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Brief paper

Quadratic stability and stabilization of bimodal piecewise linear systems[☆]

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ABSTRACT

This paper deals with quadratic stability and feedback stabilization problems for continuous bimodal piecewise linear systems. First, we provide necessary and sufficient conditions in terms of linear matrix inequalities for quadratic stability and stabilization of this class of systems. Later, these conditions are investigated from a geometric control point of view and a set of sufficient conditions (in terms of the zero dynamics of one of the two linear subsystems) for feedback stabilization are obtained.

1. Introduction

Common quadratic Lyapunov functions are among the most popular tools in the stability of linear switching systems, both for state-independent (Agrachev & Liberzon, 2001; Liberzon, 1999) and state-dependent switchings (Johansson & Rantzer, 1998). One of the main reasons behind their popularity is that (whenever exists) such Lyapunov functions can be efficiently computed via linear matrix inequalities. As such, providing sufficient conditions for stability in terms of feasibility of a set of linear matrix inequalities is highly popular in the literature of linear switching systems (Camlibel, Pang, & Shen, 2007; Pavlov, Pogromsky, Van De, & Nijmeijer, 2007). However, these conditions are rather computational in nature and often do not relate to the underlying structure of the system under study, in particular for the case of state-dependent switchings.

In this paper, we focus on a particular class of linear switching systems with state-dependent switchings, namely piecewise bimodal systems with a continuous vector field. In a way, these systems form the simplest class of piecewise affine systems. The main goal of the paper is to investigate the existence of a quadratic Lyapunov function for such systems with an eye towards the underlying geometric structure. It turns out that continuity of the underlying vector field leads to an alternative linear matrix inequality based condition for the existence of a common quadratic Lyapunov function. In turn this alternative condition enables us to look at the feedback stabilization problem from a geometric point of view. Indeed, one of the main results of the paper is to provide sufficient conditions for the existence of a stabilizing static state feedback for bimodal systems. These sufficient conditions are not of linear matrix inequality type but rather geometric conditions and involve the zero dynamics of one of the linear subsystems (and hence also the other due to continuity). We also compare the (open-loop) stabilizability conditions and those for the static state feedback stabilization.

The paper is organized as follows. In Section 2, we first introduce the class of bimodal systems as well as the quadratic stability notion under study. Then, we provide necessary and sufficient conditions for quadratic stability in terms of linear matrix inequalities. Section 3 deals with the feedback stabilization problem and provides necessary and sufficient conditions for the

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existence of a static state feedback rendering the closed-loop system quadratically stable. After comparing the existing open-loop stabilizability conditions and those presented for the feedback stabilization, we provide a set of sufficient conditions for the feedback stabilization in terms of the zero dynamics of one of the linear subsystems. Finally, the paper closes with conclusions in Section 4 and Appendix which presents a technical lemma and its proof for the sake of completeness.

2. Quadratic stability of bimodal systems

Consider the bimodal piecewise affine system given by

$$\dot{x}(t) = \begin{cases} A_1 x(t) + f + bu(t) & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) + f + bu(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and all vectors/ matrices involved are of appropriate dimensions. Throughout this paper, we assume that the right-hand side is a continuous function in x, or equivalently, there exists a vector $e \in \mathbb{R}^n$ such that

$$A_1 - A_2 = ec^T. \tag{2}$$

In this case, the right-hand side of (3) is a Lipschitz continuous function. Hence, for each initial state x_0 and locally-integrable input u there exists a unique absolutely continuous function x such that (3) holds for almost all $t \in \mathbb{R}$ and $x(0) = x_0$.

Such bimodal systems can be encountered in a variety of applications sometimes artificially as approximations of nonlinear systems and sometimes naturally due to the intrinsic piecewise affine behaviour. Next, we illustrate an example for the latter case.

Example 1. As an example, consider the mechanical system shown in Fig. 1. We assume that all the elements are linear. Let x_1 and x_2 denote the displacements of the left and right cart from the tip of the leftmost spring, respectively. Also let the masses of the carts denoted by m_1 (for the left one) and m_2 (for the other), the spring constants by k' (for the leftmost one) and k (for the other), and the damping constant by d. Then, the governing differential equations can be given by

$$m_1\ddot{x}_1 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) - k' \max(-x_1, 0) = 0$$

$$m_2\ddot{x}_2 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = F$$

where F is the force that is applied to the right cart. By denoting the velocities of the left and right cars, respectively, by x_3 and x_4, one arrives at the following bimodal piecewise linear system

$$\dot{x} = \begin{cases} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k+k')}{m_1} & \frac{k}{m_1} & \frac{-d}{m_1} & \frac{d}{m_1} \\ \frac{-k}{m_2} & \frac{k}{m_2} & \frac{-d}{m_2} & \frac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \leqslant 0 \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k}{m_1} & \frac{k}{m_1} & \frac{-d}{m_1} & \frac{d}{m_1} \\ \frac{-k}{m_2} & \frac{k}{m_2} & \frac{-d}{m_2} & \frac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \geqslant 0 \end{cases}$$

$$y = x_1$$

where $x = \text{col}(x_1, x_2, x_3, x_4)$. Note that the condition (2) is satisfied for $e = \text{col}(0, 0, -\frac{k'}{m_1}, 0)$.

