level curves of $G_{\lambda,\gamma,k}(\cdot) = 1$ at various k. Convergence of $G_{\lambda,\gamma,k}(\cdot)$ as $k \to \infty$ is observed. By Proposition 4, we conclude that the strong generating function $G_{\lambda,\gamma}(\cdot)$ is finite everywhere for $\lambda = 1.1$ and $\gamma = 8$.

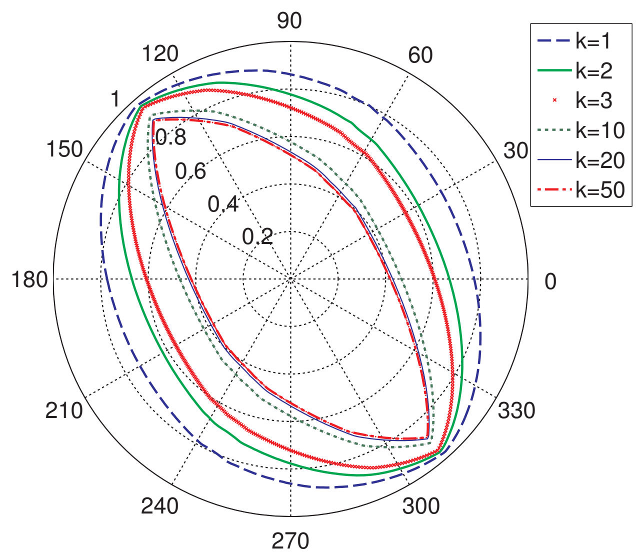

Figure 1: Level curves $G_{\lambda,\gamma,k}(\cdot) = 1$ on the unit circle for $\lambda = 1.1$, $\gamma = 8$ with k varying

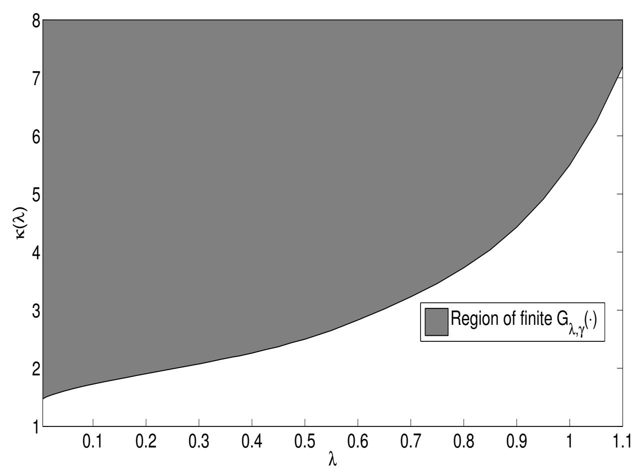

Figure 2: Plot of $\kappa(\lambda)$ vs λ for Example 1

By repeating the computations for different values of λ and γ, we can compute the generalized $\mathcal{L}_2$-gain $\kappa(\lambda)$ as a function of the discount factor λ. See Figure 2 for such a plot. The shaded region represents the region of convergence of the generating function $G_{\lambda,\gamma}(\cdot)$. From Proposition 4, the boundary curve represents the graphs of both the generalize $\mathcal{L}_2$ gain $\kappa(\lambda)$ as a function of λ, and the radius of convergence of the generating function $\lambda^*(\gamma)$ as a function of γ. At $\lambda = 1$, Figure 2 shows that the generalized $\mathcal{L}_2$-gain of the given SLS is less than 5.8. A finer estimate can be obtained using a bisection type algorithm.

Example 2. The following example illustrates the importance of having a necessary and sufficient condition in the computation of the $\mathcal{L}_2$-gain. For the 2-dimensional SLS defined by the following matrices

$$A_1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$
$$A_2 = \begin{bmatrix} 0.5 & 0.6 \\ -0.7 & -1.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

approaches based on sufficient conditions (such as the one in [2]) usually fail or give a conservative result in estimating the $\mathcal{L}_2$-gain. (The problem arises due to the lack of a common quadratic Lyapunov function ensuring asymptotic stability of the SLS in question). Using the convergence of the generating function however, we are able to estimate the (classical) $\mathcal{L}_2$-gain of the SLS as 13.6 ($G_{\lambda=1,\gamma=13.6}(\cdot) < \infty$). This bound can further be improved using a bisection approach.

