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Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.elsevier.com/locate/ejcon

Semistability of switched linear systems with application to PageRank algorithms

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ARTICLE INFO

Article history:

Received 13 June 2013

Received in revised form

16 February 2014

Accepted 8 March 2014

Recommended by Yoshito Ohta

Available online 29 March 2014

Keywords:

Semistability

Switched systems

Stability

Generating function

PageRank algorithm

ABSTRACT

This paper investigates semistability and its computation for discrete-time, switched linear systems under both deterministic and random switching policies. The notion of semistability pertains to a continuum of initial state dependent equilibria, and finds wide applications in multi-agent and distributed network systems. It is shown in this paper that exponential semistability on a common equilibrium space is equivalent to output exponential stability of a reduced switched linear system with a suitably defined output, under arbitrary and random switchings. Besides, their convergence rates are shown to be identical. A generating function based approach is proposed to compute convergence rates of the reduced switched systems under these switching rules. The obtained semistability results are applied to performance analysis of PageRank algorithms for distributed web-page systems subject to topology switching. The iteration processes of these algorithms are formulated as switched linear systems. Their equilibrium properties are studied, and convergence rates are characterized via the semistability techniques and the generating function approach.

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1. Introduction

The notion of semistability extends the regular concept of stability pertaining to a single, isolated equilibrium to a continuum of equilibria of a dynamical system. Roughly speaking, the semistability implies that any system trajectory converges to a (possibly different) stable equilibrium dependent on its initial state. Semistable dynamics have been found in mechanical systems [1,5], network systems [7], biomedical systems [13,15], and chemical kinetics [3,24]. Related semistability results in the literature include [8–11]. In particular, Refs. [10,11] study semistability of continuous-time, switched linear systems (SLSs) under arbitrary switching in the Lyapunov framework. Different from this perspective, we investigate semistability of discrete-time SLSs and characterize growth rates of system trajectories under different switching rules in this paper. Particularly, inspired by various

applications in distributed network systems subject to switching and with a continuum of equilibria, we consider the semistability of the SLSs whose trajectories converge to a common equilibrium space under two switching policies: (deterministic) arbitrary switching and random switching, as well as its applications. The contributions of the paper are summarized as follows.

1. *Semistability analysis and computation:* As a key result of the paper, we show that the original semistability problem can be converted into an equivalent, projection based, output stability problem for a reduced SLS; the semistable growth rate of the original SLS is equivalent to the exponential growth rate of the reduced SLS (cf. [Theorems 2.2](#) and [3.1](#)). This result enables us to apply (regular) stability tools for the SLSs to establish semistability under various switching policies. While the main technique in establishing this result is related to partial stability (of non-switched LTI systems) [25], the switching dynamics of the SLS render the semistability analysis nontrivial and call for new tools to handle them (cf. [Theorem 2.1](#) and [\[6, Theorem 1\]](#)).

To deal with the semistability analysis and computation under different switching rules, we exploit the recently developed generating function approach [6,18,19,21]. Informally speaking, generating functions are certain power series with coefficients determined from systems trajectories under switching rules; their convergence radii characterize growth rates of system trajectories. The generating function approach provides a unified and numerically effective framework to determine and compute stability

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¹ The research of this author is supported by the NSF Grants ECCS-0900960 and DMS-1042916.

² This material is based upon the work supported by the National Science Foundation under Grant no. 1329875.

³ This work was supported by the Defense Threat Reduction Agency, Basic Research Award #HDTRA1-10-1-0090 and Fundamental Research Award #HDTRA1-13-1-0048, to Texas Tech University.

quantities of the SLSs under several switching laws, e.g., arbitrary switching, optimal switching, and random switching [6]. This approach is used to evaluate semistability and compute convergence rates under different switchings in this paper.

2. *Applications:* As a major motivation and application of the semistability analysis, algorithm design and performance analysis of distributed network systems subject to topology switching is carried out in the current paper. Specifically, we consider PageRank algorithms for switched networks of websites. Such an algorithm is defined by an iteration processes treated as a discrete-time linear system. Furthermore, network topology of this system is often subject to switch, due to either graph topology variations or web page connection changes. Hence, the iteration process becomes a discrete-time SLS. For the PageRank application, we propose two types of SLSs that characterize the iteration dynamics of PageRank algorithms. The equilibria sets of these SLSs are studied. It is shown that while such an equilibrium set may be discrete, the semistability technique remains effective to compute convergence rates of these SLSs under suitable choice of system parameters.

The paper is organized as follows. In Sections 2 and 3, we introduce semistability and output stability notions under arbitrary switching and random switching, and we establish their equivalence. Section 4 discusses the generating function approach for computing convergence radii and system growth rates. The application to PageRank algorithms is presented in Section 5. Finally, the conclusion is drawn in Section 6. The preliminary version of this paper is reported in [20] without detailed discussions on the PageRank application presented in the current paper.

2. Semistability of switched linear systems: deterministic case

A discrete-time, autonomous switched linear system (SLS) on \mathbb{R}^n is

$$x(t+1) = A_{\sigma(t)}x(t), \quad t = 0, 1, \dots, \quad (1)$$

where its state $x(t) \in \mathbb{R}^n$ evolves by switching among a finite family of linear dynamics indexed by the index set $\mathcal{M} := \{1, \dots, M\}$, $\sigma(t) \in \mathcal{M}$ for all t , or simply σ , is the switching sequence, and $A_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{M}$, are the subsystem matrices. Denoted by $x(t; z, \sigma)$ the state trajectory of the SLS from the initial state $x(0) = z$ under the switching sequence σ . In this paper, the vector norm $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n , and the matrix norm is induced from it.

In this paper, we will focus on a semistable SLS whose state trajectories converge to equilibria (dependent on initial states and switching sequences). To characterize these equilibria, we present some technical results first. Given a switching sequence σ , define the *asymptotic index set* associated with σ :

$$\mathcal{I}_\sigma := \{i \in \mathcal{M} : \text{for any } t \in \mathbb{Z}_+, \text{ there exists } t' \geq t \text{ such that } \sigma(t') = i\}. \quad (2)$$

Since the set \mathcal{M} is finite, \mathcal{I}_σ is nonempty for any σ . Further, if $i \in \mathcal{I}_\sigma$, then σ has a constant index subsequence $\{i, \dots, i, \dots\}$. Let I_n be the $n \times n$ identity matrix, and $\mathcal{N}(\cdot)$ denote the null space of a matrix. Using this notion, we obtain a necessary condition for the semistable convergence of the SLS.

Lemma 2.1. *Given an initial state $z \in \mathbb{R}^n$ and a switching sequence σ . Suppose that $x(t; z, \sigma)$ converges to $p_* \in \mathbb{R}^n$ as $t \rightarrow \infty$. Then $p_* \in \bigcap_{i \in \mathcal{I}_\sigma} \mathcal{N}(A_i - I_n)$.*

Proof. For any fixed $i \in \mathcal{I}_\sigma$, there exists a subsequence $\{\sigma(t_k)\}$ such that $\{\sigma(t_k)\} = \{i, \dots, i, \dots\}$. Since $x(t; z, \sigma)$ converges to p_* , so do $\{x(t_k; z, \sigma)\}$ and $\{x(t_k + 1; z, \sigma)\}$. In view of $x(t_k + 1; z, \sigma) = A_i x(t_k; z, \sigma)$ for all t_k and passing the limit, we have $p_* = A_i p_*$. This shows $p_* \in \mathcal{N}(A_i - I_n)$. \square

An immediate consequence of the above lemma is that if any state trajectory of the SLS is convergent under arbitrary switching, i.e., for any nonzero $z \in \mathbb{R}^n$ and any σ , $x(t; z, \sigma)$ converges to some nonzero $p_*(z, \sigma)$ (dependent on z and σ), then $\bigcap_{i \in \mathcal{M}} \mathcal{N}(A_i - I_n)$ must be nontrivial. Define the common equilibrium subspace $\mathcal{E}_e := \bigcap_{i \in \mathcal{M}} \mathcal{N}(A_i - I_n)$ and let \mathcal{E}_e^\perp be its orthogonal complement in \mathbb{R}^n . Obviously, \mathcal{E}_e is invariant under $\{A_i\}_{i \in \mathcal{M}}$. The following standing assumption asserts that both \mathcal{E}_e and \mathcal{E}_e^\perp are nontrivial subspaces of \mathbb{R}^n .