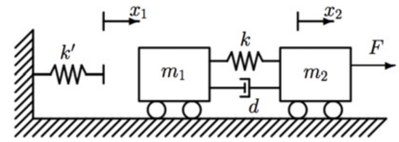

Fig. 1. Linear mechanical system with a one-sided spring.

More realistic applications of bimodal systems arising from one-sided springs can be found in for instance (Doris et al., 2008, Section 3); (Doris, van de Wouw, Heemels, & Nijmeijer, 2010, Section 4). These papers deal with observer design and disturbance attenuation problems, respectively, for a continuous bimodal system arising as a mathematical model of two steel beams, one supported at both ends by two leaf springs whereas the other (which is located parallel to the first one) clamped at both ends acting as a one-sided spring.

Other control systems applications in which bimodal systems arise intrinsically include for instance (van der Heijden, Serrarens, Camlibel, & Nijmeijer, 2007) where clutch engagement problem has been studied and Vanek, Bokor, Balas, and Arndt (2007).

In addition to engineering applications, continuous bimodal systems are also encountered in various other contexts. Examples from the area of dynamical systems include Carmona, Fernandez-Garcia, Fernandez-Sanchez, Garcia-Medina, and Teruel (2012), Carmona, Fernandez-Garcia, and Freire (2011), Michelson (1986) and Webster and Elgin (2003). In what follows, we illustrate a bimodal system arising in the study of certain partial differential equations.

Example 2. The so-called Michelson system was originally studied in Michelson (1986) in the context of the steady solutions of the Kuramoto–Sivashinsky (partial differential) equations and further studied in for instance (Carmona, Fernandez-Sanchez, & Teruel, 2008; Webster & Elgin, 2003). It can be given (after a suitable similarity transformation) as a bimodal system of the form (1) where

$$A_i = \begin{bmatrix} 0 & -1 & (-1)^i \lambda(1+\lambda^2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{for } i \in \{1, 2\},$$

$$f^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad c^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

and $\lambda \in \mathbb{R}$ is a constant. Note that the continuity assumption (2) is satisfied with $e^T = \begin{bmatrix} -2\lambda(1+\lambda)^2 & 0 & 0 \end{bmatrix}$.

Next, we focus on particular cases of (1) where $f = b = 0$, that is continuous bimodal systems of the form:

$$\dot{x}(t) = \begin{cases} A_1 x(t) & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) & \text{if } c^T x(t) \geqslant 0. \end{cases} \tag{3}$$

We say that the bimodal system (3) is *quadratically stable* if there exists a quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0$ for all $x \neq 0 \in \mathbb{R}^n$ and $\dot{V}(x(t)) < 0$ for all state trajectories x of (3) with $x(t) \not\equiv 0$. Equivalently, the system (3) is quadratically stable if and only if there exists a common quadratic Lyapunov function for the linear subsystems, that is there exists a symmetric positive definite matrix P such that

$$A_i^T P + PA_i < 0 \tag{4}$$

with $i \in \{1, 2\}$.

Note that quadratic stability of systems of the form (3) naturally yields (local) Lyapunov stability (see e.g. Khalil (2002)) of possibly non-zero equilibrium points of bimodal systems of the form:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + f & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) + f & \text{if } c^T x(t) \geqslant 0. \end{cases} \tag{5}$$

The following theorem gives an alternative characterization for the existence of a common quadratic Lyapunov function $x \mapsto \frac{1}{2}x^T Px$ satisfying (4) by exploiting the continuity condition (2).

Theorem 3. *The following statements are equivalent:*

(1) *the bimodal system (3) is quadratically stable;*
(2) *there exists a symmetric positive definite matrix P such that*

$$(A_1 - \mu e c^T)^T P + P(A_1 - \mu e c^T) < 0 \tag{6}$$

for all $\mu \in [0, 1]$*;*
(3) *there exists a symmetric positive definite matrix K such that*

$$\begin{bmatrix} A_1^T K + K A_1 & Ke - c \\ e^T K - c^T & -2 \end{bmatrix} < 0. \tag{7}$$

To prove this theorem, we need the following auxiliary result which can be derived from the proof of Theorem 1 in Willems (1973).