To demonstrate the effectiveness of the algorithm, it was tested on randomly generated SLS's with 3 stable single input subsystems of in a three dimensional state space. Generating functions were computed upto the horizon of $k = 75$ with $\lambda = 0.75, \gamma = 15$. The computations were run till a significant number (250 in total) of SLS's were found that exhibited convergent behavior for the above parameters. Figure 3 depicts the number of matrices required to completely characterize the generating function. A maximum of 19 matrices were required to do so while a majority of the generating functions could be exactly represented using 6 matrices. It must be noted that these results represent only a fraction of the SLS's with three-dimensional state space as convergence for a given λ and γ is not guaranteed for a randomly generated SLS's. Studying a larger sample size is in general computationally prohibitive. Running on an Intel Core2Duo desktop and using SeDuMi for LMI programming, computation took 3-15 minutes for $k = 75$ iterations of the generating functions for a single randomly generated SLS. Convergence was usually inferred within 30 iterations though more (≈ 100) iterations were required when the value of λ is closer to the radius of convergence.

Example 3. We consider the following three-dimensional example:

$$A_1 = \begin{bmatrix} 0.5 & 0 & -0.7 \\ 0 & 0.3 & 0 \\ 0 & -0.4 & -0.6 \end{bmatrix}, B_1 = \begin{bmatrix} 0.4 \\ 0.9 \\ 0.1 \end{bmatrix};$$
$$A_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.4 & 0.2 & 0.3 \\ 0 & 0 & 0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0.4 \\ 0.8 \\ 0 \end{bmatrix};$$
$$A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0.9 & 0.2 & 0.3 \\ -0.2 & 0.3 & -0.5 \end{bmatrix}, B_3 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

Using algorithm 1, we compute the generating function $G_{\lambda,\gamma,k}$ for $\lambda = 1.1, \gamma = 35$ and $k = 200$. The unit ball of the generating function is shown in figure 4. The generating function is completely characterized by 6-matrices instead of the theoretical 3^{200} matrices.

Example 4. To demonstrate the effect of relaxation, the following SLS is considered. This system was chosen from

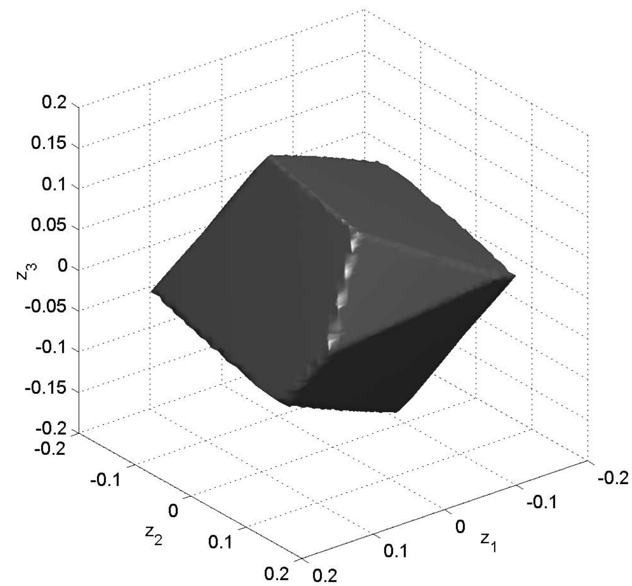

Figure 4: Unit ball of $G_{\lambda,\gamma,k}(z)$ for Example 3

the randomly generated systems used in studying the effectiveness of algorithm 1 mentioned earlier.

$$A_1 = \begin{bmatrix} 0.1515 & 0.2351 & 0.3763 \\ 0.2696 & 0.3257 & 0.1295 \\ 0.0822 & 0.2374 & 0.2100 \end{bmatrix}, B_1 = \begin{bmatrix} 0.9981 \\ 0.1132 \\ 0.3316 \end{bmatrix};$$
$$A_2 = \begin{bmatrix} 0.1719 & 0.1846 & 0.1186 \\ 0.2420 & 0.2792 & 0.3645 \\ 0.0066 & 0.3453 & 0.1936 \end{bmatrix}, B_2 = \begin{bmatrix} 0.6511 \\ 0.2015 \\ 0.7880 \end{bmatrix};$$
$$A_3 = \begin{bmatrix} 0.3249 & 0.0105 & 0.1955 \\ 0.0499 & 0.1833 & 0.3756 \\ 0.2664 & 0.0009 & 0.4905 \end{bmatrix}, B_3 = \begin{bmatrix} 0.2872 \\ 0.0415 \\ 0.6339 \end{bmatrix}.$$

For $\lambda = 0.75, \gamma = 15$ and time horizon $k = 75$, the generating functions were computed using algorithm 1 and its relaxed version. After $k = 75$ iterations, the generating function had exactly 19 non-redundant matrices. Figure 5 shows the unit ball $\{z : G_{\lambda=0.75,\gamma=15,k=75}(z) = 1\}$ obtained from the computation. Using a relaxation parameter of $\epsilon = 10^{-3}$ decreased the number of non-redundant matrices to 14, while using a $\epsilon = 10^{-2}$ cut the number of matrices to 9. In both the relaxed versions, the maximum error incurred due to the approximation was less than 10^{-3} for all initial conditions on the unit ball in $\mathbb{R}^3$. We infer that the computational savings can be significant in higher dimensional systems with a large number of subsystems. The variation of the generalized $\mathcal{L}_2$-gains as a function of the discount factor λ can be seen in Figure 6.