Assumption 2.1. The dimension and codimension of \mathcal{E}_e are both at least one.

It follows from the above assumption that there exists at least one $i \in \mathcal{M}$ such that $A_i \neq I_n$.

Definition 2.1. The SLS (1) is *exponentially semistable* under arbitrary switching if there exist constants $\rho \geq 0$ and $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$ and under any switching sequence σ , there exists a (unique) $x_e(z, \sigma) \in \mathcal{E}_e$ (dependent on z and σ) such that $\|x(t; z, \sigma) - x_e(z, \sigma)\| \leq \rho r^t \|z - x_e(z, \sigma)\|$, $\forall t \in \mathbb{Z}_+$. Here, r is called the *exponential growth rate of semistability*.

It is known that the asymptotic and exponential stabilities of the SLSs under arbitrary switching are equivalent [18]. Next, we show that the same holds for semistability and exponential semistability of the SLSs. We first introduce the following definition of semistability.

Definition 2.2. The SLS (1) is *semistable* under arbitrary switching if there exists a class \mathcal{KL} function $\alpha(\cdot, \cdot)$ such that any $z \in \mathbb{R}^n$ and under any switching sequence σ , there exists $x_e(z, \sigma) \in \mathcal{E}_e$ such that $\|x(t; z, \sigma) - x_e(z, \sigma)\| \leq \alpha(\|z - x_e(z, \sigma)\|, t)$, $\forall t \in \mathbb{Z}_+$.

The following result connects the semistability and the exponential semistability of switched linear systems.

Lemma 2.2. *The SLS (1) is exponentially semistable under arbitrary switching if and only if it is semistable under arbitrary switching.*

Proof. The “only if” part is trivial and hence is omitted. To show the other part, it follows from the standard argument, e.g., [16, Lemma 3.3], that under the assumption in Definition 2.2, the SLS (1) is uniformly asymptotically semistable under arbitrary switching. That is, for given $\delta > 0$ and $c \in (0, 1)$, there exists $T_{\delta, c} \in \mathbb{Z}_+$ (dependent on δ and c only) such that for any z and any switching sequence σ , $\|z - x_e(z, \sigma)\| \leq \delta \Rightarrow \|x(t; z, \sigma) - x_e(z, \sigma)\| \leq c\delta$, $\forall t \geq T_{\delta, c}$. This, together with the linear structure of the system, shows the exponential semistability under arbitrary switching; see [18, 22, 23] for details. \square

To characterize the exponential semistability and convergence rate of the SLS (1), we project the dynamics onto \mathcal{E}_e^\perp . Let $O \in \mathbb{R}^{n \times \ell}$ be the matrix whose columns constitute an orthonormal basis of \mathcal{E}_e^\perp (this implies that \mathcal{E}_e^\perp is of dimension ℓ), and let $P = OO^T \in \mathbb{R}^{n \times n}$ be the matrix representing the orthogonal projection onto \mathcal{E}_e^\perp . Clearly, P is idempotent, i.e., $P^2 = P$. For a given trajectory $x(t)$, let $x_{\mathcal{E}_e}(t)$ and $x_{\mathcal{E}_e^\perp}(t)$ denote the (unique) orthogonal projections of $x(t)$ onto \mathcal{E}_e and \mathcal{E}_e^\perp respectively, i.e., $x_{\mathcal{E}_e^\perp}(t) = Px(t)$ and $x_{\mathcal{E}_e}(t) = [I_n - P]x(t)$. Define the output of the SLS (1) via the projection matrix O :

$$y(t) = O^T x(t), \quad t = 0, 1, 2, \dots \quad (3)$$

Then, at any t , regardless of the current mode $\sigma(t)$, $\|y(t)\|$ is the Euclidean distance of $x(t)$ to the equilibrium subspace \mathcal{E}_e . In the following, denote by $y(t; z, \sigma)$ the output trajectory of the SLS (1) starting from the initial condition z under the switching sequence σ .

Definition 2.3. The SLS (1) with the output (3) is *output exponentially stable* under arbitrary switching if there exist constants $\kappa > 0$ and $\tilde{r} \in (0, 1)$ such that for any $z \in \mathbb{R}^n$, $\|y(t; z, \sigma)\| \leq \kappa \tilde{r}^t \|z\|$, $\forall t \in \mathbb{Z}_+$ under any σ . Here, \tilde{r} is called the *exponential growth rate of output stability*.

To simplify the subsequent development, we introduce a useful coordinate transformation as follows. Recall that \mathcal{E}_e^\perp is of dimension ℓ . Let $\tilde{O} \in \mathbb{R}^{n \times (n-\ell)}$ be the matrix whose columns constitute an orthonormal basis of \mathcal{E}_e , and define the orthogonal matrix $T = [\tilde{O} \tilde{0}]^T \in \mathbb{R}^{n \times n}$ and the state transformation $\tilde{x}(t) = T x(t)$. In the new coordinates, $\mathcal{E}_e^\perp = \mathbb{R}^\ell \times \{0\}$, $\mathcal{E}_e = \{0\} \times \mathbb{R}^{(n-\ell)}$, and the relevant matrices can be written as $\tilde{O}^T = O^T T^{-1} = [I_\ell \ 0]$,

$$\tilde{P} = \tilde{O} \tilde{O}^T = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_i = T A_i T^{-1} = \begin{bmatrix} \hat{A}_{i,11} & 0 \\ \hat{A}_{i,21} & I_{n-\ell} \end{bmatrix},$$

for all $i \in \mathcal{M}$. Furthermore, $\tilde{x}(t) = [y^T(t); \tilde{x}^T(t)]^T$, $z = [y^T(0); \tilde{x}^T(0)]^T$, $x_{\mathcal{E}_e^\perp}^T(t) = [y^T(t); 0]$, and $x_{\mathcal{E}_e}^T(t) = [0; \tilde{x}^T(t)]$, where $y(t) \in \mathbb{R}^\ell$ and $\tilde{x}(t) \in \mathbb{R}^{(n-\ell)}$ satisfy

$$y(t+1) = \hat{A}_{\sigma(t),11} y(t), \tag{4}$$

and

$$\tilde{x}(t) = \tilde{x}(0) + \sum_{\tau=0}^{t-1} \hat{A}_{\sigma(\tau),21} y(\tau). \tag{5}$$

Note that (4) yields an SLS defined by subsystem matrices $\{\hat{A}_{i,11}\}$ and is decoupled from (5). See Remark 2.1 for the geometry of the above dynamics under exponential stability conditions.

Since the state transformation does not affect the semistability and output stability as well as their growth rates, we consider the switching dynamics (4) and (5) throughout the rest of this section by dropping the notation $\hat{\cdot}$ in the equations.

2.1. The equivalence of semistability and output stability under arbitrary switching

In this section, we show the equivalence of exponential semistability and output exponential stability. To this end, we introduce some notions and a technical result. We call the SLS *convergent (to the origin)* under arbitrary switching if for any initial state z , $x(t; z, \sigma)$ converges to the origin as $t \rightarrow \infty$ under any switching sequence σ . The following result asserts the equivalence of this convergence, asymptotic and exponential stabilities for a general SLS under arbitrary switching.

Theorem 2.1. A switched linear system is convergent under arbitrary switching if and only if it is exponentially stable under arbitrary switching.

Proof. It suffices to prove the “only if”. Consider a SLS with the matrices $\{A_i\}_{i=1}^m$. The following claim holds:

Claim: If there exists $T_* \in \mathbb{N}$ (independent of z and σ) such that for any z with $\|z\| = 1$ and under any switching sequence σ , there exists $t_* \in [0, T_*]$ such that $\|x(t_*; z, \sigma)\| \leq 0.5$, then the SLS is exponentially stable under arbitrary switching.

Let $\kappa = \sum_{j=0}^{T_*} (\max_i \|A_i\|)^j$. The claim can be shown via induction that under the given conditions, $\|x(t; z, \sigma)\| \leq \kappa (0.5)^{t/T_* - 1} \|z\|$, $\forall t \in \mathbb{Z}_+$ for any z and under any switching sequence σ . This thus yields the exponential stability.

