Stability and Stabilization of Relative Equilibria of a Dumbbell Body in Central Gravity

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Abstract

A dumbbell-shaped rigid body can be used to represent certain large spacecraft or asteroids with bimodal mass distributions. Such a dumbbell body is modeled here as two identical mass particles connected by a rigid, massless link. Equations of motion for the five degrees of freedom of the dumbbell body in a central gravitational field are obtained. The equations of motion characterize three orbit degrees of freedom, two attitude degrees of freedom, and the coupling between them. The system has a continuous symmetry due to a cyclic variable associated with the angle of right ascension of the dumbbell body. Reduction with respect to this symmetry gives a reduced system with four degrees of freedom. Relative equilibria, corresponding to

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circular orbits, are obtained from these reduced equations of motion; stability of these relative equilibria is assessed. It is shown that unstable relative equilibria can be stabilized by suitable attitude feedback control of the dumbbell.

Nomenclature

r = radial distance from origin to center of mass of dumbbell body

ν = angle of right ascension of center of mass of dumbbell body

λ = angle of declination of center of mass of dumbbell body

$\vec{e_r}$ = unit vector along local vertical (radial) direction

$\vec{e_\nu}$ = unit vector along direction of increasing ν

$\vec{e_\lambda}$ = unit vector along direction of increasing λ

$\vec{e_x}$ = unit vector along longitudinal axis of dumbbell

$\vec{e_y},\ \vec{e_z}$ = orthogonal unit vectors spanning plane perpendicular to dumbbell axis

$\vec{\omega_L}$ = angular velocity vector of LVLH coordinate frame with respect to inertial frame

$\vec{\omega_I}$ = angular velocity vector of body-fixed coordinate frame with respect to inertial frame

$e_1,\ e_2,\ e_3$ = standard basis column vectors of $\mathbb{R}^3$

m = mass of each end mass of dumbbell-shaped body

$2l$ = length of rigid link connecting the two end masses of the dumbbell

μ = gravitational force constant

$$\mathrm{SO}(3) \;=\; \text{group of rigid-body rotations in } \mathbb{R}^3$$

$$R \in \mathrm{SO}(3) \;=\; \text{rotation matrix from body-fixed frame to LVLH frame}$$

$$\mathfrak{so}(3) \;=\; \text{Lie algebra of SO(3), identified with } \mathbb{R}^3$$

$$\omega \in \mathfrak{so}(3) \;=\; \text{angular velocity of dumbbell body with respect to LVLH frame}$$

$$\|\cdot\| \;=\; \text{Euclidean norm, or two-norm in } \mathbb{R}^3$$

$$\widehat{(\cdot)} \;=\; \text{adjoint representation of } \mathfrak{so}(3) \text{ as } 3\times 3 \text{ skew-symmetric matrices}$$

$$\mathbf{Q} \;=\; \text{configuration manifold for dumbbell body in central gravity}$$

$$\mathtt{T}\mathbf{Q} \;=\; \text{velocity state space for dumbbell body in central gravity}$$

$$\mathbb{S} \;=\; \text{the one-dimensional circle, or } \mathbb{R}/\{2\pi\}$$

1 Introduction

Equations of motion are derived for a dumbbell-shaped body in a central gravitational field. The equations of motion describe the translational or orbit dynamics and the rotational or attitude dynamics, and their coupling. The dumbbell consists of two ideal mass particles of identical mass m connected by a rigid, massless link of length $2l$. The dumbbell can rotate and translate in three dimensions under the action of gravity forces. A gravity force acts on each individual mass particle of the dumbbell. The differential gravity effects about the dumbbell's center of mass play a crucial role in its dynamics.

This model is similar to the dumbbell spacecraft models in [1, 2], which treat dynamics and control of an elastic dumbbell restricted to planar motion. The full dynamics of this model is treated in [1], while the reduced dynamics is treated in [2], assuming attitude and shape actuation only. These cited models include flexibility effects in the link connecting the two mass particles. For simplicity, flexibility effects are not included in the models developed in this paper. The models here are also similar to the dumbbell spacecraft model in [3], which treats the orbit and attitude dynamics of a dumbbell spacecraft moving in a plane. The dumbbell can also be considered as a special case of a "full body" treated in [4]. In this paper, we treat both the full and the reduced dynamics of a dumbbell body in three spatial dimensions.

The dumbbell can also be viewed as a model of a tethered spacecraft. Typical assumptions for tethered spacecraft include negligible elastic effects, and a taut tether corresponding to a positive tension force in the tether. Due to its relevance, some of this prior work is now described. Deployment, station keeping, and retrieval of tethers have been studied in [5]. Attitude dynamics issues for tethered spacecraft have been treated in [6, 7, 8]. Orbital dynamics issues for tethered spacecraft have been treated in [9, 10]. None of these papers provides a comprehensive model that includes both orbit and attitude degrees of freedom. This paper makes a contribution to this problem for the simplified dumbbell model.

The dumbbell model is simple, but effective in demonstrating complex dynamics that can arise when it is in orbit about a massive central spherical body. It provides a framework for studying the orbital degrees of freedom, the attitude degrees of freedom, and the coupling between them. The dynamics of large extended bodies in central gravity present significant analytical challenges. In this paper, we introduce new orbital and attitude problems that have not been previously studied in the published literature. We obtain relative equilibria for the full dynamics of the dumbbell body in a central gravitational field; these correspond to the equilibria of the reduced dynamics. The reduced dynamics are obtained by the process of Routh reduction ([11, 12]), and stability properties of the relative equilibria are obtained from the reduced dynamics. Control laws based on potential shaping ([13, 14, 15]), using attitude feedback for stabilization of the unstable relative relative equilibria, are also developed and presented.

The present paper can also be viewed as an extension of [16]. In that paper coupling between translational and rotational degrees of freedom was studied. However, [16] did not include a central body gravity field, so the results in that paper are not directly applicable to the problems considered here.

2 Equations of Motion

An inertial coordinate frame is chosen such that its origin is at the center of a large spherical central body, e.g. the Earth. This inertial coordinate frame is defined by three mutually-orthogonal axes. It is convenient to express the orbital motion in terms of spherical coordinates r, ν, and λ, for the position of the center of mass of the dumbbell in the inertial frame, as shown in Figure 1. This spherical coordinate frame is also termed the Local Vertical Local Horizontal (LVLH) coordinate frame. In the LVLH coordinate frame, $\vec{e_r}$, $\vec{e_\nu}$ and $\vec{e_\lambda}$ form a mutually orthogonal right-handed set of unit vectors.

Figure 1 gives a graphical illustration of the dumbbell in the inertial and LVLH coordinate frames. In addition, a coordinate frame is introduced that is fixed to the dumbbell; its origin is at the dumbbell center of mass. The unit vectors $\vec{e_x}$, $\vec{e_y}$ and $\vec{e_z}$ form a mutually orthogonal, body-fixed, right-handed set of unit vectors. Hence, there are three different coordinate frames, each of which consists of mutually-orthogonal axes consistent with the right hand rule. In the subsequent development, substantial care must be taken when representations in $\mathbb{R}^3$ are used to express a vector in one of these coordinate frames.

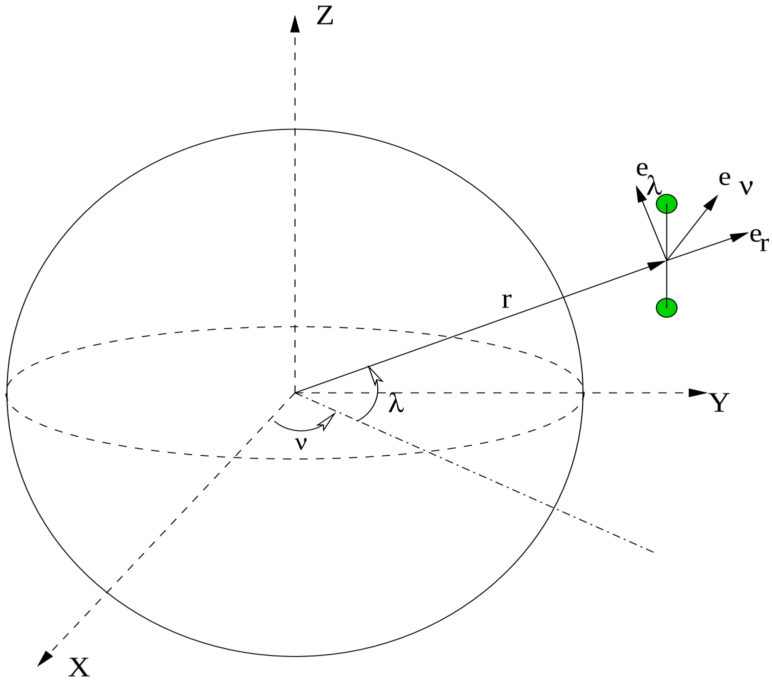

Figure 1: Dumbbell in Local Vertical Local Horizontal coordinate frame.

The angular velocity vector of the LVLH coordinate frame with respect to the inertial frame is

$$\vec{\omega}_L = \dot{\nu} \sin \lambda \vec{e}_r - \dot{\lambda} \vec{e}_\nu + \dot{\nu} \cos \lambda \vec{e}_\lambda. \tag{1}$$

The angular velocity vector of the body-fixed coordinate frame with respect to the inertial frame is denoted by $\vec{\omega}_I$. The position vectors of the two end masses are given by

$$\vec{x}_1 = \vec{x} + l\vec{e}_x,$$

$$\vec{x}_2 = \vec{x} - l\vec{e}_x,$$

where $2l$ is the length of the dumbbell.