6. CONCLUSION

We were able to derive an efficient algorithm for the estimation of $\mathcal{L}_2$-gains for discrete-time switched linear systems through the computation of the corresponding generating functions. The proposed algorithm can be further relaxed to trade off accuracy with computational complexity. Future directions include deriving tighter bounds on the relaxation

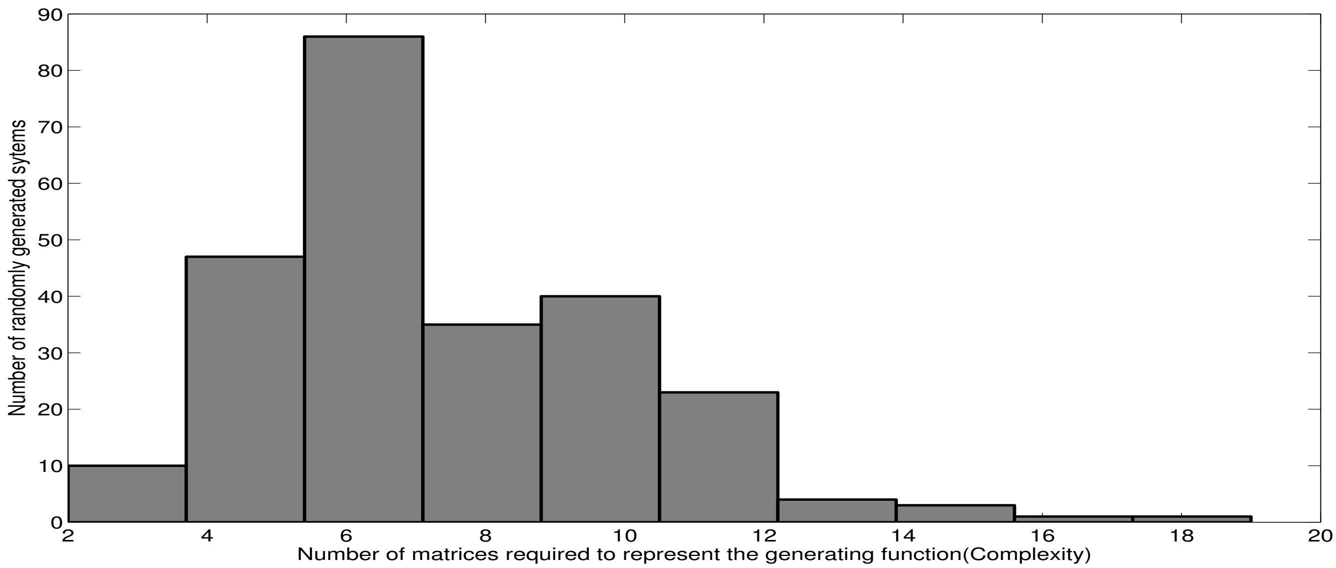

Figure 3: Distribution of non-redundant matrices for random systems

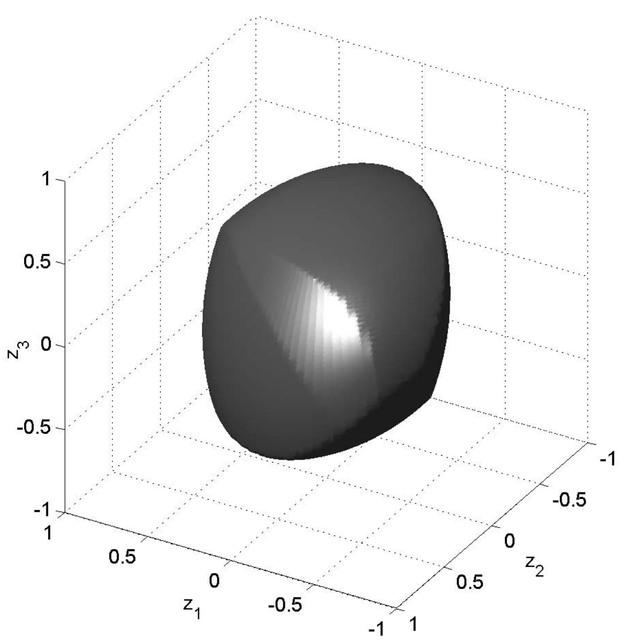

Figure 5: Unit ball of $G_{\lambda,\gamma,k}$ for Example 4

error and investigating $\mathcal{L}_2$-gains under input and switching constraints.