Now suppose that the SLS is convergent but not exponentially stable, under arbitrary switching. It follows from the above claim that there exist an initial state sequence $\{z_k\}$ with $\|z_k\| = 1$, an increasing time sequence $\{T_k\}$ with $\lim_{k \rightarrow \infty} T_k = \infty$, and a sequence of switching sequences $\{\sigma_k\}$ such that for each k , $\|x(t; z_k, \sigma_k)\| \geq 0.5$

for all $t = 0, 1, \dots, T_k$. Using a similar argument as that of [18, Theorem 3], we can construct an initial state z and a switching sequence σ such that $\|x(t; z, \sigma)\| \geq 0.5$ for all t . This is contradictory to the convergence of the SLS. \square

By making use of this theorem, we establish one of the main results as follows.

Theorem 2.2. The SLS (1) is exponentially semistable under arbitrary switching if and only if the SLS (1) with the output (3) is output exponentially stable under arbitrary switching. Further, the exponential growth rates of semistability and output stability are identical, i.e., $r \in (0, 1)$ is the exponential growth rate of semistability if and only if it is that of output stability.

Proof. “If”. Let $\kappa > 0$ and $\tilde{r} \in (0, 1)$ be such that for any $z \in \mathbb{R}^n$ and any switching sequence σ , $\|y(t; z, \sigma)\| \leq \kappa \tilde{r}^t \|z\|$ for all $t \in \mathbb{Z}_+$. For an initial state $z \in \mathbb{R}^n$ and switching sequence σ , it follows from (5) that, for any $s, w \in \mathbb{Z}_+$ with $s < w$,

$$\begin{aligned} & \|\tilde{x}(w; z, \sigma) - \tilde{x}(s; z, \sigma)\| \\ &= \left\| \sum_{\tau=s}^{w-1} A_{\sigma(\tau),21} y(\tau; z, \sigma) \right\| \leq \sum_{\tau=s}^{w-1} \|A_{\sigma(\tau),21}\| \cdot \|y(\tau; z, \sigma)\| \\ &\leq \max_i (\|A_{i,21}\|) (\kappa \tilde{r}^s + \dots + \kappa \tilde{r}^{w-1}) \|z\| \\ &\leq \max_i (\|A_{i,21}\|) \kappa \tilde{r}^s / (1 - \tilde{r}) \|z\|. \end{aligned}$$

Therefore, for all s, w sufficiently large (no matter how far they are apart), $\|\tilde{x}(w; z, \sigma) - \tilde{x}(s; z, \sigma)\|$ is sufficiently small. This shows that $\{\tilde{x}(t; z, \sigma)\}$ is a Cauchy sequence in $\mathbb{R}^{n-\ell}$ and thus converges in $\mathbb{R}^{n-\ell}$ (because $\mathbb{R}^{n-\ell}$ is complete). Since $\mathbb{R}^{n-\ell}$ is closed, $\tilde{x}_e(z, \sigma) =: \lim_{t \rightarrow \infty} \tilde{x}(t; z, \sigma) \in \mathbb{R}^{n-\ell}$. Let $x_e(z, \sigma) = [0; \tilde{x}_e^T(z, \sigma)]^T \in \mathcal{E}_e$ and $\tilde{z} = z - x_e(z, \sigma)$. It can be verified via (4) and (5) directly that

$$x(t; \tilde{z}, \sigma) = \begin{bmatrix} y(t; z, \sigma) \\ \tilde{x}(t; z, \sigma) \end{bmatrix} - \begin{bmatrix} 0 \\ \tilde{x}_e(z, \sigma) \end{bmatrix}.$$

This shows that $\|y(t; \tilde{z}, \sigma)\| = \|y(t; z, \sigma)\| \leq \kappa \tilde{r}^t \|\tilde{z}\|$. Moreover, for all t and under any σ ,

$$\begin{aligned} \|\tilde{x}(t; \tilde{z}, \sigma)\| &= \|\tilde{x}(t; z, \sigma) - \tilde{x}_e(z, \sigma)\| = \left\| \sum_{\tau=t}^{\infty} A_{\sigma(\tau),21} y(\tau; \tilde{z}, \sigma) \right\| \\ &\leq \left(\max_i \|A_{i,21}\| \right) \kappa (\tilde{r}^t + \tilde{r}^{t+1} + \dots) \|\tilde{z}\| \\ &\leq \left(\max_i \|A_{i,21}\| \right) \frac{\kappa \tilde{r}^t}{1 - \tilde{r}} \|\tilde{z}\|. \end{aligned}$$

In view of the above results and $\|x(t; \tilde{z}, \sigma)\| \leq \|y(t; \tilde{z}, \sigma)\| + \|\tilde{x}(t; \tilde{z}, \sigma)\|$, we obtain a constant $\rho > 0$, independent of z and σ , such that $\|x(t; z, \sigma) - x_e(z, \sigma)\| = \|x(t; \tilde{z}, \sigma)\| \leq \rho \tilde{r}^t \|\tilde{z}\| = \rho \tilde{r}^t \|z - x_e(z, \sigma)\|$, $\forall t \in \mathbb{Z}_+$. This gives rise to the exponential semistability and shows that the exponential growth rate \tilde{r} of output stability is also that of exponential semistability.

“Only if”: We prove this part by contradiction. Suppose that the SLS (1) is exponentially semistable but not output exponentially stable, under arbitrary switching. It can be seen that the output trajectory $y(t)$ is equivalent to the state trajectory of the SLS (4) defined by the subsystem matrices $\{A_{i,11}\}$. Hence, if the original SLS (1) is not output exponentially stable under arbitrary switching, neither is the SLS (4). We thus deduce from Theorem 2.1 that the SLS (4) is not convergent under arbitrary switching. Hence, there exist $z \in \mathbb{R}^n$ and a switching sequence σ such that $y(t; z, \sigma)$ does not converge to the origin of \mathbb{R}^ℓ . This contradicts the exponential semistability of the SLS (1) under arbitrary switching.

Finally, we show that a constant $r \in (0, 1)$ is the exponential growth rate of semistability if and only if it is that of output stability. It suffices to consider the “only if” part as the other part has been shown before. Suppose that the SLS (1) is exponentially semistable with the exponential growth rate $r \in (0, 1)$, i.e., for any

$z \in \mathbb{R}^n$ and under any σ , there exists $x_e(z, \sigma) \in \mathcal{E}_e$ such that $\|x(t; z, \sigma) - x_e(z, \sigma)\| \leq \rho r^t \|z - x_e(z, \sigma)\|$, $\forall t$ with $\rho > 0$. This implies, in light of the proof for the “only if” above, that the SLS (4) is exponentially stable with the exponential growth rate $\tilde{r} \in [0, 1)$ (not necessarily equal to r at this stage) and the parameter $\kappa > 0$, under arbitrary switching. By slightly abusing notation, we use $y(t; y(0), \sigma)$ to denote the trajectory of the SLS (4) starting from $y(0)$ under σ . Hence, we have, for any $y(0) \in \mathbb{R}^\ell$ and under any σ , $\|y(t; y(0), \sigma)\| \leq \kappa \tilde{r}^t \|y(0)\|$, $\forall t$. It thus follows from the above proof for the “if” part that for any initial state $z = [y^T(0); \tilde{x}^T(0)]^T$ and any switching sequence σ , $\tilde{x}_e(z, \sigma) = \tilde{x}(0) + \lim_{t \rightarrow \infty} \sum_{\tau=0}^{t-1} A_{\sigma(\tau), 21} y(\tau; y(0), \sigma)$ such that the latter limit exists. Moreover,

$$\begin{aligned} \|z - x_e(z, \sigma)\| &\leq \|y(0)\| + \left\| \lim_{t \rightarrow \infty} \sum_{\tau=0}^{t-1} A_{\sigma(\tau), 21} y(\tau; y(0), \sigma) \right\| \\ &\leq \|y(0)\| + \max_i \|A_{i, 21}\| \sum_{\tau=0}^{\infty} \|y(\tau; y(0), \sigma)\| \leq \|y(0)\| \\ &\quad + \max_i \|A_{i, 21}\| \left(\sum_{\tau=0}^{\infty} \kappa \tilde{r}^\tau \|y(0)\| \right) \\ &\leq \|y(0)\| + \max_i \|A_{i, 21}\| \frac{\kappa}{1 - \tilde{r}} \|y(0)\|. \end{aligned}$$

Consequently, by the exponential semistability, we have

$$\begin{aligned} \|y(t; z, \sigma)\| &\leq \|x(t; z, \sigma) - x_e(z, \sigma)\| \leq \rho r^t \|z - x_e(z, \sigma)\| \\ &\leq \tilde{\rho} r^t \|y(0)\| \leq \tilde{\rho} r^t \|z\|, \quad \forall t \in \mathbb{Z}_+, \end{aligned}$$

for some constant $\tilde{\rho} > 0$. This shows that r is the exponential growth rate of output stability. \square

Remark 2.1. The above theorem and Eqs. (4) and (5) show that under the exponential stability assumption, the SLS (1) can be thought of two dynamical processes: one is the dynamics of $y(t)$ in the fiber direction governed by the exponentially stable SLS (4) defined by the matrix set $\{A_{i, 11}\}$, and the other is the dynamics of $\tilde{x}(t)$ along the base direction that evolves by integrating $A_{\sigma(t), 21} y(t)$. The latter dynamics will move at worst in the pace proportional to $\|y(t)\|$ and thus converge to a constant vector at the same exponential rate as that of $y(t)$.