Lemma 4. *There exists a symmetric positive definite matrix P such that*

$$(A_1 - \mu e c^T)^T P + P(A_1 - \mu e c^T) < 0$$

for all $\mu \in (\alpha, \beta)$ *only if there exists* $\gamma > 0$ *such that* $Q = \gamma P$ *satisfies*

$$A_1^T Q + Q A_1 + \left(Qe - \frac{\alpha+\beta}{2}c\right)\left(Qe - \frac{\alpha+\beta}{2}c\right)^T - \alpha\beta c c^T \leqslant 0.$$

Proof of Theorem 3. 1 ⇒ 2: this follows from the observation that $(A_1 - \mu e c^T)^T P + P(A_1 - \mu e c^T)$ with $\mu \in [0, 1]$ is a convex combination of the terms $A_1^T P + P A_1$ and $A_2^T P + P A_2 = (A_1 - e c^T)^T P + P(A_1 - e c^T)$.

2 ⇒ 3: due to continuity there exists sufficiently small $\varepsilon > 0$ such that

$$(A_1 - \mu e c^T)^T P + P(A_1 - \mu e c^T) < 0 \tag{8}$$

for all $\mu \in [0, 1 + \varepsilon]$. Then, it follows from Lemma 4 that there exists $\gamma > 0$ such that $Q = \gamma P$ satisfies

$$A_1^T Q + Q A_1 + \left(Qe - \frac{1+\varepsilon}{2}c\right)\left(Qe - \frac{1+\varepsilon}{2}c\right)^T \leqslant 0.$$

By taking $K = \frac{2}{1+\varepsilon}Q$, we obtain

$$A_1^T K + K A_1 + \frac{1+\varepsilon}{2}(Ke - c)(Ke - c)^T \leqslant 0. \tag{9}$$

Since $(Ke - c)(Ke - c)^T$ is positive semi-definite and $\varepsilon > 0$, we further get

$$A_1^T K + K A_1 + \frac{1}{2}(Ke - c)(Ke - c)^T \leqslant 0. \tag{10}$$

Now, we claim that

$$A_1^T K + K A_1 + \frac{1}{2}(Ke - c)(Ke - c)^T < 0. \tag{11}$$

To see this, let $x \in \mathbb{C}^n$ such that

$$x^H\left(A_1^T K + K A_1 + \frac{1}{2}(Ke - c)(Ke - c)^T\right)x = 0. \tag{12}$$

Then, it follows from (9) that

$$\frac{\varepsilon}{2}x^H(Ke - c)(Ke - c)^T x \leqslant 0.$$

Since $\varepsilon > 0$, we can conclude that

$$(Ke - c)^T x = 0.$$

Now, it follows from (12) that

$$x^H(A_1^T K + K A_1)x = 0.$$

Since $K = \frac{2\gamma}{1+\varepsilon}P$, we obtain

$$x^H(A_1^T P + P A_1)x = 0.$$

Therefore, we get $x = 0$ since $A_1^T P + P A_1$ is negative definite due to (8) with $\mu = 0$. Hence, we showed that (11) holds. The LMI (7) readily follows from (11) by employing a Schur complement argument.

3 ⇒ 1: it follows immediately from (7) that $A_1^T K + K A_1 < 0$. Now, we claim that

$$(A_1 - e c^T)^T K + K(A_1 - e c^T) < 0.$$

To show this, we first obtain

$$A_1^T K + K A_1 + \frac{1}{2}(Ke - c)(Ke - c)^T < 0$$

by taking the Schur complement with respect to -2 of the left hand side of (7). Note that

$$\begin{aligned} 0 &> A_1^T K + K A_1 + \frac{1}{2}(Ke - c)(Ke - c)^T \\ &= (A_1 - e c^T)^T K + K(A_1 - e c^T) + \frac{1}{2}(Ke + c)(Ke + c)^T \\ &\geqslant (A_1 - e c^T)^T K + K(A_1 - e c^T). \end{aligned}$$

Therefore, we get $(A_1 - e c^T)^T K + K(A_1 - e c^T) < 0$. ■

Note that Theorem 3 shows that one needs to solve the $(n+1) \times (n+1)$ LMI (7) in order to check the existence of a common Lyapunov function (4) given by two $n \times n$ LMIs. It also shows that the existence of a common quadratic Lyapunov function is intimately related to a certain type of passivity of the linear system given by the quadruple $(A_1, e, c, 1)$. More interestingly, Theorem 3 leads to a number of geometric sufficient conditions for the feedback stabilization of bimodal systems as discussed in what follows.