7. ACKNOWLEDGMENTS

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8. REFERENCES

[1] P. Colaneri, P. Bolzern, and J. Geromel. Root mean square gain of discrete-time switched linear systems

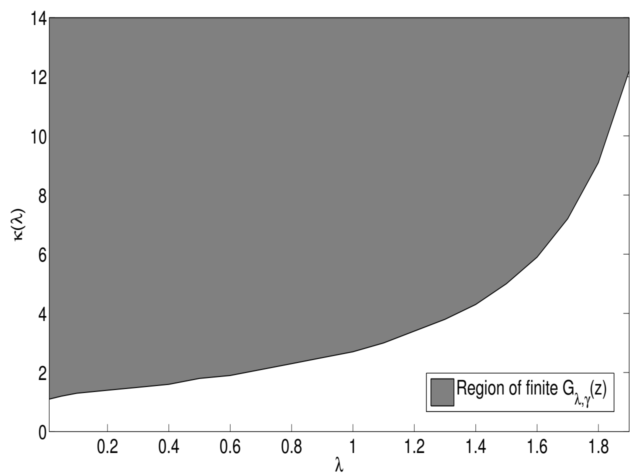

Figure 6: Plot of $\kappa(\lambda)$ vs λ for Example 4

under dwell time constraints. *Automatica*, 47(8):1677 – 1684, 2011.

[2] P. Colaneri and J. Geromel. RMS gain with dwell time for discrete-time switched linear systems. In *Control and Automation, 2008 16th Mediterranean Conference on*, pages 107–112, june 2008.

[3] S. Ge, Z. Sun, and T. Lee. Reachability and controllability of switched linear discrete-time systems. *Automatic Control, IEEE Transactions on*, 46(9):1437–1441, 2002.

[4] J. Hespanha. Root-mean-square gains of switched linear systems. *Automatic Control, IEEE Transactions on*, 48(11):2040–2045, 2003.

[5] J. Hespanha. $\mathcal{L}_2$-induced gains of switched linear systems. In V. Blondel and A. Megretski, editors, *Unsolved problems in mathematical systems and*

control theory, pages 131–133. Princeton University Press, 2004.

[6] J. Hespanha and A. Morse. Input-output gains of switched linear systems. *Open problems in mathematical systems and control theory*, page 121, 1999.

[7] K. Hirata and J. P. Hespanha. $\mathcal{L}_2$-induced gain analysis for a class of switched systems. In *Proceedings of the 48th Conf. on Decision and Control*, Dec. 2009.

[8] K. Hirata and J. P. Hespanha. $\mathcal{L}_2$-induced gain analysis of switched linear systems via finitely parametrized storage functions. In *Proceedings of the 2010 American Control Conference*, June 2010.

[9] J. Hu, J. Shen, and W. Zhang. Generating functions of switched linear systems: analysis, computation, and stability applications. *Automatic Control, IEEE Transactions on*, 56(5):1059–1074, 2011.

[10] J. Lee and G. E. Dullerud. Optimal disturbance attenuation for Discrete-Time switched and markovian jump linear systems. *SIAM Journal on Control and Optimization*, 45:1329, 2006.

[11] D. Liberzon. *Switching in systems and control*. Springer, 2003.

[12] H. Lin and P. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *Automatic Control, IEEE Transactions on*, 54(2):308–322, 2009.

[13] M. Margaliot and J. Hespanha. Root-mean-square gains of switched linear systems: A variational approach. *Automatica*, 44(9):2398–2402, 2008.

[14] V. Putta, G. Zhu, J. Shen, and J. Hu. A study of the generalized input-to-state $\mathcal{L}_2$-gain of discrete-time switched linear systems. In *50th Conf. on Decision and Control*, pages 435–440, Dec. 2011.

[15] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM review*, 49(4):545, 2007.

[16] G. Zhai, B. Hu, K. Yasuda, and A. Michel. Qualitative analysis of discrete-time switched systems. In *Proceedings of the 2002 American Control Conference*, volume 3, pages 1880 – 1885, 2002.

[17] W. Zhang, J. Hu, and A. Abate. On the value functions of the discrete-time switched LQR problem. *Automatic Control, IEEE Transactions on*, 54(11):2669–2674, 2009.

[18] J. Zhao and D. Hill. On stability, $\mathcal{L}_2$-gain and H_∞ control for switched systems. *Automatica*, 44(5):1220–1232, 2008.