3. Semistability of switched linear systems: random case

The notion of semistability can be defined for a random SLS that evolves at each time by a subsystem matrix selected randomly from the matrix set $\{A_i\}_{i \in \mathcal{M}}$ according to a stationary distribution. In this case, the system state is a stochastic process $X(t)$ with the dynamics $X(t+1) = \mathbf{A}(t)X(t)$, $t = 0, 1, 2, \dots$,

where at each time t , $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$ is drawn independently randomly from the matrix set $\{A_i\}_{i \in \mathcal{M}}$ with the probability $\mathbb{P}\{\mathbf{A}(t) = A_i\} = p_i, i \in \mathcal{M}$ for some probability distribution $p = \{p_i\}_{i \in \mathcal{M}}$ with $\sum_{i \in \mathcal{M}} p_i = 1$ and $p_i \geq 0$. For a given probability distribution p , denote by $X(t; z, p)$ the stochastic trajectory of the random SLS (6) starting from a deterministic initial state $X(0) = z$, and denote by \mathbb{E} the expectation operator.

Definition 3.1. The random SLS (6) is *mean square exponentially semistable* if there exist constants $\rho \geq 0$ and $r \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, there exists a random vector $X_e(z, p) \in \mathcal{E}_e$ such that

$$\mathbb{E}[\|X(t; z, p) - X_e(z, p)\|^2] \leq \rho r^t \|z - \mathbb{E}[X_e(z, p)]\|^2, \quad \forall t \in \mathbb{Z}_+. \quad (7)$$

Here, r is called the *exponential growth rate of mean square semistability*.

Remark 3.1. One can also define the mean square exponential semistability by replacing the right-hand side of (7) by a mean square, i.e., $\mathbb{E}[\|X(t; z, p) - X_e(z, p)\|^2] \leq \rho r^t \mathbb{E}[\|z - X_e(z, p)\|^2]$. It can be shown, in view of $\|z - \mathbb{E}[X_e(z, p)]\|^2 \leq \mathbb{E}[\|z - X_e(z, p)\|^2]$, that the

latter definition preserves the same qualitative properties as that defined in (7), such as those stated in Theorem 3.1. In what follows, we consider the mean square semistability defined in (7).

Let O be the projection matrix used in Section 2. Define the output of the random SLS (6) by

$$Y(t) = O^T X(t). \quad (8)$$

For a given probability distribution p , denote by $Y(t; z, p)$ the stochastic output trajectory of the SLS (6) from the deterministic initial state $X(0) = z$.

Definition 3.2. The random SLS (6) with the output (8) is *mean square output exponentially stable* if there exist constants $\kappa \geq 0$ and $\tilde{r} \in [0, 1)$ such that for any $z \in \mathbb{R}^n$, $\mathbb{E}[\|Y(t; z, p)\|^2] \leq \kappa \tilde{r}^t \|z\|^2$, $\forall t \in \mathbb{Z}_+$. Here, \tilde{r} is called the *exponential growth rate of mean square output stability*.

We adopt the same (deterministic) state transformation introduced in Section 2. Therefore, $\mathbf{A}(t)$ can be written as $\mathbf{A}(t) = \begin{bmatrix} A_{11}(t) & 0 \\ A_{21}(t) & I_{n-\ell} \end{bmatrix}$, where at each t , $A_{11}(t) \in \mathbb{R}^{\ell \times \ell}$ and $A_{21}(t) \in \mathbb{R}^{(n-\ell) \times \ell}$ are drawn independently randomly from the matrix tuple sets $\{(A_{i, 11}, A_{i, 21})\}_{i \in \mathcal{M}}$. Further, let $X^T(t) = [Y^T(t); \tilde{X}^T(t)]$, and $X_{e_e}^T(t) = [0; \tilde{X}^T(t)]$. Hence,

$$Y(t+1) = \mathbf{A}_{11}(t)Y(t), \quad (9)$$

and

$$\tilde{X}(t) = \tilde{X}(0) + \sum_{\tau=0}^{t-1} \mathbf{A}_{21}(\tau)Y(\tau). \quad (10)$$

3.1. The equivalence of semistability and output stability

The following theorem, as a counterpart of Theorems 2.2, asserts the equivalence of mean square exponential semistability and mean square output exponential stability.

Theorem 3.1. *The random SLS (6) is mean square exponentially semistable if and only if the SLS (6) with the output (8) is mean square output exponentially stable. Besides, a constant $r \in [0, 1)$ is the exponential growth rate of mean square exponential semistability if and only if it is that of mean square output exponential stability.*

Proof. “If”. Let $\kappa > 0$ and $\tilde{r} \in [0, 1)$ be such that for any $z \in \mathbb{R}^n$, $\mathbb{E}[\|Y(t; z, p)\|^2] \leq \kappa \tilde{r}^t \|z\|^2$ for all $t \in \mathbb{Z}_+$. Since $\mathbf{A}_{11}(t)$ in (9) is identically distributed, the matrix $V := \mathbb{E}[\mathbf{A}_{11}(t)]$ is independent of t . We claim that V is a stable matrix in the discrete sense, i.e., the spectral radius of V is strictly less than 1. To see this, note that the mean square exponential stability implies that $\mathbb{E}[\|Y(t; z, p)\|^2]$, hence $\mathbb{E}[Y(t; z, p)]$, converges to zero as $t \rightarrow \infty$, for all z . Since $\mathbb{E}[Y(t+1; z, p)] = V \cdot \mathbb{E}[Y(t; z, p)]$, the spectral radius of V must be strictly less than 1.

Given an initial state z and a probability distribution p , let $\tilde{X}(t; z, p)$ denote the stochastic state trajectory of $\tilde{X}(t)$ in (10) and define $L := \max_{i \in \mathcal{M}} \|A_{i, 21}\|$. For arbitrary $0 \leq s < w$,

$$\begin{aligned} \|\tilde{X}(w; z, p) - \tilde{X}(s; z, p)\|^2 &\leq L^2 \|Y(s; z, p) + \dots + Y(w-1; z, p)\|^2 \\ &\leq L^2 \left(\sum_{k=s}^{w-1} \|Y(k; z, p)\|^2 + 2 \sum_{s \leq j < k \leq w-1} Y^T(j; z, p)Y(k; z, p) \right). \end{aligned}$$

Since $Y(k; z, p) = \mathbf{A}_{11}(k-1) \dots \mathbf{A}_{11}(j)Y(j; z, p)$ for $j < k$ and V is a stable matrix, we have

$$\begin{aligned} \mathbb{E}[Y^T(j; z, p)Y(k; z, p)] &= \text{tr}(\mathbb{E}[\mathbf{A}_{11}(k-1) \dots \mathbf{A}_{11}(j)] \mathbb{E}[Y(j; z, p)Y^T(j; z, p)]) \\ &= \text{tr}(V^{k-j} \mathbb{E}[Y(j; z, p)Y^T(j; z, p)]) \\ &\leq \alpha \nu^{k-j} \mathbb{E}[\|Y(j; z, p)\|^2] \end{aligned}$$

for some constants $\alpha > 0$ and $\nu \in [0, 1)$ dependent on V and its order ℓ only. By the mean square exponential stability of $Y(t; z, p)$,

we further have

$$\begin{aligned} & \mathbb{E}[\|\tilde{X}(w; z, p) - \tilde{X}(s; z, p)\|^2] \\ & \leq L^2 \left(\sum_{k=s}^{w-1} \kappa \tilde{r}^k \|z\|^2 + 2 \sum_{s \leq j < k \leq w-1} \alpha \nu^{k-j} \tilde{r}^j \|z\|^2 \right) \\ & \leq \kappa L^2 \left(\frac{\tilde{r}^s}{1-\tilde{r}} + 2\alpha \sum_{s \leq j < k \leq w-1} \nu^{k-j} \tilde{r}^j \right) \|z\|^2. \end{aligned}$$