The inertial velocities of the end masses in the LVLH frame are

$$\dot{\vec{x}}_1 = \dot{\vec{x}} + l(\vec{\omega_I} \times \vec{e_x}),$$

$$\dot{\vec{x}}_2 = \dot{\vec{x}} - l(\vec{\omega_I} \times \vec{e_x}).$$

The kinetic energy is given by

$$T = \frac{m}{2}(\|\dot{\vec{x}}_1\|^2 + \|\dot{\vec{x}}_2\|^2).$$

Using the expressions for $\dot{\vec{x}}_1$ and $\dot{\vec{x}}_2$ we have

$$T = m\Big(\|\dot{\vec{x}}\|^2 + \|l(\vec{\omega_I} \times \vec{e_x})\|^2\Big) \tag{2}$$

Since $\vec{x} = r\vec{e_r}$, it follows that

$$\begin{aligned} \dot{\vec{x}} &= \dot{r}\vec{e_r} + r(\vec{\omega_L} \times \vec{e_r}) \\ &= \dot{r}\vec{e_r} + r\dot{\nu}\cos\lambda\vec{e}_\nu + r\dot{\lambda}\vec{e}_\lambda \end{aligned}$$

and

$$\|\dot{\vec{x}}\|^2 = \dot{r}^2 + r^2(\dot{\nu}^2\cos^2\lambda + \dot{\lambda}^2).$$

In the subsequent development, we represent $\vec{\omega}_L$ and $\vec{x}$ in terms of column vectors ω_L and x in $\mathbb{R}^3$ with respect to the basis vectors $\vec{e}_r$, $\vec{e}_\nu$ and $\vec{e}_\lambda$ in the LVLH frame. The notation ω_B in $\mathbb{R}^3$ is used to express the components of the angular velocity vector $\vec{\omega_I}$ in the body-fixed coordinate frame. The standard basis vectors in $\mathbb{R}^3$ are denoted by $e_1 = [1 \;\; 0 \;\; 0]^\top$, $e_2 = [0 \;\; 1 \;\; 0]^\top$ and $e_3 = [0 \;\; 0 \;\; 1]^\top$. We also introduce the rotation matrix, denoted by $R \in \mathrm{SO}(3)$, that maps the representation of a vector in the body-fixed coordinate frame into the representation in the LVLH frame. We use the notation $\widehat{\cdot}: \mathfrak{so}(3) \to \mathfrak{so}(3)$ or $\widehat{(\cdot)}: \mathfrak{so}(3) \to \mathfrak{so}(3)$ to denote the adjoint representation of $\mathfrak{so}(3)$ (identified with $\mathbb{R}^3$), given by

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

This allows us to write

$$\|\vec{\omega_I} \times \vec{e_x}\|^2 = \omega_B^\top \widehat{e_1}^\top \widehat{e_1} \omega_B = \omega_B^\top (I_3 - e_1 e_1^\top)\omega_B,$$

so that the kinetic energy is

$$T = \frac{1}{2} m[\dot{x}_1^\top \dot{x}_1 + \dot{x}_2^\top \dot{x}_2] = [m\dot{x}^\top \dot{x} + \omega_B^\top J \omega_B],$$

where

$$J = ml^2(I_3 - e_1 e_1^\top) \tag{3}$$

is the constant inertia matrix of the dumbbell. The definition of the dumbbell as a rigid connection of two ideal mass particles leads to the fact that $\mathrm{rank}(J) = 2$. The implications of this assumption are discussed in a later section.

We use ω to denote the components of the angular velocity of the body fixed frame relative to the LVLH frame, expressed in the body fixed frame. Thus

$$\omega_B = R^\top \omega_L + \omega$$

and the kinetic energy can be written as

$$T = m[\dot{r}^2 + r^2\dot{\nu}^2 \cos^2\lambda + r^2\dot{\lambda}^2] + (R^\top \omega_L + \omega)^\top J (R^\top \omega_L + \omega). \tag{4}$$

The exact potential energy of the two mass particles that define the dumbbell is

$$-\frac{\mu m}{\|\vec{x_1}\|}-\frac{\mu m}{\|\vec{x_2}\|}$$

where

$$\|\vec{x_1}\| = \sqrt{r^2+2rle_1^\top Re_1+l^2},\ \|\vec{x_2}\| = \sqrt{r^2-2rle_1^\top Re_1+l^2}.$$

In our subsequent analysis, we assume $r > 0$ and $\frac{l}{r} \ll 1$. Since $\frac{2rle_1^\top Re_1+l^2}{r^2} \ll 1$ and $\frac{2rle_1^\top Re_1-l^2}{r^2} \ll 1$, we can use the second order approximation for the gravitational potential energy

$$V_g = -\frac{\mu m}{r}\Big(2-\frac{l^2}{r^2}\big(1-3(e_1^\top Re_1)^2\big)\Big). \tag{5}$$

Note that the potential energy of the dumbbell depends only on the radial position r of the center of mass of the dumbbell and the direction of the dumbbell axis Re_1 in the LVLH frame. The Lagrangian is thus obtained as

$$\begin{aligned}&\mathcal{L}(r,\lambda,R,\dot r,\dot\nu,\dot\lambda,\omega) = T - V_g = m[\dot r^2 + r^2(\dot\nu^2\cos^2\lambda+\dot\lambda^2)]\\ &+(R^\top\omega_L+\omega)^\top J(R^\top\omega_L+\omega)+\frac{\mu m}{r}\Big(2-\frac{l^2}{r^2}\big(1-3(e_1^\top Re_1)^2\big)\Big)\end{aligned} \tag{6}$$

The attitude kinematics of the dumbbell is given by

$$\dot R = R\widehat{\omega}. \tag{7}$$

The orbital equations of motion are given by the ordinary Euler-Lagrange equations obtained from the Lagrangian (6) for these degrees of freedom. The configuration manifold of the system is denoted by $\mathbf{Q}$. The configuration is specified by the translation, represented by the local coordinates (r,ν,λ), and the attitude, represented by the rotation matrix R.

We define

$$f(\lambda) = \sin\lambda e_1 + \cos\lambda e_3, \quad g(\lambda) = \cos\lambda e_1 - \sin\lambda e_3,$$

so that:

$$\frac{\partial\omega_L}{\partial\dot{\nu}} = f(\lambda), \quad \frac{d}{dt}\Big(\frac{\partial\omega_L}{\partial\dot{\nu}}\Big) = \dot{\lambda}g(\lambda), \quad \frac{\partial\omega_L}{\partial\lambda} = \dot{\nu}g(\lambda).$$

The orbital equations of motion can be expressed as

$$\ddot{r} - r\dot{\nu}^2\cos^2\lambda - r\dot{\lambda}^2 + \frac{\mu}{r^2} - \frac{3\mu l^2}{2r^4}(1 - 3(e_1^\top Re_1)^2) = 0, \tag{8}$$

$$m[(r^2\ddot{\nu}\cos\lambda + 2r\dot{r}\dot{\nu}\cos\lambda - 2r^2\dot{\nu}\dot{\lambda}\sin\lambda)\cos\lambda] + \dot{\lambda}g(\lambda)^\top RJ(R^\top\omega_L + \omega) +$$

$$f(\lambda)^\top R\widehat{\omega}J(R^\top\omega_L + \omega) + f(\lambda)^\top RJ(R^\top\dot{\omega}_L + \dot{\omega} - \widehat{\omega}R^\top\omega_L) = 0, \tag{9}$$

$$m[r(r\ddot{\lambda} + 2\dot{r}\dot{\lambda} + r\dot{\nu}^2\sin\lambda\cos\lambda)] - e_2^\top R\widehat{\omega}J(R^\top\omega_L + \omega) + e_2^\top RJ(\widehat{\omega}R^\top\omega_L - R^\top\dot{\omega}_L - \dot{\omega})$$

$$-\dot{\nu}g(\lambda)^\top RJ(R^\top\omega_L + \omega) = 0. \tag{10}$$

In each of these scalar equations, the first set of terms are Keplerian terms expressed in spherical coordinates. The additional terms represent perturbations that arise from the attitude dynamics.

The attitude equations of motion are obtained as a modification of the Euler-Poincaré equations, obtained by applying the variational principle to the Lagrangian (6), as in [11, 12]. If we define the conjugate momentum

$$\Pi = \Big(\frac{\partial\mathcal{L}}{\partial\omega}\Big)^\top = 2J(R^\top\omega_L + \omega),$$

then the attitude equation of motion is given by

$$\dot{\Pi} + (\omega + R^\top\omega_L)\times\Pi - \frac{6\mu ml^2}{r^3}\big(e_1^\top Re_1\big)e_1 \times (R^\top e_1) = 0. \tag{11}$$

Substituting for Π, we obtain the following attitude equation of motion

$$J(\dot{\omega} + R^\top \dot{\omega}_L - \widehat{\omega} R^\top \omega_L) + (\widehat{\omega} + \widehat{R^\top \omega_L}) J(\omega + R^\top \omega_L) - \frac{3\mu m l^2}{r^3}\big(e_1^\top R e_1\big)\widehat{e_1} R^\top e_1 = 0, \quad (12)$$

The derivation of equation (11) is given in Appendix A. This vector equation describes the attitude dynamics including perturbations that arise from the orbit dynamics. In particular, the last term in equation (12) is the familiar gravity gradient term. Equations of motion (8)-(10), along with equation (11) or (12) describe the full dynamics of the system in $\mathtt{T}\mathbf{Q}$.