3. Quadratic feedback stabilization of bimodal systems

We turn our attention to bimodal piecewise linear systems with inputs of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + bu(t) & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) + bu(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{13}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, and all the matrices involved are of appropriate dimensions. We assume that the right-hand side of (13) is continuous in both x and u, i.e. the continuity condition (2) holds. As such, for each initial state x_0 and locally-integrable input u there exists a unique absolutely continuous function $x^{x_0,u}$ such that (13) holds for almost all $t \in \mathbb{R}$ and $x^{x_0,u}(0) = x_0$.

The problem we will address is under what conditions there exists a state feedback of the form $u = k^T x$ which renders the closed-loop bimodal system

$$\dot{x}(t) = \begin{cases} (A_1 + bk^T)x(t) & \text{if } c^T x(t) \leqslant 0 \\ (A_2 + bk^T)x(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{14}$$

quadratically stable. In the case such a feedback exists, we say that the bimodal system (13) is *quadratically feedback stabilizable*.

An intimately related concept is the open-loop stabilizability. We say that the bimodal system (13) is *(open-loop) stabilizable* if for each initial state x_0 there exists a locally-integrable input u such that $x^{x_0,u}(t)$ asymptotically vanishes as t tends to infinity. Necessary and sufficient conditions for the stabilizability of bimodal systems were presented in Camlibel, Heemels, and Schumacher (2008b, Thm. 2.3).

Theorem 5 (Camlibel et al., 2008b, Thm. 2.3)**.** *Suppose that the transfer function $c^T(sI - A_1)^{-1}b$ is not identically zero. Then, the bimodal system (13) is stabilizable if and only if*

(1) *the pair $(A_1, \begin{bmatrix} b & e \end{bmatrix})$ is stabilizable; and*
(2) *the implication*

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0, \quad 0 \leqslant \lambda \in \mathbb{R}, \ v \neq 0, \ i = 1, 2 \\ \implies \mu_1 \mu_2 > 0$$

holds.

In the following theorem, we state necessary and sufficient conditions for feedback stabilization in terms of linear matrix inequalities. Later, we will investigate geometric sufficient conditions based on this theorem. To state the theorem, we first introduce a notational convention: for a symmetric matrix M and a subspace $\mathcal{W}$ of the underlying linear space, we write $M \overset{\mathcal{W}}{<} 0$ meaning that $w^T M w < 0$ for all nonzero $w \in \mathcal{W}$.

Theorem 6. *The following statements are equivalent:*

(1) *the bimodal system (13) is quadratically feedback stabilizable;*
(2) *there exist k and $P = P^T > 0$ such that*

$$\begin{bmatrix} (A_1 + bk^T)^T P + P(A_1 + bk^T) & Pe - c \\ e^T P - c^T & -2 \end{bmatrix} < 0; \tag{15}$$

(3) *there exists $Q = Q^T > 0$ such that*

$$\begin{bmatrix} A_1 Q + Q A_1^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} \overset{\mathcal{W}}{<} 0 \tag{16}$$

where $\mathcal{W} = \ker b^T \times \mathbb{R}$.

If the statement 3 holds, one can choose $k^T = -\alpha b^T Q^{-1}$ for some sufficiently large $\alpha > 0$.

Proof of Theorem 6. 1 ⇔ 2: this readily follows from the application of Theorem 3 to the bimodal system (14).

2 ⇒ 3: by pre- and post-multiplying (15) by $\begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix}$ where $Q = P^{-1}$, we obtain

$$\begin{bmatrix} A_1 Q + Q A_1^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} + \begin{bmatrix} bk^T Q + Qkb^T & 0 \\ 0 & 0 \end{bmatrix} < 0. \tag{17}$$

Then, it follows from (17) that

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A_1 Q + Q A_1^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} < 0 \tag{18}$$

for all $x \in \ker b^T$ and $u \in \mathbb{R}$.