Using the pattern of the summation above in the last inequality, we have

$$\begin{aligned} & \sum_{s \leq j < k \leq w-1} \nu^{k-j} \tilde{r}^j = \tilde{r}^{s-1} \sum_{i=1}^{w-s} \tilde{r}^i \left(\sum_{k=1}^{w-s+1-i} \nu^k \right) \\ & \leq \tilde{r}^{s-1} \frac{\nu}{1-\nu} \sum_{i=1}^{w-s} \tilde{r}^i \leq \tilde{r}^s \frac{\nu}{(1-\nu)(1-\tilde{r})}. \end{aligned}$$

This implies that

$$\mathbb{E}[\|\tilde{X}(w; z, p) - \tilde{X}(s; z, p)\|^2] \leq \tilde{L} \tilde{r}^s \|z\|^2, \quad (11)$$

where $\tilde{L} > 0$ is a constant independent of w, s and z . By letting $s=0$, we conclude that $X(t; z, p)$ has bounded second moment. Hence, $\{\tilde{X}(t; z, p)\}_{t=0,1,\dots}$ is a Cauchy sequence of random vectors in the L^2 -space with respect to the underlying probability measure. Since the L^2 -space is complete, $\tilde{X}(t; z, p)$ converges in mean square to some random vector $\tilde{X}_e(z, p)$ (with the finite second moment) as $t \rightarrow \infty$. Let $\tilde{z} = z - [0; \mathbb{E}[\tilde{X}_e^T(z, p)]]^T$. Hence, $Y(t; \tilde{z}, p) = Y(t; z, p)$ (in distribution), and this, together with (10), shows that $\tilde{X}(t; \tilde{z}, p) = \tilde{X}(t; z, p) - \mathbb{E}[\tilde{X}_e(z, p)]$. Further, as $t \rightarrow \infty$, $\tilde{X}(t; \tilde{z}, p)$ converges to the random vector $\tilde{X}_e(\tilde{z}, p) = \tilde{X}_e(z, p) - \mathbb{E}[\tilde{X}_e(z, p)]$ (with zero mean) in mean square. Let $X_e^T(z) = [0; \tilde{X}_e^T(z)]$ and $X_e^T(\tilde{z}, p) = [0; \tilde{X}_e^T(\tilde{z}, p)]$. Then

$$\begin{aligned} & \mathbb{E}[\|X(t; z, p) - X_e(z, p)\|^2] = \mathbb{E}[\|X(t; \tilde{z}, p) - X_e(\tilde{z}, p)\|^2] \\ & = \mathbb{E}[\|Y(t; \tilde{z}, p)\|^2] + \mathbb{E}[\|\tilde{X}(t; \tilde{z}, p) - \tilde{X}_e(\tilde{z}, p)\|^2]. \end{aligned}$$

By letting $s=t$, $w=\infty$, and replacing z by \tilde{z} in (11), we see $\mathbb{E}[\|\tilde{X}(t; \tilde{z}, p) - \tilde{X}_e(\tilde{z}, p)\|^2] \leq \tilde{L} \tilde{r}^t \|\tilde{z}\|^2$. Along with the mean square output exponential stability, we obtain

$$\begin{aligned} & \mathbb{E}[\|X(t; z, p) - X_e(z, p)\|^2] \leq (\kappa + \tilde{L}) \tilde{r}^t \|\tilde{z}\|^2 \\ & \leq (\kappa + \tilde{L}) \tilde{r}^t \|z - \mathbb{E}[X_e(z, p)]\|^2, \quad \forall t, z. \end{aligned}$$

This yields the mean square exponential semistability and shows that the exponential growth rate \tilde{r} of the mean square output stability is also that of the mean square semistability.

“Only if”. Suppose that the random SLS is mean square exponentially semistable but not mean square output exponentially stable. In light of (9) and (10), we deduce, via the equivalence of mean square asymptotic stability and mean square exponential stability of randomly jumped linear systems [4, Theorem 4.1.1], that the random SLS (9) is not mean square asymptotically stable. This yields a contradiction, using a similar argument of Theorem 2.2.

To complete the proof, we only need to show that the exponential growth rate $r \in [0, 1)$ of mean square exponential semistability is that of mean square output exponential stability. By observing the above “only if” part, we deduce that $\mathbb{E}[\|y(t; z, p)\|^2] \leq \tilde{\kappa} \tilde{r}^t \|z\|^2$ for all t and z , for some constants $\tilde{\kappa} > 0$ and $\tilde{r} \in [0, 1)$ (not necessarily equal to r at this stage). It follows from the above “if” part that $\tilde{X}(t; z, p)$ converges to $\tilde{X}_e(z, p)$ in mean square as $t \rightarrow \infty$. By (10), we have that $\sum_{\tau=0}^t \mathbf{A}_{21}(\tau) Y(\tau; z, p)$ also converges in the L^2 -space as $t \rightarrow \infty$. Consequently,

$$\|\tilde{X}(0) - \mathbb{E}[\tilde{X}_e(z, p)]\|^2 = \left\| \mathbb{E} \left[\sum_{\tau=0}^{\infty} \mathbf{A}_{21}(\tau) Y(\tau; z, p) \right] \right\|^2$$

$$\begin{aligned} & \leq \mathbb{E} \left[\left\| \sum_{\tau=0}^{\infty} \mathbf{A}_{21}(\tau) Y(\tau; z, p) \right\|^2 \right] \\ & \leq \mathbb{E}[\|\tilde{X}_e(z, p) - \tilde{X}(0)\|^2] \leq \tilde{L} \|z\|^2, \end{aligned}$$

where the last inequality follows from (11) by letting $s=0$ and $w=\infty$. Finally, recalling that $X_e(z, p) = [0; \tilde{X}_e^T(z, p)]^T$, we deduce the existence of some constant $\tilde{\rho} > 0$ such that

$$\begin{aligned} & \mathbb{E}[\|Y(t; z, p)\|^2] \leq \mathbb{E}[\|X(t; z, p) - X_e(z, p)\|^2] \leq \rho r^t \|z - \mathbb{E}[X_e(z, p)]\|^2 \\ & \leq \rho r^t (\|Y(0)\|^2 + \tilde{L} \|z\|^2) \leq \tilde{\rho} r^t \|z\|^2, \quad \forall t \in \mathbb{Z}_+. \end{aligned}$$

This yields the desired growth rate for the mean square output exponential stability. \square

4. Numerical verification of semistability of the SLSs via generating functions

It is shown in the preceding sections that semistability and convergence rate of the SLS boils down to regular stability and convergence rate of the reduced SLS, for which various numerical approaches have been developed. In this section, we consider generating function based algorithms recently proposed in [6]. An advantage of the generating function approach is that it yields a unified analytic framework and effective numerical schemes for computing exponential growth rates under different switching rules [6]. Since the generating function approach is relatively new, we present concise discussions of essential elements of this approach and its semistability application in order to be self-contained.

4.1. Verification of semistability via strong generating functions

Consider the semistability under arbitrary switching. In view of Theorem 2.2, the maximum exponential growth rate of the semistable SLS (1) is completely determined by that of the reduced SLS (4), under arbitrary switching. The latter exponential growth rate, denoted by r^* , may serve as a quantitative measure of the semistability of the SLS (1). Indeed, the SLS (1) is exponentially semistable if and only if $r^* < 1$. In what follows, we characterize r^* , and thus exponential semistability, using the strong generating function [6,18,19,21].