The total energy

$$\begin{aligned} &E = T + V_g = m[\dot{r}^2 + r^2(\dot{\nu}^2 \cos^2 \lambda + \dot{\lambda}^2)] \\ &+(R^\top \omega_L + \omega)^\top J (R^\top \omega_L + \omega) - \frac{\mu m}{r}\Big(2 - \frac{l^2}{r^2}\big(1 - 3(e_1^\top R e_1)^2\big)\Big) \end{aligned} \quad (13)$$

is conserved along the flow defined by equations (8)-(10) and (12), as shown in Appendix B. Also note that the variable $\nu \in \mathbb{S}$ is a cyclic variable for the Lagrangian (6), and corresponds to a symmetry in the system. This gives rise to the following result.

Proposition 1. *: The conjugate momentum*

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\nu}} = 2mr^2 \dot{\nu} \cos^2 \lambda + 2f(\lambda)^\top R J (R^\top \omega_L + \omega) \quad (14)$$

is conserved along the flow defined by equations (8)-(10) and (12).

It is easy to differentiate p with respect to time, and confirm that $\dot{p} = 0$ is equivalent to equation (9).

The complexity of the above equations reflects the complex coupling that arises between the orbit and attitude degrees of freedom for the physically simple dumbbell body. These equations of motion are especially suited for analysis of the full body dynamics of dumbbell-like asteroids or dumbbell-like spacecraft.

3 Routh Reduction and Reduced Equations of Motion

In this section, we obtain the reduced equations of motion obtained by eliminating the degree of freedom associated with the (cyclic) symmetry variable $\nu \in \mathbb{S}$. Stability analysis of the relative equilibira of the system is done using the reduced dynamics, since they correspond to the equilibria of the reduced dynamics. Let S_p denote the momentum level set in the configuration space of the dumbbell, corresponding to the constant angular momentum value p. The classical Routhian [11, 12] is obtained from the Lagrangian in (6) by the partial Legendre transform

$$\mathcal{R}(r, \lambda, R, \dot{r}, \dot{\lambda}, \omega) = \{L - \dot{\nu}p\}\big|_{S_p},$$

where $\dot{\nu}$ is obtained from (14) for constant p. Carrying out this substitution to eliminate $\dot{\nu}$, we obtain the following expression for the Routhian:

$$\begin{aligned}
&\mathcal{R}(r, \lambda, R, \dot{r}, \dot{\lambda}, \omega) = m[\dot{r}^2 + r^2\dot{\lambda}^2] + (\omega - \dot{\lambda}R^\top e_2)^\top J(\omega - \dot{\lambda}R^\top e_2) \\
&-\big(f(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\big)^2 U_p(r, \lambda, R) + pf(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)U_p(r, \lambda, R) \\
&-V_p(r, \lambda, R), \qquad (15)
\end{aligned}$$

where $V_p(r, \lambda, R)$ is the *amended potential energy* given by

$$V_p(r, \lambda, R) = V_g(r, R) + \frac{p^2}{4}U_p(r, \lambda, R), \qquad (16)$$

and V_g is the gravitational potential expressed as in (5). The function $U_p(r, \lambda, R)$ is given by

$$U_p(r, \lambda, R) = \frac{1}{mr^2\cos^2\lambda + f(\lambda)^\top RJR^\top f(\lambda)}.$$

We assume that $\frac{f(\lambda)^\top RJR^\top f(\lambda)}{mr^2\cos^2\lambda} \ll 1$ and the declination angle λ is bounded away from $\pm\frac{\pi}{2}$ radians. Then we can approximate U_p as

$$U_p(r,\lambda,R) = \frac{1}{mr^2}\Big\{\sec^2\lambda - \frac{1}{mr^2} f(\lambda)^\top RJR^\top f(\lambda)\sec^4\lambda\Big\}. \tag{17}$$

We use this approximation for the function $U_p(r,\lambda,R)$ in equations (15) and (16). The configuration space for the reduced dynamics is $\mathbf{Q}/\mathbb{S}$, and the configuration is represented by (r,λ) for the orbital motion, and the rotation matrix R for the attitude. The equations of motion for the orbital degrees of freedom are obtained by using the Routhian in place of the Lagrangian in the Euler-Lagrange equations of motion. The attitude equations of motion are obtained from the variational principle by substituting the Routhian in place of the Lagrangian.

The orbital equations of motion for the reduced dynamics are obtained as

$$\begin{aligned}
&2m\ddot{r} - 2mr\dot{\lambda}^2 + \big(f(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\big)^2\frac{\partial U_p}{\partial r} - pf(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\frac{\partial U_p}{\partial r}\\
&+\frac{2\mu m}{r^2} - \frac{3\mu m l^2}{r^4}(1 - 3(e_1^\top Re_1)^2) + \frac{p^2}{4}\frac{\partial U_p}{\partial r} = 0,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&m_{\lambda\lambda}(r,\lambda,R)\ddot{\lambda} + m_{\lambda\omega}(r,\lambda,R)\dot{\omega} + \frac{\partial m_{\lambda\lambda}}{\partial r}\dot{r}\dot{\lambda} + \frac{1}{2}\frac{\partial m_{\lambda\lambda}}{\partial\lambda}\dot{\lambda}^2 + m_{\lambda\omega}^R(\lambda,R,\omega)\omega + \frac{\partial m_{\lambda\omega}}{\partial r}\dot{r}\omega\\
&+p\Big\{\frac{\partial m_{p\lambda}}{\partial r}\dot{r} + m_{p\lambda}^R(r,\lambda,R,\omega)\Big\} - \frac{1}{2}\omega^\top\frac{\partial M_{\omega\omega}}{\partial\lambda}\omega - p\frac{\partial m_{p\omega}}{\partial\lambda}\omega + \frac{\partial V_p}{\partial\lambda} = 0,
\end{aligned} \tag{19}$$

where

$$m_{\lambda\lambda}(r,\lambda,R) = 2mr^2 + 2e_2^\top RJR^\top e_2 - 2\big(f(\lambda)^\top RJR^\top e_2\big)^2 U_p(r,\lambda,R),$$

$$m_{\lambda\omega}(r,\lambda,R) = -2e_2^\top RJ + 2U_p(r,\lambda,R)\big(f(\lambda)^\top RJR^\top e_2\big)f(\lambda)^\top RJ,$$

$$M_{\omega\omega}(\lambda,R) = 2J - 2JR^\top f(\lambda)f(\lambda)^\top RJU_p(r,\lambda,R),$$

$$m_{p\lambda}(r,\lambda,R) = f(\lambda)^\top RJR^\top e_2 U_p(r,\lambda,R),$$

$$m_{p\omega}(r,\lambda,R) = f(\lambda)^\top RJU_p(r,\lambda,R),$$

and $a^R(r,\lambda,R,\omega) = \frac{d}{dt}\Big|_{(r,\lambda)} a(r,\lambda,R)$ denotes the time derivative obtained by varying R and holding r and λ constant.

The attitude equations of motion for the reduced system are expressed in terms of

$$\widetilde{\Pi} = \left(\frac{\partial \mathcal{R}}{\partial \omega}\right)^\top = 2J\omega - 2\dot{\lambda}JR^\top e_2 + \big(p - 2f(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\big)JR^\top f(\lambda)U_p.$$

One can verify that

$$\widetilde{\Pi} = \Pi\big|_{S_p}.$$

In terms of this momentum $\widetilde{\Pi}$, the attitude equations of motion are

$$\dot{\widetilde{\Pi}} + (\omega - \dot{\lambda}R^\top e_2)\times\widetilde{\Pi} - \big\{p - 2f(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\big\}\widehat{R^\top f(\lambda)}J(\omega - \dot{\lambda}R^\top e_2)U_p + \mathfrak{v}^\top = 0, \quad (20)$$

where

$$\mathfrak{v} = \frac{6\mu m l^2}{r^3}\big(e_1^\top Re_1\big)e_1^\top R\widehat{e_1} + \frac{p^2}{4m^2r^4}f(\lambda)^\top R\{\big(JR^\top f(\lambda)\big)\widehat{} - J\widehat{R^\top f(\lambda)}\}\sec^4\lambda.$$

The derivation of this equation is provided in Appendix A. Equations (18)-(20) describe the reduced dynamics of the system in $\mathtt{T}(\mathbf{Q}/\mathbb{S})$.

4 Relative Equilibria for the Orbit and Attitude Dynamics

In this section, we study certain dynamics of the orbit and attitude degrees of freedom of the dumbbell. Three categories of relative equilibria are identified. Stability of each relative equilibrium is studied.