3 ⇒ 2: take $k^T = -\alpha b^T Q^{-1}$. It follows from Finsler's lemma (see e.g. Boyd, El Ghaoui, Feron, and Balakrishnan (1994)) that

$$\begin{bmatrix} A_1 Q + Q A_1^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} + \begin{bmatrix} bk^T Q + Qkb^T & 0 \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} A_1 Q + Q A_1^T - 2\alpha bb^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} < 0 \tag{19}$$

for all sufficiently large $\alpha > 0$. By pre- and post-multiplying this LMI by $\begin{bmatrix} Q^{-1} & 0 \\ 0 & -1 \end{bmatrix}$ and defining $P = Q^{-1}$, we obtain

$$\begin{bmatrix} (A_1 + bk^T)^T P + P(A_1 + bk^T) & Pe - c \\ e^T P - c^T & -2 \end{bmatrix} < 0. \quad \blacksquare$$

Remark 7. Although they are different in nature, conditions of Theorem 5 are necessary for the feedback stabilization and hence should imply those of Theorem 6. To see this implication, note first that the LMI (16) readily implies that (A_1, b) and hence $(A_1, \begin{bmatrix} b & e \end{bmatrix})$ is stabilizable. To see that the second condition of Theorem 5 also follows from (16), let $\lambda \geqslant 0$, $v \neq 0$ and μ_i with $i = 1, 2$ satisfy

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0.$$

This yields that $v \in \ker b^T$, $v^T A_i + \mu_i c^T = \lambda v^T$, and $\mu_2 = \mu_1 + v^T e$. As such, we have

$$\begin{bmatrix} v \\ \mu_1 \end{bmatrix}^T \begin{bmatrix} A_1 Q + Q A_1^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} \begin{bmatrix} v \\ \mu_1 \end{bmatrix} = 2\lambda v^T Q v - 2\mu_1 \mu_2.$$

Since the right-hand side is negative, λ is nonnegative, and Q is positive definite, one can conclude that $\mu_1 \mu_2 > 0$.

Theorem 6 provides necessary and sufficient conditions for the quadratic feedback stabilizability in terms of certain linear matrix inequalities. Next, we further investigate these linear matrix inequalities with an eye towards the geometric structure of the linear subsystems of the bimodal systems (13). To do so, we first quickly introduce some notation.

Consider the linear system $\Sigma(A, b, c)$

$$\dot{x} = Ax + bu \tag{20}$$

$$y = c^T x \tag{21}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output.

A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called *controlled invariant* if there exists $f \in \mathbb{R}^n$ such that $(A - bf^T)\mathcal{V} \subseteq \mathcal{V}$. Let $\mathcal{V}^*(A, b, c^T)$ be the largest controlled invariant subspace that is contained in $\ker c^T$. A subspace $\mathcal{T}$ is called *conditioned invariant* if there exists $g \in \mathbb{R}^n$ such that $(A - gc^T)\mathcal{T} \subseteq \mathcal{T}$. Let $\mathcal{T}^*(A, b, c^T)$ be the smallest conditioned invariant subspace that contains $\operatorname{im} b$.

These two subspaces play a key role in the study of linear systems with geometric approach (see e.g. Trentelman, Stoorvogel, & Hautus, 2001). The subspace $\mathcal{V}^*(A, b, c^T)$ consists of initial states for which one can keep the output of the system (20) identically zero for all times. Also, the subspace $\mathcal{T}^*(A, b, c^T)$ admits a characterization in terms of system trajectories. However, it is rather involved (see e.g. Trentelman et al., 2001, Section 8.3).

It is well-known (see e.g. Aling & Schumacher, 1984, Prop. 4) that the transfer function $c^T(sI - A)^{-1}b$ is invertible as a rational matrix if and only if $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$, and $b \neq 0 \neq c$.

The continuity condition (2) has a number of intriguing and useful consequences. Indeed, it can be easily verified (see e.g. Camlibel, Heemels, and Schumacher (2008a, Prop. II.1)) that

$$\mathcal{V}^*(A_1, b, c^T) = \mathcal{V}^*(A_2, b, c^T)$$
$$\mathcal{T}^*(A_1, b, c^T) = \mathcal{T}^*(A_2, b, c^T).$$

Together with the invertibility conditions, these equalities imply that the transfer function $c^T(sI - A_1)^{-1}b$ is invertible if and only if so is $c^T(sI - A_2)^{-1}b$.

With this preparation, we are ready to provide geometric sufficient conditions for quadratic feedback stabilization.

Lemma 8. *Suppose that the transfer function $c^T(sI - A_1)^{-1}b$ is not identically zero and $\mathcal{V}^*(A_1, b, c^T) = \{0\}$. Then, the bimodal system (13) is quadratically feedback stabilizable.*

Proof. Since the transfer function $c^T(sI - A_1)^{-1}b$ is not identically zero and $\mathcal{V}^*(A_1, b, c^T) = \{0\}$, one has $c^T(sI - A_1)^{-1}b = p_0/(s^n + q_{n-1}s^{n-1} + \cdots + q_1 s + q_0)$ where p_0 and q_i with $i = 0, 1, \ldots, n-1$ are some real numbers. Then, one can take

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{n-1} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p_0 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

without loss of generality.