Let $y(t; v, \sigma)$ denote the trajectory of the reduced SLS (4) starting from the initial state $v \in \mathbb{R}^\ell$ under the switching sequence σ . The strong generating function $G: \mathbb{R}_+ \times \mathbb{R}^\ell \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ of the SLS (4) is defined as

$$G(\lambda, v) =: G_\lambda(v) = \sup_{\sigma} \sum_{t=0}^{\infty} \lambda^t \|y(t; v, \sigma)\|^2, \quad v \in \mathbb{R}^\ell, \lambda \geq 0, \quad (12)$$

where the supremum is taken over all switching sequences σ . A wealth of analytic properties of the generating functions can be found in [6]. For the strong generating function $G_\lambda(\cdot)$, the radius of strong convergence λ^* is defined as $\lambda^* =: \sup\{\lambda > 0 : G_\lambda(v) < \infty, \forall v \in \mathbb{R}^\ell\}$. The following theorem characterizes the exponential stability of the SLS (4) via λ^* :

Theorem 4.1 (Hu et al. [6, Theorem 2]). *The SLS (4) is exponentially stable under arbitrary switching if and only if its radius of strong convergence $\lambda^* > 1$.*

Moreover, as shown in [6, Corollary 1], the maximum exponential growth rate of the SLS (4) is given by $r^* = (\lambda^*)^{-1/2}$. Consequently, we obtain the following corollary:

Corollary 4.1. *The SLS (1) is exponentially semistable under arbitrary switching if and only if the radius of strong convergence of the reduced SLS (4) satisfies $\lambda^* > 1$. Further, the maximum exponential growth rate of the semistable trajectories of the SLS (1) is given by $(\lambda^*)^{-1/2}$.*

In order to compute the strong generating function and the radius of strong convergence for the SLS (4), we approximate G_λ , which is the value function of an infinite horizon optimal control problem, by a sequence of finite horizon approximations. Specifically, define $G_\lambda^k(v) = \max_{\sigma} \sum_{t=0}^k \lambda^t \|Y(t; v, \sigma)\|^2$, $v \in \mathbb{R}^\ell$, $k = 0, 1, \dots$. Then the functions $G_\lambda^k(v)$ can be computed recursively by $G_\lambda^0(v) = \|v\|^2$ and $G_\lambda^k(v) = \|v\|^2 + \lambda \max_{i \in \mathcal{M}} G_\lambda^{k-1}(A_{i,11}v)$, $k \in \mathbb{N}$. Based on the Bellman equation and the sub-additivity of G_λ , an algorithm is developed to obtain increasingly accurate estimates of G_λ on a grid of the unit sphere; see [6, Section III] for details.

Remark 4.1. As shown in the above corollary, finding the exponential growth rate of semistable trajectories of the SLS under arbitrary switching amounts to finding the radius of strong convergence. The latter problem is known to be NP-hard. For a given k , the generating function based algorithm has linear complexity with respect to the number of subsystems (4). Therefore, the proposed algorithm is suitable for a reduced SLS with a relatively large number of subsystems (4) but a smaller state dimension, namely, ℓ ; its overall complexity is exponential [6].

4.2. Verification of mean square semistability via mean generating functions

Theorem 3.1 enables us to characterize the mean square exponential semistability and its maximum exponential growth rate by using the mean generating function of the reduced random SLS (9). Let $Y(t; v, p)$ denote the stochastic state trajectory of the random SLS (9) starting from the deterministic initial state $v \in \mathbb{R}^\ell$ under the switching probability distribution p . The mean generating function $F: \mathbb{R}_+ \times \mathbb{R}^\ell \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ of the random SLS (9) is defined as

$$F(\lambda, z) = F_\lambda(z) = \mathbb{E} \left[\sum_{t=0}^{\infty} \lambda^t \|Y(t; v, p)\|^2 \right] = \sum_{t=0}^{\infty} \lambda^t \mathbb{E} [\|Y(t; v, p)\|^2]. \quad (13)$$

The mean generating function F_λ shares the similar properties of the strong generating function G_λ , and a collection of its properties can be found in [6, Proposition 13]. The radius of convergence of the mean generating function F_λ is defined as

$$\lambda_p^* = \sup \{ \lambda \geq 0 : F_\lambda(v) < \infty, \forall v \in \mathbb{R}^\ell \}. \quad (14)$$

This quantity can be used to determine the mean square exponential stability as shown below.

Lemma 4.1 (Hu et al. [6, Theorem 4]). *The random SLS (9) is mean square exponentially stable if and only if the radius of convergence of the mean generating function F_λ satisfies $\lambda_p^* > 1$.*

It also follows from a similar argument of [6, Corollary 1] that $(\lambda_p^*)^{-1/2}$ is the maximum exponential growth rate of the random SLS (9). In view of this and Theorem 3.1, we have:

Corollary 4.2. *The random SLS (6) is mean square exponentially semistable if and only if the radius of convergence λ_p^* of the reduced random SLS (9) satisfies $\lambda_p^* > 1$. Further, the maximum exponential growth rate of the mean square semistable trajectories of the SLS (6) is $(\lambda_p^*)^{-1/2}$.*

The mean generating function F_λ and its radius of convergence λ_p^* can be efficiently computed using the finite horizon approximations of F_λ given by $F_\lambda^k(v) = \sum_{t=0}^k \lambda^t \mathbb{E} [\|Y(t; v, p)\|^2]$. Further, $F_\lambda^k(v)$ can be computed recursively by $F_\lambda^0(v) = \|v\|^2$ and $F_\lambda^k(v) = \|v\|^2 + \lambda \sum_{i \in \mathcal{M}} p_i F_\lambda^{k-1}(A_{i,11}v)$, $k \in \mathbb{N}$. The details can be found in [6, Section V].

4.3. Numerical example

To illustrate the generating function approach, we consider the SLS motivated by Hui [12, Example 3.1] that consists of two subsystems whose matrices are given by

$$A_1 = \begin{bmatrix} 0.7 & 0.0707 & 0.0707 \\ 0.0707 & 0.85 & -0.15 \\ 0.0707 & -0.15 & 0.85 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.5 & 0.2121 & 0.2121 \\ 0.2121 & 0.75 & -0.25 \\ 0.2121 & -0.25 & 0.75 \end{bmatrix}.$$

The common equilibrium subspace is $\mathcal{E}_e = \cap_{i=1}^2 \mathcal{N}(A_i - I) = \text{span}\{v\}$, where $v = (0, -1/\sqrt{2}, 1/\sqrt{2})^T$. Hence, $\ell = 2$. Using the transformation for (4) and (5), we obtain

$$\hat{A}_1 = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.1 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0.5 & 0.3 & 0 \\ 0.3 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This gives rise to the two subsystem matrices in (4):

$$\hat{A}_{1,11} = \begin{bmatrix} 0.7 & 0.1 \\ 0.1 & 0.7 \end{bmatrix} \quad \text{and} \quad \hat{A}_{2,11} = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}.$$

In what follows, we compute the exponential semistability convergence rates r^* under two switching rules using the proposed method:

- (1) Arbitrary switching: the radius of convergence λ^* of the strong generating function is computed to be within the interval (1.561, 1.564); hence $r^* \in (0.7996, 0.8004)$.
- (2) Random switching with the distribution $\mathbb{P}(\mathbf{A}(t) = A_1) = \frac{1}{3}$ and $\mathbb{P}(\mathbf{A}(t) = A_2) = \frac{2}{3}$; it is computed that $\lambda_p^* \in (1.562, 1.563)$; hence $r^* \in (0.7999, 0.8001)$.

In this example, note that $\hat{A}_{1,11}$ and $\hat{A}_{2,11}$ commute so that they can be simultaneously diagonalized as

$$\hat{A}_{1,11} = V \begin{bmatrix} 0.6 & \\ & 0.8 \end{bmatrix} V^{-1}, \quad \hat{A}_{2,11} = V \begin{bmatrix} 0.2 & \\ & 0.8 \end{bmatrix} V^{-1},$$

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

As a result, the semistability convergence rates under arbitrary and random switching can be found explicitly as 0.8, which is very close to the numerical results obtained above.

5. Application to the PageRank algorithm

The PageRank algorithm is one of the main tools at Google to evaluate web page relevance by quantifying the importance of each web page [2,14]. The key idea of this algorithm is that a web page having links from important web pages is assigned a higher value based on page interconnection structure. Such a value is determined as the sum of the contributions from all the other web pages connecting to it [14]. For a fixed link topology, the PageRank problem boils down to an eigenvector problem for a stochastic matrix. The latter problem can be solved by an iterative update scheme formulated as a discrete-time linear system; see [2,14] and the references therein.