We first identify the natural relative equilibria that correspond to circular orbits in a fixed orbital plane for the dumbbell. The relative equilibria are equilibria for the reduced equations, and satisfy :

$$\ddot{r} = \dot{r} = 0, \quad \omega_L = \dot{\nu} e_3, \quad \ddot{\nu} = 0, \quad \dot{\lambda} = 0, \quad \omega = 0.$$

We assume that the inclination of the orbital plane, $\lambda = 0$. We use the subscript 'e' to denote quantities evaluated at a relative equilibrium. Substituting into the reduced equations of motion we obtained in the last section, we see that the relative equilibria are zeros of the gradient of the modified potential, namely:

$$\widetilde{\nabla} V_p \equiv \begin{bmatrix} \frac{\partial V_p}{\partial r} \\ \frac{\partial V_p}{\partial \lambda} \\ \mathfrak{v}^\top \end{bmatrix} = 0, \tag{21}$$

The radial part of the gradient of the modified potential (21) gives:

$$\frac{2\mu m}{r^2} - \frac{3\mu m l^2}{r^4}(1 - 3(e_1^\top R e_1)^2) - \frac{p^2}{2mr_e^3} + \frac{p^2}{m^2 r_e^5} e_3^\top R_e J R_e^\top e_3 = 0, \tag{22}$$

at a relative equilibrium, using (17) to approximate $U_p(r, \lambda, R)$. This can also be expressed in terms of the orbtial rate at the relative equilibrium, $\dot{\nu}_e$. The horizontal equation of motion (9) at a relative equilibrium is trivially satisfied. The second term of equation (21), at a relative equilibrium, gives:

$$2m\dot{\nu}_e^2 e_1^\top R_e J R_e^\top e_3 = \frac{p^2}{2m^2 r_e^4} e_1^\top R_e J R_e^\top e_3 = 0, \tag{23}$$

using (17) to approximate $U_p(r, \lambda, R)$. The third term of equation (21), when evaluated at a relative equilibrium, gives:

$$\frac{6\mu m l^2}{r_e^3}\left(e_1^\top R_e e_1\right)\widehat{e_1} R_e^\top e_1 = \frac{p^2}{2m^2 r_e^4}\widehat{R_e^\top e_3} J R_e^\top e_3, \tag{24}$$

using (17) to approximate $U_p(r, \lambda, R)$. This again, can also be expressed in terms of the orbtial rate at the relative equilibrium, $\dot{\nu}_e$.

Let R_e denote the attitude at a relative equilibrium and $R_e^\top = [u_1\ u_2\ u_3]$, where $u_i^\top u_i = 1$ and $u_i^\top u_j = 0$ for $i \neq j$, $i, j \in \{1, 2, 3\}$. Then substituting for J from equation (3) into equations (23) and (24), we obtain three different conditions for relative equilibria of the dumbbell body in orbit:

(a) $u_{11} = 0$ and $u_{31} = 0$, OR

(b) $u_3 = e_1$, OR

(c) $u_1 = e_1$,

where $u_1 = [u_{11}\ u_{12}\ u_{13}]^\top$ and $u_3 = [u_{31}\ u_{32}\ u_{33}]^\top$. The only rotation matrices that satisfy at least one of these conditions are given by:

$$1.)\ R_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},\quad 2.)\ R_e = \begin{bmatrix} 0 & \cos\alpha & \sin\alpha \\ 1 & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix},$$

$$\text{and } 3.)\ R_e = \begin{bmatrix} 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \\ 1 & 0 & 0 \end{bmatrix},$$

where α is an arbitrary angle which represents rotations about the longitudinal axis of the dumbbell. This gives us three different types of relative equilibria for this body. We now look at how the relative equilibrium conditions (22) simplify at these three types of relative equilibria. Note that, since the longitudinal axis is an axis of symmetry for the dumbbell body, arbitrary rotations about this axis at any relative equilibrium also gives another relative equilibrium of the same type. Also note that due to the equal masses at the ends of the dumbbell body, there is a discrete ($\mathbb{Z}^2$) symmetry. An instantaneous rotation by π radians about an axis perpendicular to the longitudinal axis of the dumbbell does not affect the dynamics. Hence, there are only three relative equilibria, instead of a possible six in the case of the end masses being unequal.

The first type of relative equilibria corresponds to an orientation in which the dumbbell has its longitudinal axis aligned with the local vertical (radial) direction. This class of relative equilibria satisfies:

$$R_e e_1 = e_1, \text{ and } \dot{\nu}_e^2 = \frac{\mu}{r_e^3} + \frac{3\mu l^2}{r_e^5}. \tag{25}$$

The constant angular rate at which the dumbbell revolves around the central body is given by $\dot{\nu}_e$.

The second type of relative equilibria corresponds to the longitudinal axis of the dumbbell being aligned with the local horizontal direction in the plane of the orbit. This class of relative equilibria satisfies:

$$R_e e_1 = e_2, \text{ and } \dot{\nu}_e^2 = \frac{\mu}{r_e^3} - \frac{3\mu l^2}{2r_e^5}, \tag{26}$$

where $\dot{\nu}_e$ is the constant angular rate at which the dumbbell revolves around the central body.

The third type of relative equilibria corresponds to the longitudinal axis of the dumbbell orthogonal to the orbital plane. This class of relative equilibria satisfies:

$$R_e e_1 = e_3, \text{ and } \dot{\nu}_e^2 = \frac{\mu}{r_e^3} - \frac{3\mu l^2}{2r_e^5}, \tag{27}$$

where $\dot{\nu}_e$ is the constant angular rate at which the dumbbell revolves around the central body.

Each of the above relative equilibrium solutions corresponds to a particular attitude of the dumbbell body with respect to the LVLH frame, and an orbital frequency that differs from the Keplerian orbital frequency by a factor dependent on the size of the dumbbell body and its attitude.

4.1 Stability of the Relative Equilibria

A sufficient condition for the stability of a relative equilibrium of the dumbbell is given by the *Routh stability criterion*, which is based on the energy-momentum method (see [11, 12]). This result is based on the reduced dynamics obtained from Routh reduction.

Stability of a relative equilibrium of the dumbbell is expressed in terms of a modification of the Hessian of the amended potential, given by

$$\widetilde{\nabla}^2 V_p(r,\lambda,R) = \begin{bmatrix} \frac{\partial^2 V_p}{\partial r^2} & \frac{\partial^2 V_p}{\partial r \partial\lambda} & \frac{\partial \mathfrak{v}}{\partial r} \\ \frac{\partial^2 V_p}{\partial r \partial\lambda} & \frac{\partial^2 V_p}{\partial \lambda^2} & \frac{\partial \mathfrak{v}}{\partial \lambda} \\ \left(\frac{\partial \mathfrak{v}}{\partial r}\right)^\top & \left(\frac{\partial \mathfrak{v}}{\partial \lambda}\right)^\top & \mathcal{V} \end{bmatrix}, \tag{28}$$

where

$$\begin{aligned} \mathcal{V} &= \frac{6\mu m l^2}{r^3}\Big[\widehat{e_1} R^\top e_1 e_1^\top R \widehat{e_1} - (e_1^\top R e_1)\widehat{e_1}\widehat{R^\top e_1}\Big] + \frac{p^2}{2m^2 r^4}\Big[\widehat{R^\top f(\lambda)} J \widehat{R^\top f(\lambda)} \\ &\quad - \widehat{\big(JR^\top f(\lambda)\big)}\widehat{R^\top f(\lambda)}\Big]\sec^2\lambda. \end{aligned} \tag{29}$$

Note that the rank of the matrix $\mathcal{V}$ is at most two for the inertia matrix J given by (3), which also has rank two, since e_1 is an eigenvector with zero eigenvalue. The computation of this Hessian is shown in Appendix C.

The following theorem is based on the Routh stability criterion.

Theorem 1. *A relative equilibrium is stable if the modification of the Hessian of the amended potential given by (29), evaluated at the relative equilibrium, is positive semi-definite with rank deficiency one. It is unstable if this Hessian has negative eigenvalue(s).*

The kernel of $\widetilde{\nabla}^2 V_p$ has dimension of at least one, and its third row and column are zero. The first statement of the theorem follows from Routh's stability criterion. Note that if the quantity $\widetilde{\nabla}^2 V_p$ evaluated at a relative equilibrium has negative eigenvalues, then the linearization of the reduced dynamics is unstable. Hence the system is formally unstable (see [11], pp. 39-42) in this case.

Using Theorem 1 above, we verify the stability of relative equilibria of the dumbbell, when the axis of the dumbbell is aligned with the local vertical. We have:

$$R_e e_1 = e_1, \quad \dot{\nu}_e^2 = \frac{\mu}{r_e^3} + \frac{3\mu l^2}{r_e^5}, \text{ and } p^2 = 4m^2\Big(\mu r_e + \frac{5\mu l^2}{r_e}\Big).$$

Corollary 1. *The first class of relative equilibria of the dumbbell, where the axis of the dumbbell is aligned with the local vertical, is stable.*

The modified Hessian evaluated at such a relative equilibrium is

$$\widetilde{\nabla}^2 V_p\Big|_1 = \begin{bmatrix} \frac{2\mu m}{r_e^3} - \frac{14\mu m l^2}{r_e^5} & 0 & 0_{1\times 3} \\ 0 & 2m\Big(\frac{\mu}{r_e} + \frac{5\mu l^2}{r_e^3}\Big)\Big(1 - \frac{l^2}{r_e^2}\Big) & -2ml^2\Big(\frac{\mu}{r_e^3} + \frac{5\mu l^2}{r_e^5}\Big)e_2^\top \\ 0_{3\times 1} & -2ml^2\Big(\frac{\mu}{r_e^3} + \frac{5\mu l^2}{r_e^5}\Big)e_2 & 2ml^2\Big(\frac{\mu}{r_e^3} + \frac{5\mu l^2}{r_e^5}\Big)E_1 - \frac{6\mu m l^2}{r_e^3}\widehat{e_1}^2 \end{bmatrix} \tag{30}$$

where

$$E_1 = -\widehat{e_3}^2 - \widehat{e_3}\widehat{e_1}^2\widehat{e_3}.$$

This modified Hessian has one zero eigenvalue (the third row and third column are zeros), and the remaining eigenvalues are always positive since $\frac{l}{r_e} \ll 1$, according to symbolic calculations using Mathematica. This proves that the first class of relative equilibria given by (25), with the axis of the dumbbell aligned with the local radial direction, is stable.