In view of Theorem 6, it is enough to show that there exists a positive definite matrix Q such that (16) holds. Note that any vector $v \in \ker b^T$ is of the form $v^T = (v_1, v_2, \ldots, v_{n-1}, 0)^T$. Let $\tilde{v} = (v_1, v_2, \ldots, v_{n-1})$. Straightforward calculations yield that

$$\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} A_1Q + QA_1^T & Qc - e \\ c^TQ - e^T & -2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \tilde{v} \\ w \end{bmatrix}^T \begin{bmatrix} \tilde{Q} + \tilde{Q}^T & \tilde{q} - \tilde{e} \\ \tilde{q}^T - \tilde{e}^T & -2 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ w \end{bmatrix}$$

where $\tilde{Q} = Q_{\{2,3,\ldots,n\},\{1,2,\ldots,n-1\}}$, $\tilde{q} = Q_{\{1,2,\ldots,n-1\},\{1\}}$, $\tilde{e} = e_{\{1,2,\ldots,n-1\}}$.[2] Then, the application of Lemma 11 with $m = n$, $\eta = \tilde{e}$, $\alpha = \beta = 1$, and $\gamma = 2$ yields a symmetric matrix Q such that

$$Q > \alpha I \quad \text{and} \quad \begin{bmatrix} \tilde{Q} + \tilde{Q}^T & \tilde{q} - \tilde{e} \\ \tilde{q}^T - \tilde{e}^T & -2 \end{bmatrix} < 0.$$

Therefore, we have

$$\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} A_1Q + QA_1^T & Qc - e \\ c^TQ - e^T & -2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} < 0$$

for all $v \in \ker b^T$. Then, the claim follows from Theorem 6. ■

Since the transfer function $c^T(sI - A_1)^{-1}b$ is not identically zero due to the hypotheses of the lemma above, it must be invertible as rational function. Together with $\mathcal{V}^* = \{0\}$, this implies that $c^T(sI - A_1)^{-1}b$ has no zeros. In the following theorem, we show that quadratic feedback stabilization can be achieved in the case all zeros are on the open left half plane, that is when the system $\Sigma(A_1, b, c^T)$ is minimum phase.

Theorem 9. *Suppose that the transfer function $c^T(sI - A_1)^{-1}b$ is not identically zero. Let $\mathcal{V}^* = \mathcal{V}^*(A_1, b, c^T)$ and f be such that $(A_1 - bf^T)\mathcal{V}^* \subseteq \mathcal{V}^*$. Suppose that $(A_1 - bf^T)|_{\mathcal{V}^*}$ is Hurwitz. Then, the bimodal system (13) is quadratically feedback stabilizable.*

Proof. Our first aim is to put the system (13) into a certain canonical form. By applying the feedback law $u = -f^Tx + v$, where v is the new input, we get

$$\dot{x}(t) = \begin{cases} (A_1 - bf^T)x(t) + bv(t) & \text{if } c^Tx(t) \leqslant 0 \\ (A_2 - bf^T)x(t) + bv(t) & \text{if } c^Tx(t) \geqslant 0. \end{cases} \tag{22}$$

Clearly, this bimodal system is quadratically feedback stabilizable if and only if so is (13). Let $\mathcal{T}^* = \mathcal{T}^*(A_1, b, c^T)$. Since the transfer function $c^T(sI - A_1)^{-1}b$ is invertible, $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$. Let the dimensions of the subspaces $\mathcal{V}^*$ and $\mathcal{T}^*$ be n_1 and n_2, respectively. Also let the vectors $\{x_1, x_2, \ldots, x_n\}$ be a basis for $\mathbb{R}^n$, such that the first n_1 vectors form a basis for $\mathcal{V}^*$ and the last n_2 for $\mathcal{T}^*$. Let g be such that $(A_1 - gc^T)\mathcal{T}^* \subseteq \mathcal{T}^*$. Then, one immediately gets

$$b = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

in the new coordinates as $\mathcal{V}^* \subseteq \ker c^T$ and $\operatorname{im} b \subseteq \mathcal{T}^*$. Note that $(A_1 - bf^T - gc^T)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $(A_1 - bf^T - gc^T)\mathcal{T}^* \subseteq \mathcal{T}^*$ according to Camlibel et al. (2008a, Prop. II.1). Therefore, the matrix $(A_1 - bf^T - gc^T)$ should be of the form $\left[\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right]$ in the new coordinates where the row (column) blocks have n_1 and n_2 rows (columns), respectively. With the above partitions, one gets

$$A_1 - bf^T = \begin{bmatrix} A_{11} & g_1c_2^T \\ 0 & A_{22} \end{bmatrix}.$$

In view of Theorem 6, it suffices to prove the statement by showing the existence of a positive definite matrix Q such that