Specifically, consider a network of n web pages that is described by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ denotes the index set of the web pages and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the links between the web pages. The notation $(i, j) \in \mathcal{E}$ means that page i has an outgoing link to page j . For each $i \in \mathcal{V}$, let $\mathcal{L}_i = \{k : (k, i) \in \mathcal{E}\}$,

and let n_i be the total number of outgoing links of page i . The corresponding link matrix is $L = (\ell_{ij}) \in \mathbb{R}^{n \times n}$, where

$$\ell_{ij} = \begin{cases} \frac{1}{n_j}, & j \in \mathcal{L}_i, \\ 0, & \text{otherwise,} \end{cases}$$

where we assume that L is a column stochastic matrix. The PageRank problem is to find a probability vector p , i.e., $p \in \mathbb{R}_+^n$ with $\mathbf{1}^T p = 1$, such that $Lp = p$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector of ones. This eigenvector problem may have multiple solutions in general. To overcome this problem, a modified version is used by replacing L with $(1-m)L + (m/n)S$, where $m \in (0, 1)$ is a parameter and S is an $n \times n$ matrix of ones, i.e., $S = \mathbf{1}\mathbf{1}^T$. The following iterative update scheme is employed to solve the modified eigenvector problem:

$$x(t+1) = \left[(1-m)L + \frac{m}{n}S \right] x(t) = (1-m)Lx(t) + \frac{m}{n}\mathbf{1}, \quad t \in \mathbb{Z}_+,$$

where an initial state $x(0)$ is a probability vector. The convergence rate of this scheme is $(1-m)$.

In this section, we study the update dynamics of the PageRank Algorithm on a switched link topology from the point view of semistability. This scenario occurs when web page connections are subject to change (a different scenario also occurs in a distributed randomized approach for solving the eigenvector problem [14]). Due to the topology change, the iterative update scheme gives rise to an SLS whose subsystem matrices are column stochastic matrices, and finding a desired eigenvector leads to a semistability-like problem of the SLS. It should be pointed out that because of state constraint and other properties of the PageRank problem, the common equilibrium set may not exist or, if exists, becomes a discrete or even a singleton set. We introduce two update schemes in this section, one of which is based on link matrices and the other based on the modified matrices with the parameters m_i . While these schemes are restricted to switching topologies that admit a common equilibrium set, it is shown that the convergence rates of certain classes of update schemes can be effectively computed using the semistability techniques and the generating function approach.

Consider the case where there are M link topologies $\mathcal{E}_i, i \in \mathcal{M} := \{1, \dots, M\}$, each of which is characterized by the link matrix L_i and the parameter $m_i \in (0, 1)$ (for the modified version). Two update schemes for the switching topology are as follows.

1. The PR-SLS of type I, where the link matrices L_i are used as subsystem matrices:

$$x(t+1) = L_{\sigma(t)}x(t),$$

where each L_i is a nonnegative matrix and is assumed to be a column stochastic matrix. In this case, if $\cap_{i \in \mathcal{M}} \mathcal{N}(L_i - I_n)$ is non-trivial, then the proposed semistability results can be applied. Some properties of this SLS are as follows:

- (1) Let $\mathcal{P} := \{z \in \mathbb{R}_+^n : \mathbf{1}^T z = 1\}$ denote the set of probability vectors. For any $z \in \mathcal{P}$ and any switching sequence $\sigma, x(t; z, \sigma) \in \mathcal{P}$ for any t . (In other words, the SLS is positively invariant on \mathcal{P} under arbitrary switching.) Hence, the common equilibrium set is given by $\cap_{i \in \mathcal{M}} \mathcal{N}(L_i - I_n) \cap \mathcal{P}$.
- (2) Suppose that $\cap_{i \in \mathcal{M}} \mathcal{N}(L_i - I_n)$ is of dimension $\ell \geq 2$. Since \mathcal{P} is an $(n-1)$ -dimensional polytope, the common equilibrium set $\cap_{i \in \mathcal{M}} \mathcal{N}(L_i - I_n) \cap \mathcal{P}$ is a polytope of dimension $\ell - 1$.

2. The PR-SLS of type II, where we use the matrices

$$A_i = (1-m_i)L_i + \frac{m_i}{n}S, \quad i \in \mathcal{M}$$

as subsystem matrices. Then the update scheme is given by the SLS:

$$x(t+1) = A_{\sigma(t)}x(t), \tag{15}$$

where the initial state is a probability vector z . It is ready to see that each A_i is a (strictly) positive and column stochastic matrix, and the SLS is positively invariant on \mathcal{P} under arbitrary switching. Equilibria properties of the SLS (15) are given below.

Lemma 5.1. *Given a switching sequence σ , assume that $x(t; z, \sigma)$ converges under σ for some $z \in \mathcal{P}$ as $t \rightarrow \infty$. Then for any $z \in \mathcal{P}$, $x(t; z, \sigma)$ converges to a unique vector in \mathcal{P} under the same σ . In other words, the limiting point of $x(t; z, \sigma)$ only depends on σ but is independent of the initial state z .*

Proof. To see this, note that the SLS can be written as

$$x(t+1) = (1-m_{\sigma(t)})L_{\sigma(t)}x(t) + \frac{m_{\sigma(t)}}{n}\mathbf{1}.$$

Therefore,

$$\begin{aligned} x(t+1) &= (1-m_{\sigma(t)})(1-m_{\sigma(t-1)})L_{\sigma(t)}L_{\sigma(t-1)}x(t-1) + \frac{m_{\sigma(t)}}{n}\mathbf{1} \\ &\quad + (1-m_{\sigma(t)})\frac{m_{\sigma(t-1)}}{n}L_{\sigma(t)}\mathbf{1} \\ &= \underbrace{\left(\prod_{i=0}^t (1-m_{\sigma(i)}) \right) L_{\sigma(t)}L_{\sigma(t-1)} \cdots L_{\sigma(0)} z}_{w(t+1; z, \sigma)} \\ &\quad + \underbrace{\left\{ m_{\sigma(t)}I + \sum_{i=1}^t \left(m_{\sigma(t-i)} \prod_{j=1}^i (1-m_{\sigma(j)}) \right) L_{\sigma(t)}L_{\sigma(t-1)} \cdots L_{\sigma(t-i+1)} \right\}}_{v(t+1; \sigma)} \frac{\mathbf{1}}{n}, \end{aligned}$$

where we drop z in v since it is independent of z . We study the dynamic behavior of $w(t; z, \sigma)$ as follows. It is easy to see that $w(t; z, \sigma) \in \mathbb{R}_+^n$ for any t . Hence, observing that L_i is a column stochastic matrix, we have

$$\begin{aligned} \|w(t; z, \sigma)\|_1 &= \mathbf{1}^T w(t; z, \sigma) = \left(\prod_{i=0}^{t-1} (1-m_{\sigma(i)}) \right) \mathbf{1}^T z \\ &= \prod_{i=0}^{t-1} (1-m_{\sigma(i)}) \leq r^t, \quad \forall t \in \mathbb{Z}_+, \end{aligned}$$

where $r := \max_{i \in \mathcal{M}} (1-m_i) \in (0, 1)$. This shows that for any $z \in \mathcal{P}$, $w(t; z, \sigma) \rightarrow 0$ as $t \rightarrow \infty$ under the given (in fact, any) σ . Hence, the limit of $x(t; z, \sigma)$ is completely determined by the limit of $v(t; \sigma)$ that is independent of $z \in \mathcal{P}$. \square

In view of Lemma 5.1, we call a switching sequence σ satisfying the assumption stated in the lemma a *switching sequence of convergence* (a word of caution on terminology: σ itself is not necessarily convergent but it yields a converging state trajectory for any initial state). For such a switching sequence σ , let $p_*(\sigma)$ be the unique limit of a state trajectory under σ . Then we have

$$p_*(\sigma) \in \{ \cap_{i \in \mathcal{I}_\sigma} \mathcal{N}(A_i - I_n) \} \cap \mathcal{P},$$

where \mathcal{I}_σ is the asymptotic index set associated with σ defined in (2). This result follows directly from Lemma 2.1 and the fact that each trajectory state is in \mathcal{P} which is closed. The following proposition presents more equilibrium properties of the SLS (15).