Now we assess the stability of relative equilibria of the dumbbell, when the axis of the dumbbell is aligned with the local horizontal direction in the plane of a circular orbit. For the second class of relative equilibria, we have:

$$R_e e_1 = e_2, \;\; \dot{\nu}_e^2 = \frac{\mu}{r_e^3} - \frac{3\mu l^2}{2r_e^5}, \text{ and } p^2 = 2m^2\Big(2\mu r_e - \frac{\mu l^2}{r_e}\Big).$$

Corollary 2. *The second class of relative equilibria of the dumbbell, where the axis of the dumbbell is aligned with the local horizontal direction in the plane of a circular orbit, is unstable.*

The modified Hessian evaluated at such a relative equilibrium is

$$\widetilde{\nabla}^2 V_p\Big|_2 = \begin{bmatrix} \frac{2\mu m}{r_e^3} - \frac{5\mu m l^2}{r_e^5} & 0 & 0_{1\times 3} \\ 0 & \frac{2\mu m}{r_e} - \frac{3\mu m l^2}{r_e^3} & 0_{1\times 3} \\ 0 & 0 & \Big(\frac{2\mu m l^2}{r_e^3} - \frac{\mu m l^4}{r_e^5}\Big)E_2 - \frac{6\mu m l^2}{r_e^3} e_2 e_2^\top \end{bmatrix}, \tag{31}$$

where

$$E_2 = -\widehat{e_2}^2 - \widehat{e_2}\widehat{e_1}^2\widehat{e_2}.$$

This modified Hessian has one zero eigenvalue (the third row and third column are zeros), and there is a negative eigenvalue, namely $\frac{-6\mu m l^2}{r_e^3}$. Using Theorem 1, we conclude that the second class of relative equilibria given by (26), with the dumbbell axis aligned to the local in-plane horizontal, is unstable.

The stability of the third class of relative equilibria of the the dumbbell, when the axis of the dumbbell is aligned to be orthogonal to the plane of a circular orbit, can be assessed using Theorem 1. For the third class of relative equilibria, we have:

$$R_e e_1 = e_3, \quad \dot{\nu}_e^2 = \frac{\mu}{r_e^3} - \frac{3\mu l^2}{2r_e^5}, \text{ and } p^2 = 2m^2\Big(2\mu r_e - \frac{3\mu l^2}{r_e}\Big).$$

Corollary 3. *The third class of relative equilibria of the dumbbell, with the axis of the dumbbell aligned to be orthogonal to the plane of the circular orbit, is unstable.*

The modified Hessian evaluated at such a relative equilibrium is

$$\widetilde{\nabla}^2 V_p\Big|_3 = \begin{bmatrix} \frac{2\mu m}{r_e^3} + \frac{3\mu m l^2}{r_e^5} & 0 & 0_{1\times 3} \\ 0 & \frac{2\mu m}{r_e} - \frac{5\mu m l^2}{r_e^3} & \Big(\frac{2\mu m l^2}{r_e^3} - \frac{3\mu m l^4}{r_e^5}\Big) e_3^\top \\ 0_{3\times 1} & \Big(\frac{2\mu m l^2}{r_e^3} - \frac{3\mu m l^4}{r_e^5}\Big) e_3 & -\frac{6\mu m l^2}{r_e^3} e_3 e_3^\top - \Big(\frac{2\mu m l^2}{r_e^3} - \frac{3\mu m l^4}{r_e^5}\Big) \widehat{e_1}^4 \end{bmatrix}. \tag{32}$$

This modified Hessian has one zero eigenvalue (the third row and third column are zeros), and the eigenvalue $-\Big(\frac{2\mu m l^2}{r_e^3} - \frac{3\mu m l^4}{r_e^5}\Big)$ is negative since $\frac{l}{r_e} \ll 1$. Using Theorem 1, we conclude that the third class of relative equilibria given by (27) is unstable.

5 Stabilization of Unstable Relative Equilibria

In this section we assume the attitude of the dumbbell body can be controlled through a moment vector expressed in the body fixed coordinate frame. Based on this control assumption, the conjugate momentum corresponding to the cyclic variable ν remains conserved.

Consequently, the reduced equations can be obtained as previously, resulting in

$$
\begin{aligned}
&2m\ddot{r} - 2mr\dot{\lambda}^2 + \big(f(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\big)^2 \frac{\partial U_p}{\partial r} - pf(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\frac{\partial U_p}{\partial r} \\
&+\frac{2\mu m}{r^2} - \frac{3\mu m l^2}{r^4}(1 - 3(e_1^\top Re_1)^2) + \frac{p^2}{4}\frac{\partial U_p}{\partial r} = 0,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&m_{\lambda\lambda}(r,\lambda,R)\ddot{\lambda} + m_{\lambda\omega}(r,\lambda,R)\dot{\omega} + \frac{\partial m_{\lambda\lambda}}{\partial r}\dot{r}\dot{\lambda} + \frac{1}{2}\frac{\partial m_{\lambda\lambda}}{\partial \lambda}\dot{\lambda}^2 + m^R_{\lambda\omega}(\lambda,R,\omega)\omega + \frac{\partial m_{\lambda\omega}}{\partial r}\dot{r}\omega \\
&+p\Big\{\frac{\partial m_{p\lambda}}{\partial r}\dot{r} + m^R_{p\lambda}(r,\lambda,R,\omega)\Big\} - \frac{1}{2}\omega^\top\frac{\partial M_{\omega\omega}}{\partial \lambda}\omega - p\frac{\partial m_{p\omega}}{\partial \lambda}\omega + \frac{\partial V_p}{\partial \lambda} = 0,
\end{aligned}
\tag{34}
$$

and

$$
\dot{\widetilde{\Pi}} + (\omega - \dot{\lambda}R^\top e_2)\times\widetilde{\Pi} - \big\{p - 2f(\lambda)^\top RJ(\omega - \dot{\lambda}R^\top e_2)\big\}\widehat{R^\top f(\lambda)}J(\omega - \dot{\lambda}R^\top e_2)U_p + \mathfrak{v}^\top = \tau, \tag{35}
$$

where τ is the control moment vector. This control moment can be used to influence the attitude dynamics and, indirectly, the orbit dynamics of the dumbbell.

The control moment is used here to stabilize the relative equilibria that, if uncontrolled, would be unstable. The approach is to select the control moment to modify the amended potential so that the unstable relative equilibria are made Lyapunov stable. This approach is referred to as potential shaping. Note that the feedback control moment depends on only attitude feedback.

The idea of potential shaping is not new, and [13] and [14] consider the interesting case of asymptotic stabilization of underactuated Hamiltonian systems. Potential shaping has also been used in conjunction with controlled Lagrangian techniques in [15] to asymptotically stabilize equilibria of Hamiltonian systems. In our application, we use this technique to modify the amended potential to stabilize unstable relative equilibria of the reduced system of the dumbbell in three-dimensional motion in a central gravitational field. The feedback moment maintains the Hamiltonian structure of the system, so the feedback system is also conservative, and we obtain Lyapunov stability, which can be verified by applying the Routh stability criterion (Theorem 1). In addition to the potential shaping attitude feedback control presented here, one may apply Rayleigh dissipation to the system by angular velocity feedback, to make the system asymptotically stable.

5.1 Potential Shaping for Dumbbell in Space

We observe, from equations (31) and (32), that the unstable modes at the unstable relative equilibria (26) and (27) are due to the attitude degrees of freedom only. Therefore, a feedback control law that stabilizes an unstable relative equilibrium may be obtained by adding an artificial potential $V_a(R)$, that depends on the attitude only, so that the Hessian of the total amended potential $V(r, \lambda, R) = V_p(r, \lambda, R) + V_a(R)$, is positive semi-definite with one zero eigenvalue corresponding to the eigenvector representing the axial direction of the dumbbell body in the body frame. This property of the Hessian of the total potential, also ensures that the feedback does not create a moment about this axial direction. The attitude feedback stabilizing control law, $\tau(R)$, is then obtained from the first variation of the artificial potential $V_a(R)$. Note that, this artificial potential does not break the symmetry due to the cyclic variable ν, since it does not depend on it, and hence does not act on the ν dynamics. This is unlike the application in [15], where potential shaping is carried out to break existing symmetries in a mechanical system.