$$\begin{bmatrix} (A_1 - bf^T)Q + Q(A_1 - bf^T)^T & Qc - e \\ c^TQ - e^T & -2 \end{bmatrix} \overset{\mathcal{W}}{<} 0 \tag{23}$$

where $\mathcal{W} = \ker b^T \times \mathbb{R}$. Let a symmetric matrix Q (partitioned accordingly) be of the form $Q = \left[\begin{smallmatrix} Q_1 & 0 \\ 0 & Q_2 \end{smallmatrix}\right]$. Straightforward calculations yield

$$\left[\begin{array}{c|c} (A_1 - bf^T)Q + Q(A_1 - bf^T)^T & Qc - e \\ \hline c^TQ - e^T & -2 \end{array}\right] = \left[\begin{array}{cc|c} A_{11}Q_1 + Q_1A_{11}^T & g_1c_2^TQ_2 & -e_1 \\ Q_2c_2g_1^T & A_{22}Q_2 + Q_2A_{22}^T & Q_2c_2 - e_2 \\ \hline -e_1^T & c_2^TQ_2 - e_2 & -2 \end{array}\right].$$

In the new coordinates, $v \in \ker b^T$ if and only if $v = \operatorname{col}(v_1, v_2)$ and $v_2 \in \ker b_2^T$. Note that $\mathcal{V}^*(A_{22}, b_2, c_2^T) = \{0\}$. Then, Lemma 8 implies that there exists a symmetric positive definite matrix Q_2 such that

$$Q_2 > 0$$

$$\begin{bmatrix} A_{22}Q_2 + Q_2A_{22}^T & Q_2c_2 - e_2 \\ c_2^TQ_2 - e_2 & -2 \end{bmatrix} \overset{\mathcal{W}}{<} 0.$$

Note that $(A_1 - bf^T)|_{\mathcal{V}^*}$ can be identified with A_{11} and hence A_{11} is Hurwitz due to the hypotheses. Therefore, for any $R = R^T < 0$ one can find $Q_1 = Q_1^T > 0$ such that $A_{11}Q_1 + Q_1A_{11}^T = R$. Then, it follows from a Schur complement argument that we can choose Q_1 and hence Q such that (23) holds. ■

Remark 10. Stability of possibly discontinuous piecewise linear systems has been addressed in Iwatani and Hara (2006). Among many other useful and interesting results, the paper (Iwatani &

[2] For a matrix $M \in \mathbb{R}^{k\times\ell}$, we write $M_{\alpha,\beta}$ where $\alpha \subseteq \{1, 2, \ldots, k\}$ and $\beta \subseteq \{1, 2, \ldots, \ell\}$ to denote the submatrix of M formed by selecting the rows indexed by α and columns by β.

Hara, 2006) provides sufficient conditions for the feedback stabilization of possibly discontinuous bimodal systems in Proposition 20. As a special case, these results can be employed for continuous bimodal systems of the form (13). However, Proposition 20 requires the conditions

i. $\min\{r \mid c^T A_1^{r-1} b \neq 0\} \leqslant 2$; and
ii. at least one of the pairs (A_i, b) with $i \in \{1, 2\}$ is controllable

to be met for the system (13). None of these conditions are necessary to apply our results in Lemma 8 and Theorem 9. Further, note that the first condition is satisfied if and only if

$$(c^T b \neq 0) \quad \text{or} \quad (c^T b = 0 \text{ and } c^T A_1 b \neq 0).$$

However, Lemma 8 proves that the system is feedback stabilizable if

$$c^T b = c^T A_1 b = \cdots = c^T A_1^{n-2} b = 0 \quad \text{and} \quad c^T A_1^{n-1} b \neq 0.$$

In addition, Lemma 8 and Theorem 9 require only stabilizability of $(A_1, \begin{bmatrix} b & e \end{bmatrix})$ as discussed in Remark 7. This requirement is weaker than one of the pairs being controllable. As such, the results we presented in this paper are stronger than those of Iwatani and Hara (2006, Prop. 20) in the context of continuous bimodal systems. Nevertheless, Iwatani and Hara (2006, Prop. 20) can deal with discontinuous bimodal systems which we do not consider in this paper.

4. Conclusions

In this paper, we investigated the quadratic stability of continuous bimodal piecewise linear systems. After establishing necessary and sufficient conditions for quadratic stability and feedback stabilization in terms of linear matrix inequalities, we provide sufficient conditions for feedback stabilization in terms of the zero dynamics of one of the two linear subsystems. Also we discuss the relations between the existing open-loop stabilizability conditions and those for the feedback stabilization. Based on the approach and the results of this paper, several further research possibilities arise: (i) relaxing the continuity assumption; (ii) extending the results to multi-modal piecewise linear systems; and (iii) investigating piecewise quadratic stability.