Proposition 5.1. *The following statements hold:*

- (1) Let σ be a switching sequence of convergence. Then the subspace $\cap_{i \in \mathcal{I}_\sigma} \mathcal{N}(A_i - I_n)$ is of dimension one, and the set $\{ \cap_{i \in \mathcal{I}_\sigma} \mathcal{N}(A_i - I_n) \} \cap \mathcal{P}$ contains a unique positive vector.
- (2) The SLS (15) has at most M limiting points, where M is the number of link topologies.
- (3) If each state trajectory of the SLS (15) is convergent under arbitrary σ , then the SLS (15) has a unique equilibrium in $\{ \cap_{i \in \mathcal{M}} \mathcal{N}(A_i - I_n) \} \cap \mathcal{P}$, which is independent of z and σ .

Proof. (1) Let $i \in \mathcal{I}_\sigma$. Since each A_i is a strictly positive and column stochastic matrix, the spectral radius $\rho(A_i) = 1$. It thus follows from

Perron's Theorem [17, Section 8.2] that there is a unique positive vector p such that $A_i p = p$ and $\|p\|_1 = \mathbf{1}^T p = 1$. Hence, the dimension of $\mathcal{N}(A_i - I_n)$ is one. The nonemptiness of $\{\cap_{i \in \mathcal{I}_\sigma} \mathcal{N}(A_i - I_n)\} \cap \mathcal{P}$ is obvious. To see the singleton property, let $p \in \{\cap_{i \in \mathcal{I}_\sigma} \mathcal{N}(A_i - I_n)\} \cap \mathcal{P}$, i.e., p is a probability vector such that $A_i p = p$ for all $i \in \mathcal{I}_\sigma$. The previous argument based on Perron's Theorem shows that p is the unique positive vector such that $A_i p = p$ and $\|p\|_1 = \mathbf{1}^T p = 1$. Consequently, $p = p_*(\sigma) > 0$.

(2) Let Σ denote the set of all switching sequences of convergence. Let $\sigma, \sigma' \in \Sigma$. It follows from (1) that if $\mathcal{I}_\sigma \cap \mathcal{I}_{\sigma'}$ is nonempty, then $p_*(\sigma) = p_*(\sigma')$. In view of this, we call two index sets \mathcal{I}_σ and \mathcal{I}_{π} with $\sigma, \pi \in \Sigma$ similar, denoted by $\mathcal{I}_\sigma \sim \mathcal{I}_{\pi}$, if there exists a (finite) sequence $\{\mathcal{I}_{\sigma_k}\}_{k=1}^\ell$ with $\sigma_k \in \Sigma$ such that $\mathcal{I}_\sigma \cap \mathcal{I}_{\sigma_1} \neq \emptyset$, $\mathcal{I}_{\sigma_1} \cap \mathcal{I}_{\sigma_2} \neq \emptyset$, ..., and $\mathcal{I}_{\sigma_\ell} \cap \mathcal{I}_{\pi} \neq \emptyset$. It is clear that \sim defines an equivalent relation on $\{\mathcal{I}_\sigma : \sigma \in \Sigma\}$, i.e., it is reflexive, symmetric, and transitive. This yields a finite collection of (disjoint and distinct) equivalence classes $\{\mathcal{H}_\sigma\}$ of $\{\mathcal{I}_\sigma : \sigma \in \Sigma\}$. Each equivalent class corresponds to a unique limit $p_*(\sigma)$ for any σ with $\mathcal{I}_\sigma \in \mathcal{H}_\sigma$. Therefore, the number of the equivalent classes \mathcal{H}_σ is exactly the total number of limiting points of all convergent state trajectories in \mathcal{P} . Since there are at most M equivalent classes, the statement holds.

(3) In this case, note that $\Sigma = \mathcal{M} \times \dots \times \mathcal{M} \times \dots$. Therefore, $\mathcal{I}_\sigma = \mathcal{M}$ for some σ . By (1), we see that the SLS has a unique equilibrium in $\{\cap_{i \in \mathcal{M}} \mathcal{N}(A_i - I_n)\} \cap \mathcal{P}$, which is independent of z and σ . \square

The above proposition indicates that the set of limiting points of the SLS (15) is discrete and contains at most M points. Under the assumption of convergence under arbitrary switching, this set shrinks to a singleton set which contains a unique common equilibrium point. While this property is different from the original setting of the semistability analysis, the following example demonstrates that the convergence rates of the two types of PR-SLSs can be effectively computed using the semistability techniques and the generating function approach under suitable choice of system parameters, e.g., m_i .

Example 5.1. For illustration, we consider a three-webpage network with two topologies whose link matrices are given by

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}.$$

The two update schemes under arbitrary switching are as follows:

- (1) The PR-SLS of type I: $x(t+1) = L_{\sigma(t)} x(t)$. The common equilibrium subspace $\mathcal{E}_e = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, and the common equilibrium set $\mathcal{E}_e \cap \mathcal{P}$ is the polytope $\{z \in \mathbb{R}_+^3 : z_1 + z_2 = 1, z_3 = 0\}$. Further, the output SLS defined in (4) is a scalar system (i.e., $\ell = 1$) with $\hat{A}_{1,11} = 1/2$ and $\hat{A}_{2,11} = 2/3$. Hence, it is easy to obtain the convergence rate as $2/3$. Given an initial state $z = (z_1, z_2, z_3)^T \in \mathcal{P}$, it can be shown that under any σ , the state trajectory converges to $x_e = (z_1, 1 - z_1, 0)^T \in \mathcal{P}$, which is dependent on z but independent of σ .
- (2) The PR-SLS of type II: $x(t+1) = A_{\sigma(t)} x(t)$, where $A_i = (1 - m_i)L_i + m_i/3 \cdot S$ and the parameters $m_i \in (0, 1)$ are to be determined. A direct computation shows that

$$\mathcal{N}(A_1 - I_3) = \text{span} \left\{ \begin{bmatrix} \frac{1+m_1}{2m_1} \\ \frac{1}{m_1} \\ 1 \end{bmatrix} \right\},$$

$$\mathcal{N}(A_2 - I_3) = \text{span} \left\{ \begin{bmatrix} \frac{1+2m_2}{3m_2} \\ \frac{2+m_2}{3m_2} \\ 1 \end{bmatrix} \right\}.$$

Hence, the two null spaces have nontrivial intersection if and only if

$$\frac{3}{m_1} = \frac{2}{m_2} + 1.$$

We choose, for example, $m_2 = 1/2$; this leads to $m_1 = 3/5$. The unique equilibrium point is given by $p_* = (\frac{1}{3}, \frac{5}{12}, \frac{1}{4})^T \in \cap_{i=1,2} \mathcal{N}(A_i - I_3) \cap \mathcal{P}$. Using the transformation introduced in (4), we have $\ell = 2$ and

$$\hat{A}_i = \begin{bmatrix} \hat{A}_{i,11} & 0 \\ \hat{A}_{i,21} & 1 \end{bmatrix}, \quad i = 1, 2,$$

where

$$\hat{A}_{1,11} = \begin{bmatrix} 0.3291 & 0.0546 \\ 0.1677 & 0.2709 \end{bmatrix}, \quad \hat{A}_{2,11} = \begin{bmatrix} 0.4409 & 0.0455 \\ 0.1398 & 0.3924 \end{bmatrix},$$

$$\hat{A}_{1,21}^\top = \begin{bmatrix} -0.0849 \\ 0.1414 \end{bmatrix}, \quad \hat{A}_{2,21}^\top = \begin{bmatrix} -0.0707 \\ 0.1179 \end{bmatrix}.$$

The convergence rate of this update scheme under arbitrary switching is that of the output SLS with the subsystem matrices $\{\hat{A}_{1,11}, \hat{A}_{2,11}\}$ that can be computed using the generating function approach as $\lambda^* \in (3.99, 4.01)$. Therefore, the exponential convergence rate $r^* = (\lambda^*)^{-1/2} \in (0.4994, 0.5006)$. Numerical issues for a large size network follow from Remark 4.1.

6. Conclusion

This paper performs semistability analysis of the SLSs under both deterministic and random switchings. It is shown that exponential semistability is equivalent to exponential stability of some reduced SLS under given switching rules; the generating functions are exploited to compute convergence rates. These results are used to determine performance of the update schemes of PageRank algorithms subject to switching topology.

Acknowledgments

The authors would like to thank the reviewers and Associate Editor for many helpful comments that improve the organization and quality of the paper.

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