The artificial potential is chosen to be of the form

$$V_a(R) = -\frac{1}{2}c^\top RJR^\top c + \frac{1}{2}ml^2\eta(e_1^\top Re_1)^2, \tag{36}$$

where $c \in \mathbb{R}^3$ is a constant vector and η is a constant non-negative real scalar. Note that the vector c has units of angular velocity, and can be thought of as an "artificial angular velocity" induced by the feedback control. The first term in (36) can therefore be described as an "artificial amended potential." The second term can clearly be described as an "artificial gravity potential," when compared with the natural gravitational potential in (5).

With this choice of artificial potential, the total potential $V(r, \lambda, R) = V_p(r, \lambda, R) + V_a(R)$, has a Hessian whose structure is given by

$$\widetilde{\nabla}^2 V(r, \lambda, R) = \begin{bmatrix} \frac{\partial^2 V_p}{\partial r^2} & \frac{\partial^2 V_p}{\partial r \partial \lambda} & \frac{\partial \mathfrak{v}}{\partial r} \\ \frac{\partial^2 V_p}{\partial r \partial \lambda} & \frac{\partial^2 V_p}{\partial \lambda^2} & \frac{\partial \mathfrak{v}}{\partial \lambda} \\ \left(\frac{\partial \mathfrak{v}}{\partial r}\right)^\top & \left(\frac{\partial \mathfrak{v}}{\partial \lambda}\right)^\top & \mathcal{V} + \mathcal{V}_a \end{bmatrix}, \tag{37}$$

with zeros in the third row and third column, corresponding to a single zero eigenvalue. Here $\mathcal{V}_a$ is the Hessian of the artificial potential, and it is obtained from the second variation of the artificial potential (36). From the given form of the artificial potential (36), we obtain the feedback control moment

$$\tau = \widehat{R^\top c} J R^\top c + ml^2 \eta (e_1^\top R e_1) \widehat{e_1} R^\top e_1, \tag{38}$$

and the Hessian

$$\mathcal{V}_a = \widehat{R^\top c}\big[J \widehat{R^\top c} - \widehat{J R^\top c}\big] + ml^2 \eta \Big[(e_1^\top R e_1) \widehat{e_1} \widehat{R^\top e_1} - \widehat{e_1} R^\top e_1 e_1^\top R \widehat{e_1}\Big]. \tag{39}$$

The derivation of these quantities is shown in Appendix C.

The closed-loop dynamics of the dumbbell in a central gravitational potential is also Hamiltonian. Hence, we can apply the Routh stability criterion (Theorem 1) to the closed-loop dynamics of the dummbbell body. If the quantity $\widetilde{\nabla}^2 V$ evaluated at a relative equilibrium has negative eigenvalues, then the linearization of the closed-loop reduced dynamics is unstable. We now apply Theorem 1 to stabilize the unstable relative equilibria of the dumbbell body in space using attitude feedback.

5.2 Stabilization of Horizontal In-Plane Relative Equilibria

For the unstable relative equilibria given by (26) with the dumbbell axis pointing along the horizontal in-plane direction, we have from (31) for the free dynamics

$$\mathcal{V} = \Big(\frac{2\mu m l^2}{r_e^3} - \frac{\mu m l^4}{r_e^5}\Big)\big(-\widehat{e_2}^2 - \widehat{e_2}\widehat{e_1}^2\widehat{e_2}\big) - \frac{6\mu m l^2}{r_e^3} e_2 e_2^\top.$$

In matrix form,

$$\mathcal{V} = ml^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & q \end{bmatrix}, \quad n = -\frac{6\mu}{r_e^3}, \quad q = \Big(\frac{2\mu}{r_e^3} - \frac{\mu l^2}{r_e^5}\Big). \tag{40}$$

We choose an artificial potential of the form (36) with c given by

$$c = c_1 e_1 + c_3 e_3,$$

where c_1 and c_3 are real scalars, and $\eta = 0$. The control law obtained from this artificial potential using (38) is

$$\tau = R^\top \begin{bmatrix} c_1 \\ 0 \\ c_3 \end{bmatrix} \times JR^\top \begin{bmatrix} c_1 \\ 0 \\ c_3 \end{bmatrix}. \tag{41}$$

The Hessian of the artificial potential, evaluated using (39), is

$$\mathcal{V}_a = \big[\widehat{R^\top c}J - \widehat{JR^\top c}\big]\widehat{R^\top c}, \quad c = [c_1 \ 0 \ c_3]^\top. \tag{42}$$

Evaluated at the relative equilibria given by (26), this Hessian gives

$$\mathcal{V}_a = ml^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_1^2 & -c_1c_3 \\ 0 & -c_1c_3 & c_3^2 \end{bmatrix}. \tag{43}$$

The closed-loop system is obtained by using the feedback control moment (41) as an input to the attitude equation of motion (35) for the reduced dynamics. The following result gives a sufficient condition for the stability of the closed-loop system based on Theorem 1.

Corollary 4. *The second class of relative equilibria of the dumbbell, given by equation (26), is stable with the feedback control moment given by (41) if*

$$nq + nc_3^2 + qc_1^2 > 0 \text{ and } q + n + c_1^2 + c_3^2 > 0, \tag{44}$$

where

$$n = -\frac{6\mu}{r_e^3}, \quad q = \left(\frac{2\mu}{r_e^3} - \frac{\mu l^2}{r_e^5}\right).$$

In this case, one can verify that

$$\mathcal{V} + \mathcal{V}_a \geq 0, \quad \text{and } \mathrm{Ker}(\mathcal{V} + \mathcal{V}_a) = \{e_1\}.$$

This makes the Hessian of the total potential, (37), positive semi-definite with one zero eignevalue, and the result follows. If we choose the specific constants

$$c_1 = \sqrt{\frac{9\mu}{r_e^3}}, \text{ and } c_3 = \sqrt{\frac{\mu l^2}{2r_e^5}}, \tag{45}$$

which satisfy (44), then we obtain a control law from (41) that stabilizes the unstable horizontal in-plane relative equilibrium of the dumbbell body, given by $r = r_e$, $\lambda = 0$, and $R = R_e$ such that $R_e e_1 = e_2$.

5.3 Stabilization of Horizontal Out-of-Plane Relative Equilibria

At the unstable relative equilibria given by (27) with the dumbbell axis pointing along the horizontal out-of-plane direction, the attitude submatrix of the Hessian matrix of the modified potential is given by (32) as

$$\mathcal{V} = -\frac{6\mu m l^2}{r_e^3} e_3 e_3^\top - \Big(\frac{2\mu m l^2}{r_e^3} - \frac{3\mu m l^4}{r_e^5}\Big)\widehat{e_1}^4.$$

In matrix form,

$$\mathcal{V} = ml^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & 0 & n_2 \end{bmatrix}, \quad n_1 = -\Big(\frac{2\mu}{r_e^3} - \frac{3\mu l^2}{r_e^5}\Big), \quad n_2 = -\Big(\frac{8\mu}{r_e^3} - \frac{3\mu l^2}{r_e^5}\Big). \tag{46}$$

We choose an artificial potential of the form (36) with c given by

$$c = c_1 e_1 + c_2 e_2,$$

where c_1 and c_2 are real scalars, and $\eta > 0$. The control law obtained from this artificial potential is obtained using (38) as

$$\tau = R^\top \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} \times J R^\top \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} + ml^2\eta\big(e_1^\top R e_1\big) e_1 \times R^\top e_1. \tag{47}$$

The Hessian of the artificial potential is evaluated using (39) as

$$\begin{aligned} \mathcal{V}_a &= \big[\widehat{R^\top c} J - \widehat{J R^\top c}\big]\widehat{R^\top c} + ml^2\eta\big[\big(e_1^\top R e_1\big)\widehat{R^\top e_1}\widehat{e_1} - \widehat{e_1} R^\top e_1 e_1^\top R \widehat{e_1}\big], \\ c &= [c_1 \; c_2 \; 0]^\top, \; \eta > 0. \end{aligned} \tag{48}$$

Evaluated at the relative equilibria given by (27), this Hessian gives

$$\mathcal{V}_a = ml^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2^2 & -c_1 c_2 \\ 0 & -c_1 c_2 & c_1^2 + \eta \end{bmatrix}. \tag{49}$$

The closed-loop system is obtained by using the feedback control moment (47) as an input to the attitude equation of motion (35) for the reduced dynamics. The following result gives a sufficient condition for the stability of the closed-loop system based on Theorem 1.