Appendix

Lemma 11. *For any integer $m \geqslant 1$, any vector $\eta \in \mathbb{R}^{m-1}$, and any positive real numbers α, β, γ, there exists a symmetric matrix $\Lambda \in \mathbb{R}^{m\times m}$ such that*

$$\Lambda > \alpha I \tag{A.1}$$

$$\tilde{\Lambda} + \tilde{\Lambda}^T < -\beta I \tag{A.2}$$

$$\begin{bmatrix} \tilde{\Lambda} + \tilde{\Lambda}^T & \tilde{\lambda} - \eta \\ \tilde{\lambda}^T - \eta^T & -\gamma \end{bmatrix} < 0 \tag{A.3}$$

where $\tilde{\Lambda} = \Lambda_{\{2,3,\ldots,m\},\{1,2,\ldots,m-1\}}$ and $\tilde{\lambda} = \Lambda_{\{1,2,\ldots,m-1\},\{1\}}$.

Proof. It trivially holds for $m = 1$. Suppose that it holds for $m = \ell$. Take a vector $\zeta \in \mathbb{R}^{\ell}$. Let $\tilde{\zeta} = \zeta_{\{1,2,\ldots,\ell-1\}}$. Since this is a $(m-1)$-vector, there exists a symmetric positive definite matrix $\Lambda \in \mathbb{R}^{m\times m}$ such that

$$\Lambda > \alpha I \tag{A.4}$$

$$\tilde{\Lambda} + \tilde{\Lambda}^T < -\beta I \tag{A.5}$$

$$\begin{bmatrix} \tilde{\Lambda} + \tilde{\Lambda}^T & \tilde{\lambda} - \tilde{\zeta} \\ \tilde{\lambda}^T - \tilde{\zeta}^T & -\gamma \end{bmatrix} < 0. \tag{A.6}$$

Now, define $\Theta \in \mathbb{R}^{(\ell+1)\times(\ell+1)}$ as follows

$$\Theta = \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\Lambda} & \begin{matrix}\hat{\lambda} \\ \mu\end{matrix} \\ \hdashline \hat{\lambda}^T & \mu & \rho \end{array}\right]$$

where $\hat{\lambda} = -\Lambda_{\{2,3,\ldots,\ell\},\{\ell\}}$, μ and ρ are real numbers. Let $\tilde{\Theta} = \Theta_{\{2,3,\ldots,\ell+1\},\{1,2,\ldots,\ell\}}$ and $\tilde{\theta} = \Theta_{\{1,2,\ldots,\ell\},\{1\}}$. It suffices to prove that μ and ρ can be chosen so that

$$\Theta > \alpha I \tag{A.7}$$

$$\tilde{\Theta} + \tilde{\Theta}^T < -\beta I \tag{A.8}$$

$$\begin{bmatrix} \tilde{\Theta} + \tilde{\Theta}^T & \tilde{\theta} - \zeta \\ \tilde{\theta}^T - \zeta^T & -\gamma \end{bmatrix} < 0. \tag{A.9}$$

Note that

$$\tilde{\Theta} + \tilde{\Theta}^T = \begin{bmatrix} \tilde{\Lambda} + \tilde{\Lambda}^T & 0 \\ 0 & 2\mu \end{bmatrix} \tag{A.10}$$

and that

$$\left[\begin{array}{c:c} \tilde{\Theta} + \tilde{\Theta}^T & \tilde{\theta} - \zeta \\ \hdashline \tilde{\theta}^T - \zeta^T & -\gamma \end{array}\right] = \left[\begin{array}{cc:c} \tilde{\Lambda} + \tilde{\Lambda}^T & 0 & \tilde{\lambda} - \tilde{\zeta} \\ 0 & 2\mu & \Lambda_{1n} - \zeta_m \\ \hdashline \tilde{\lambda}^T - \tilde{\zeta}^T & \Lambda_{1n} - \zeta_m & -\gamma \end{array}\right].$$

It follows from (A.6) that this matrix can be made negative definite by choosing a negative μ sufficiently small. Once μ is fixed, one can choose ρ sufficiently large to satisfy $\Theta > \alpha I$ as $\Lambda > \alpha I$. ■

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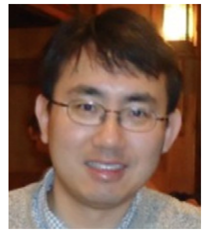

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