Corollary 5. *Assume c_1^2, c_2^2 and η are all of the order of $\frac{\mu}{r_e^3}$. The third class of relative equilibria of the dumbbell, given by equation (27), is stable with the feedback control moment given by (47) if*

$$n_1 n_2 + n_1 c_1^2 + n_2 c_2^2 + \eta(n_1 + c_2^2) > 0 \text{ and } n_1 + n_2 + c_1^2 + c_2^2 + \eta > 0, \tag{50}$$

where

$$n_1 = -\Big(\frac{2\mu}{r_e^3} - \frac{3\mu l^2}{r_e^5}\Big), \quad n_2 = -\Big(\frac{8\mu}{r_e^3} - \frac{3\mu l^2}{r_e^5}\Big).$$

In this case, both the modes obtained from the attitude degrees of freedom at relative equilibria given by (27) are unstable. With the feedback control torque given by (47), we can verify that

$$\mathcal{V} + \mathcal{V}_a \geq 0, \quad \text{and } \mathrm{Ker}(\mathcal{V} + \mathcal{V}_a) = \{e_1\}.$$

The 3×3 submatrix of the Hessian (37) of the feedback system, obtained by eliminating the first and third rows and columns, given by

$$ml^2 \begin{bmatrix} p & 0 & -n_1 \\ 0 & n_1 + c_2^2 & 0 \\ -n_1 & 0 & n_3 + c_1^2 + \eta \end{bmatrix},$$

where $p = \frac{2\mu}{r_e l^2} - \frac{5\mu}{r_e^3}$, is positive definite, which makes the Hessian (37) positive semi-definite with one zero eigenvalue. If we make the specific choices

$$c_1 = \sqrt{\frac{\mu}{r_e^3}}, \quad c_2 = \sqrt{\frac{3\mu}{r_e^3}} \text{ and } \eta = \frac{12\mu}{r_e^3}, \tag{51}$$

which satisfy (50), we obtain a control law from (47) that stabilizes the unstable horizontal out-of-plane relative equilibrium of the dumbbell body, given by $r = r_e$, $\lambda = 0$, and $R = R_e$ such that $R_e e_1 = e_3$.

6 Conclusions

We have extended some of the results of our earlier work, which treat the dynamics of a dumbbell-shaped body in planar motion in a central gravitational field, to motion in three dimensional space. The system of the dumbbell body in three-dimensional motion in a central gravity field consists of three orbital degrees of freedom and two attitude degrees of freedom, since the inertia about the longitudinal axis of the dumbbell is ignored. We represent the orbital degrees of freedom using spherical coordinates, defined by the Local Vertical Local Horizontal (LVLH) coordinates; the attitude is represented globally by a rotation matrix from a body-fixed coordinate frame to the LVLH frame. We obtain the equations of motion representing the full orbit and attitude dynamics.

We obtain the equations of motion representing the reduced dynamics using Routh reduction. The reduced system has four degrees of freedom; the orbit degrees of freedom are represented by the radial distance and the angle of declination. The attitude is represented by the rotation matrix from the body-fixed frame to the LVLH frame. We obtain the relative equilibria, which correspond to local extrema of the modified potential for the reduced dynamics. These relative equilibria correspond to circular orbits, with fixed orbital rate and fixed attitude.

Since the two end masses of the dumbbell model are equal, the system also has a discrete symmetry. This gives rise to three types of relative equilibria: one in which the dumbbell axis is aligned with the radial (local vertical) direction, another in which the axis is aligned with the local horizontal direction in the plane of the circular orbit, and a third in which the axis is aligned with the local horizontal direction out of the plane of the orbit. The first two types are identical to those obtained for the dumbbell in planar motion, dealt with in our previous work. We analyze the stability of these three types of relative equilibria using the Routh stability criterion. The first type of relative equilibria is found to be locally (Lyapunov) stable, while the other two types of relative equilibria are unstable.

In the final part of the paper, we use attitude feedback control based on potential shaping to stabilize the unstable relative equilibria of the dumbbell body. This is based on the fact that the unstable modes at the unstable relative equilibria are due to the attitude, rather than the orbital degrees of freedom. Hence, potential shaping with attitude feedback is adequate for stabilizing these relative equilibria. To do this, we create an artificial potential depending on the attitude, that is similar to the modified potential of the natural reduced dynamics of the dumbbell in central gravity. This artificial potential has two terms, one of which is similar to the gravity potential, and the other is similar to the amendment in the modified potential. The feedback torques for stabilization of an unstable relative equilibrium are obtained by computing the first variation of this artificial potential with respect to the attitude. The stability of the feedback controlled system is analyzed by applying the Routh stability criterion to the the Hessian of the total potential, which is the sum of the modified and artificial potentials, at that relative equilibrium. We find that to stabilize the unstable relative equilibria where the axis is aligned with the local horizontal direction in the plane of the circular orbit, we need to use feedback control based only the term of the artificial potential that is similar to the amendment. However, to stabilize the unstable relative equilibria where the axis is aligned with the local horizontal direction out of the plane of the circular orbit, we need to use feedback control based on both terms of the artificial potential.

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Appendices

Appendix A

Here we show the derivation of the attitude equations of motion for the full and reduced dynamics of the dumbbell body in central gravity. We define the quantity $\Sigma \in \mathfrak{so}(3)$ (given in [11, 12]) so that the attitude and angular velocity variations are

$$\delta R = R\widehat{\Sigma}, \quad \delta\omega = \dot{\Sigma} + \widehat{\omega}\Sigma.$$

We then apply standard variational arguments to the Lagrangian of the full dynamics, assuming zero initial and final values of Σ. This leads to the equation

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\omega}\Big) = \frac{\partial\mathcal{L}}{\partial\omega}\widehat{\omega} + \mathfrak{r}, \tag{A}$$

where $\mathfrak{r}$ is the $\mathfrak{so}(3)$-valued one-form obtained such that

$$-\frac{1}{2}\langle\langle \mathfrak{r}, \Sigma \rangle\rangle$$

is the variation of the Lagrangian with respect to the rotation matrix R, holding other quantities constant. Here $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the Killing form in $\mathfrak{so}(3)$ [17, 18], given by

$$\langle\langle \mathfrak{a}, \mathfrak{c} \rangle\rangle = \mathrm{trace}\big(\ \widehat{\mathfrak{a}}\ \widehat{\mathfrak{c}}\ \big).$$

We denote one-forms like $\mathfrak{r} \in \mathfrak{so}(3)^\star$ by row vectors, to distinguish them from elements in $\mathfrak{so}(3)$, which are denoted by column vectors. Substituting the Lagrangian $\mathcal{L}$ given by equation (6) into equation (A), and defining the conjugate momentum $\Pi = \left(\frac{\partial\mathcal{L}}{\partial\omega}\right)^\top$, we obtain equation (11) for the attitude dynamics of the dumbbell body.

The reduced equations of motion are obtained by applying standard variational techniques to the Routhian, instead of the Lagrangian. Hence, we obtain the following equation, which is similar to equation (A):

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{R}}{\partial \omega}\Big) = \frac{\partial \mathcal{R}}{\partial \omega}\widehat{\omega} + \mathfrak{s}, \tag{B}$$

where $\mathfrak{s}$ is the $\mathfrak{so}(3)$-valued one-form obtained such that $-\frac{1}{2}\langle\langle \mathfrak{s}, \Sigma\rangle\rangle$ is the variation of the Routhian with respect to the rotation matrix R, holding other quantities constant. The one-form $\mathfrak{s}$ for the Routhian $\mathcal{R}$ given by equation (15), is expressed as the row vector

$$\begin{aligned}
\mathfrak{s} \;&=\; -2\dot\lambda(\omega-\dot\lambda R^\top e_2)^\top J\widehat{R^\top e_2} - \big\{p-2f(\lambda)^\top RJ(\omega-\dot\lambda R^\top e_2)\big\}\big\{(\omega-\dot\lambda R^\top e_2)^\top J\widehat{R^\top f(\lambda)} \\
&\qquad -\dot\lambda f(\lambda)^\top RJ\widehat{R^\top e_2}\big\}U_p - \frac{6\mu m l^2}{r^3}\big(e_1^\top Re_1\big)e_1^\top R\widehat{e_1} - \frac{p^2}{4}\mathfrak{u} \\
&= -\dot\lambda\widetilde{\Pi}^\top\widehat{R^\top e_2} - \big\{p-2f(\lambda)^\top RJ(\omega-\dot\lambda R^\top e_2)\big\}(\omega-\dot\lambda R^\top e_2)^\top J\widehat{R^\top f(\lambda)}U_p - \mathfrak{v},
\end{aligned}$$

where $\widetilde{\Pi} = \Big(\frac{\partial\mathcal{R}}{\partial\omega}\Big)^\top$ is the restriction of the angular momentum Π to the conjugate momentum level set S_p, and $\mathfrak{u}$ and $\mathfrak{v}$ are $\mathfrak{so}(3)$-valued one-forms, with

$$\mathfrak{v} = \frac{6\mu m l^2}{r^3}\big(e_1^\top Re_1\big)e_1^\top R\widehat{e_1} + \frac{p^2}{4}\mathfrak{u}, \quad \mathfrak{u} = \frac{1}{m^2r^4}f(\lambda)^\top R\big\{\big(JR^\top f(\lambda)\big)\widehat{\ } - J\widehat{R^\top f(\lambda)}\big\}\sec^4\lambda$$

as defined in Section 3, and $-\frac{1}{2}\langle\langle \mathfrak{v}, \Sigma\rangle\rangle$ is the variation of the amended potential V_p with respect to R. Substituting the Routhian $\mathcal{R}$ given by equation (15) into equation (B), and using the above expression for $\mathfrak{s}$, we obtain equation (20) for the attitude dynamics of the dumbbell body in terms of $\widetilde{\Pi} = \Big(\frac{\partial\mathcal{R}}{\partial\omega}\Big)^\top$.

Appendix B

To show that the total energy for the dumbbell system in central gravity, given by (13), is conserved, we evaluate its time derivative along the flow of the system. We write the energy expression again as

$$E = m[\dot r^2 + r^2(\dot\nu^2\cos^2\lambda + \dot\lambda^2)] + (R^\top\omega_L+\omega)^\top J(R^\top\omega_L+\omega) - \frac{\mu m}{r}\Big(2 - \frac{l^2}{r^2}\big(1-3(e_1^\top Re_1)^2\big)\Big).$$

We have

$$\omega_B = R^\top \omega_L + \omega.$$

The time derivative of E is then given by

$$\begin{aligned}
\frac{dE}{dt} &= m[2\dot{r}\ddot{r} + 2r\dot{r}(\dot{\nu}^2\cos^2\lambda + \dot{\lambda}^2) + 2r^2(\dot{\nu}\ddot{\nu}\cos^2\lambda + \dot{\lambda}\ddot{\lambda} - \dot{\nu}^2\dot{\lambda}\cos\lambda\sin\lambda)] + 2\omega_B J\dot{\omega}_B \\
&+\frac{\mu m}{r^2}\dot{r}\Big(2 - \frac{l^2}{r^2}\big(1 - 3(e_1^\top R e_1)^2\big)\Big) - \frac{2\mu m l^2 \dot{r}}{r^4}\big(1 - 3(e_1^\top R e_1)^2\big) - \frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big)e_1^\top R\widehat{\omega}e_1 \\
&= 2m\dot{r}\Big[\ddot{r} - r(\dot{\nu}^2\cos^2\lambda + \dot{\lambda}^2) + \frac{\mu}{r^2} - \frac{3\mu l^2}{r^4}\big(1 - 3(e_1^\top R e_1)^2\big)\Big] + 2m\dot{\nu}[r^2\ddot{\nu}\cos^2\lambda \\
&+2r\dot{r}\dot{\nu}\cos^2\lambda - 2r^2\dot{\nu}\dot{\lambda}\sin\lambda\cos\lambda] + 2m\dot{\lambda}[r^2\ddot{\lambda} + 2r\dot{r}\dot{\lambda} + r^2\dot{\nu}^2\sin\lambda\cos\lambda] + 2\omega_B J\dot{\omega}_B \\
&+\frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big)e_1^\top R\widehat{e_1}\omega. \qquad \text{(C)}
\end{aligned}$$

On substituting the equations of motion (8), (9), and (10) into the right hand side of equation (C), we obtain

$$\begin{aligned}
\frac{dE}{dt} &= -2\dot{\nu}[\dot{\lambda}g(\lambda)^\top RJ\omega_B + f(\lambda)^\top R\widehat{\omega}J\omega_B + f(\lambda)^\top RJ\dot{\omega_B}] - 2\dot{\lambda}[-e_2^\top R\widehat{\omega}J\omega_B - e_2^\top RJ\dot{\omega}_B \\
&\quad -\dot{\nu}g(\lambda)^\top RJ\omega_B] + 2\omega_B^\top J\dot{\omega}_B - \frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big)\omega^\top\widehat{e_1}R^\top e_1 \\
&= -2(\dot{\nu}f(\lambda) - \dot{\lambda}e_2)^\top R\widehat{\omega}J\omega_B - 2(\dot{\nu}f(\lambda) - \dot{\lambda}e_2)^\top RJ\dot{\omega}_B + 2\omega_B^\top J\dot{\omega}_B \\
&\quad -\frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big)\omega^\top\widehat{e_1}R^\top e_1 \\
&= 2\omega_B^\top J\dot{\omega}_B - 2\omega_L^\top R(\widehat{\omega}J\omega_B + J\dot{\omega}_B) - \frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big)\omega^\top\widehat{e_1}R^\top e_1 \\
&= 2\omega_B^\top J\dot{\omega}_B + 2\omega^\top\widehat{R^\top\omega_L}J\omega_B - \frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big)\omega^\top\widehat{e_1}R^\top e_1 \\
&= 2\omega^\top\Big(J\dot{\omega}_B + (\widehat{\omega} + \widehat{R^\top\omega_L})J\omega_B - \frac{3\mu m l^2}{r^3}\big(e_1^\top R e_1\big)\widehat{e_1}R^\top e_1\Big) \\
&= 0,
\end{aligned}$$

using equation (12) at the last step.

Appendix C

Here we obtain equations (28) and (29), which give the Hessian of the amended potential of the reduced dynamics, as well as the equations for feedback torque (38) and Hessian (39) obtained from the artificial potential. The top left 2×2 submatrix of the Hessian (28) is obtained from the second pertial derivatives and mixed derivative of the amended potential with respect to the coordinates r and λ. In Appendix A, we obtained the one-form $\mathfrak{v}$ from the first variation of the amended potential $V_p(r, \lambda, R)$. For convenience, we write down this expression again

$$\mathfrak{v} = \frac{6\mu m l^2}{r^3}\big(e_1^\top R e_1\big) e_1^\top R \widehat{e_1} + \frac{p^2}{4}\frac{1}{m^2 r^4} f(\lambda)^\top R\big\{\big(JR^\top f(\lambda)\big)^\wedge - J\widehat{R^\top f(\lambda)}\big\}\sec^4\lambda.$$

The partial derivatives of $\mathfrak{v}$ with respect to r and λ give the (1,3), (3,1), (2,3) and (3,2) blocks of the Hessian matrix in (28). The (3,3) block, which is obtained from the second variation of V_p with respect to the attitude R, is given by the matrix $\mathcal{V}$, such that

$$\langle\langle \mathcal{V}\Sigma, \Sigma\rangle\rangle = -2V_p^{R^2}(r, \lambda, R, \Sigma), \tag{D}$$

and $V_p^{R^2}(r, \lambda, R, \Sigma)$ is the second variation of V_p holding r and λ constant and varying R. The quantity $\mathcal{V}$ is also given by

$$\Sigma^\top \mathcal{V} = \mathfrak{v}^R(r, \lambda, R, \Sigma),$$

the first variation of $\mathfrak{v}$ with respect to R. For the amended potential V_p given by (16), we can use (D) or the above result to evaluate $\mathcal{V}$. We then obtain $\mathcal{V}$ as given by equation (29).

The artificial potential is given by (36), which we write down below for convenience

$$V_a(R) = -\frac{1}{2}c^\top R J R^\top c + \frac{1}{2} m l^2 \eta (e_1^\top R e_1)^2.$$

The first variation of this gives the feedback torque τ as follows:

$$
\begin{aligned}
V_a^R(R,\Sigma) &= \frac{1}{2}\Big[c^\top RJ\widehat{\Sigma}R^\top c - c^\top R\widehat{\Sigma}JR^\top c\Big] + ml^2\eta(e_1^\top Re_1)e_1^\top R\widehat{\Sigma}e_1 \\
&= \frac{1}{2}\Big[\Sigma^\top \widehat{R^\top c}JR^\top c - c^\top RJ\widehat{R^\top c}\Sigma\Big] - ml^2\eta(e_1^\top Re_1)e_1^\top R\widehat{e_1}\Sigma \\
&= -c^\top RJ\widehat{R^\top c}\Sigma - ml^2\eta(e_1^\top Re_1)e_1^\top R\widehat{e_1}\Sigma = -\frac{1}{2}\langle\langle\tau,\Sigma\rangle\rangle \\
&\Rightarrow\ \tau = \widehat{R^\top c}JR^\top c + ml^2\eta(e_1^\top Re_1)\widehat{e_1}R^\top e_1,
\end{aligned}
$$

as given in equation (38). The second variation of the artificial potential (36) gives the Hessian $\mathcal{V}_a$ in (39), as follows:

$$
\begin{aligned}
V_a^{R^2}(R,\Sigma) &= -\frac{1}{2}\langle\langle\mathcal{V}_a\Sigma,\Sigma\rangle\rangle \\
&= c^\top RJ\widehat{\widehat{\Sigma}R^\top c} - c^\top R\widehat{\Sigma}J\widehat{R^\top c} + ml^2\eta\Big[e_1^\top R\widehat{\Sigma}e_1(\widehat{e_1}R^\top e_1) - (e_1^\top Re_1)\widehat{e_1}\widehat{\Sigma}R^\top e_1\Big] \\
&= c^\top R\widehat{\Sigma}\widehat{JR^\top c} + \Sigma^\top\widehat{R^\top c}J\widehat{R^\top c} + ml^2\eta\Sigma^\top\Big[(e_1^\top Re_1)\widehat{e_1}\widehat{R^\top e_1} - \widehat{e_1}R^\top e_1e_1^\top R\widehat{e_1}\Big] \\
&= \Sigma^\top\widehat{R^\top c}\widehat{JR^\top c} + \Sigma^\top\widehat{R^\top c}J\widehat{R^\top c} + ml^2\eta\Sigma^\top\Big[(e_1^\top Re_1)\widehat{e_1}\widehat{R^\top e_1} - \widehat{e_1}R^\top e_1e_1^\top R\widehat{e_1}\Big] \\
&\Rightarrow\ \mathcal{V}_a = \widehat{R^\top c}\widehat{JR^\top c} + \widehat{R^\top c}J\widehat{R^\top c} + ml^2\eta\Big[(e_1^\top Re_1)\widehat{e_1}\widehat{R^\top e_1} - \widehat{e_1}R^\top e_1e_1^\top R\widehat{e_1}\Big].
\end{aligned}
$$